Hindawi Publishing Corporation
ISRN Combinatorics
Volume 2013, Article ID 363724, 7 pages
http://dx.doi.org/10.1155/2013/363724

*Research Article*

# An Extension of a Congruence by Tauraso

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Received 23 September 2012; Accepted 10 October 2012

Academic Editors: C. da Fonseca and B. Wu



For a positive integer $n$ let $H_n = \sum_{k=1}^{n} 1/n$ be the $n$th harmonic number. In this paper we prove that, for any prime $p \geq 7$, $\sum_{k=1}^{p-1} H_k/k^2 \equiv \sum_{k=1}^{p-1} H_k^2/k \equiv (3/2p)\sum_{k=1}^{p-1} 1/k^2 (\mathrm{mod}\, p^2)$. Notice that the first part of this congruence is proposed in 2008 by Tauraso. In our elementary proof of the second part of the above congruence we use certain classical congruences modulo a prime and the square of a prime, some congruences involving harmonic numbers, and a combinatorial identity due to Hernández. Our auxiliary results contain many interesting combinatorial congruences involving harmonic numbers.

## 1. Introduction and Main Results

Given positive integers $n$ and $m$, the *harmonic numbers of order m* are those rational numbers $H_{n,m}$ defined as

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{1}$$

For simplicity, we will denote by

$$H_n := H_{n,1} = \sum_{k=1}^{n} \frac{1}{k} \tag{2}$$

the $n$th *harmonic number* (we assume in addition that $H_{0,m} = H_0 = 0$).

Usually, here as always in the sequel, we consider the congruence relation modulo a prime $p$ extended to the ring of rational numbers with denominators not divisible by $p$. For such fractions we put $m/n \equiv r/s \pmod{p}$ if and only if $ms \equiv nr \pmod{p}$, and the residue class of $m/n$ is the residue class of $mn'$ where $n'$ is the inverse of $n$ modulo $p$.

By a problem proposed by Tauraso in [1] and recently solved by Tyler [2], for any prime $p \geq 7$,

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k} \left(\mathrm{mod}\, p^2\right). \tag{3}$$

Further, Tauraso [3, Theorem 2.3] proved

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv -\frac{3}{p^2} \sum_{k=1}^{p-1} \frac{1}{k} \left(\mathrm{mod}\, p^2\right). \tag{4}$$

Tauraso's proof of (4) is based on an identity due to Hernández [4] (see Lemma 8) and the congruence for triple harmonic sum modulo a prime due to Zhao [5] (see (64) of Remarks in Section 2). In this paper, we give an elementary proof of (4) and its extension as follows.

**Theorem 1**. *If $p \geq 7$ is a prime, then*

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv -\frac{3}{p^2} \sum_{k=1}^{p-1} \frac{1}{k} \equiv \frac{3}{2p} \sum_{k=1}^{p-1} \frac{1}{k^2} \left(\mathrm{mod}\, p^2\right). \tag{5}$$

Recall that Sun in [6] established basic congruences modulo a prime $p \geq 5$ for several sums of terms involving harmonic numbers. In particular, Sun established $\sum_{k=1}^{p-1} H_k^r \pmod{p^{4-r}}$ for $r = 1, 2, 3$. Further generalizations of these congruences are recently obtained by Tauraso in [7].

Recall that the *Bernoulli numbers* $B_k$ are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}. \tag{6}$$

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for odd $n \geq 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$. Applying a congruence given in [8, Theorem 5.1(a)] related to the sum $\sum_{k=1}^{p-1} 1/k^2$ modulo $p^3$, the congruence (5) in terms of Bernoulli numbers may be written as follows.

**Corollary 2**. *Let $p \geq 7$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv 3\left(\frac{B_{2p-4}}{2p-4} - \frac{2B_{p-3}}{p-3}\right)\left(\bmod p^2\right). \quad (7)$$

*In particular, one has*

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv B_{p-3}\left(\bmod p\right). \quad (8)$$

*Remark 3.* Notice that the second congruence of (8) was obtained by Sun and Tauraso [9, the congruence (5.4)] by using a standard technique expressing sum of powers in terms of Bernoulli numbers.

Our proof of the second part of the congruence (5) given in the next section is entirely elementary and it is combinatorial in spirit. It is based on certain classical congruences modulo a prime and the square of a prime, two simple congruences given by Sun [6], and two particular cases of a combinatorial identity due to Hernández [4].

## 2. Proof of Theorem 1

The following congruences given by Sun in his recent paper [6] are needed in the proof of Theorem 1.

**Lemma 4**. *Let $p \geq 7$ be a prime. Then*

$$H_{p-k} \equiv H_{k-1}\left(\bmod p\right), \quad (9)$$

$$(-1)^k \binom{p-1}{k} \equiv 1 - pH_k + \frac{p^2}{2}\left(H_k^2 - H_{k,2}\right)\left(\bmod p^3\right) \quad (10)$$

*for every $k = 1, 2, \ldots, p-1$.*

*Proof.* The congruences (9) and (10) are in fact the congruences (2.1) and (2.2) in [6, Lemma 2.1], respectively. □

The following well-known result is a generalization of *Wolstenholme's theorem* (see, e.g., [10, Theorem 1] or [11]).

**Lemma 5** (see [12, Theorem 3]). *Let $m$ be a positive integer, and let $p$ be a prime such that $p \geq m+3$. Then*

$$H_{p-1,m} \equiv \begin{cases} 0\left(\bmod p\right) & \text{if } m \text{ is even} \\ 0\left(\bmod p^2\right) & \text{if } m \text{ is odd.} \end{cases} \quad (11)$$

*In particular, for any prime $p \geq 5$,*

$$H_{p-1} \equiv 0\left(\bmod p^2\right) \quad (\textit{Wolstenholme's theorem}), \quad (12)$$

*and for any prime $p \geq 7$, $H_{p-1,3} \equiv 0(\bmod p^2)$ and $H_{p-1,2} \equiv H_{p-1,4} \equiv 0(\bmod p)$.*

**Lemma 6**. *Let $p \geq 7$ be a prime. Then*

$$\sum_{k=1}^{n} \frac{H_{k-1}}{k^3} \equiv \sum_{k=1}^{n} \frac{H_k}{k^3} \equiv 0\left(\bmod p\right), \quad (13)$$

$$\sum_{k=1}^{p-1} \frac{H_k^3}{k} \equiv \frac{3}{2}\sum_{k=1}^{p-1} \frac{H_k^2}{k^2}\left(\bmod p\right), \quad (14)$$

$$\sum_{k=1}^{n} \frac{H_{k-1,3}}{k} \equiv \sum_{k=1}^{n} \frac{H_{k,3}}{k} \equiv 0\left(\bmod p\right), \quad (15)$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k}\left(\bmod p^2\right). \quad (16)$$

*Proof.* By the congruence (9) from Lemma 4, $H_k \equiv H_{p-k-1}\ (\bmod p)$ for each $k = 1, 2, \ldots, p-1$ (notice that this is true for $k = p-1$ because $p \mid H_{p-1}$), and therefore

$$\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^3} = \sum_{k=1}^{p-1} \frac{H_{(p-k)-1}}{(p-k)^3} \equiv \sum_{k=1}^{p-1} \frac{H_k}{(p-k)^3} \equiv -\sum_{k=1}^{p-1} \frac{H_k}{k^3}\left(\bmod p\right). \quad (17)$$

Furthermore, using (11) with $m = 4$ we get

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_{k-1}}{k^3} &= \sum_{k=1}^{p-1} \frac{\left(H_k - (1/k)\right)}{k^3} \\ &= \sum_{k=1}^{p-1} \frac{H_k}{k^3} - \sum_{k=1}^{p-1} \frac{1}{k^4} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^3}\left(\bmod p\right). \end{aligned} \quad (18)$$

From (17) and (18) it follows that

$$\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^3} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^3} \equiv 0\left(\bmod p\right), \quad (19)$$

which is actually (13).

Since $H_k = H_{k-1} + 1/k$, for each $k = 1, 2, \ldots, p-1$, we have

$$\begin{aligned} \frac{H_k^3}{k} - \frac{H_{k-1}^3}{k} &= \frac{1}{k}\left(H_k - H_{k-1}\right) \\ &\quad \times \left(H_k^2 + H_k\left(H_k - \frac{1}{k}\right) + \left(H_k - \frac{1}{k}\right)^2\right) \\ &= \frac{1}{k^2}\left(3H_k^2 - 3\frac{H_k}{k} + \frac{1}{k^2}\right) = 3\frac{H_k^2}{k^2} - 3\frac{H_k}{k^3} + \frac{1}{k^4}. \end{aligned} \quad (20)$$

The above identity and (13) and (11) of Lemma 5 with $m = 4$ yield

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^3}{k} - \sum_{k=1}^{p-1} \frac{H_{k-1}^3}{k} &= 3\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} - 3\sum_{k=1}^{p-1} \frac{H_k}{k^3} + \sum_{k=1}^{p-1} \frac{1}{k^4} \\ &\equiv 3\sum_{k=1}^{p-1} \frac{H_k^2}{k^2}\left(\bmod p\right). \end{aligned} \quad (21)$$

On the other hand, since by (9) from Lemma 4, $H_k \equiv H_{p-k-1} \pmod{p}$ for each $k = 1, 2, \ldots, p-1$, then

$$\sum_{k=1}^{p-1} \frac{H_{k-1}^3}{k} = \sum_{k=1}^{p-1} \frac{H_{(p-k)-1}^3}{p-k} \equiv \sum_{k=1}^{p-1} \frac{H_k^3}{p-k} \equiv -\sum_{k=1}^{p-1} \frac{H_k^3}{k} \pmod{p}. \tag{22}$$

Taking (22) into (21) gives

$$\sum_{k=1}^{p-1} \frac{H_k^3}{k} \equiv \frac{3}{2} \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \pmod{p}, \tag{23}$$

which proves (14).

Proof of the congruence (15) is completely analogous to the previous proof using the fact that, by Lemma 5, $H_{p-1,3} \equiv 0 \pmod{p}$ and therefore, for each $k = 1, \ldots, p-1$

$$\begin{aligned} H_{k-1,3} &= \sum_{i=1}^{k-1} \frac{1}{i^3} = H_{p-1,3} - \sum_{j=1}^{p-k} \frac{1}{(p-j)^3} \\ &\equiv \sum_{j=1}^{p-k} \frac{1}{j^3} = H_{p-k,3} \pmod{p}. \end{aligned} \tag{24}$$

Finally (cf. [2]), from the identity

$$H_{k-1}^3 = \left(H_k - \frac{1}{k}\right)^3 = H_k^3 - 3\frac{H_k^2}{k} + 3\frac{H_k}{k^2} - \frac{1}{k^3} \tag{25}$$

immediately follows that

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} = \frac{1}{3}\left(H_{p-1}^3 - H_{p-1,3}\right). \tag{26}$$

Inserting in the right hand side of the identity (26) the congruences $H_{p-1} \equiv H_{p-1,3} \equiv 0 \pmod{p^2}$ given in Lemma 5, we immediately obtain (16). This completes the proof. □

**Lemma 7**. *Let $p \geq 7$ be a prime. Then*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k \cdot H_{k,2}}{k} &\equiv \sum_{1 \leq i < j < k \leq p-1} \frac{1}{ij^2k} \\ &\quad + \sum_{1 \leq i < j < k \leq p-1} \frac{1}{i^2jk} \pmod{p}. \end{aligned} \tag{27}$$

*Proof.* Since $H_k = H_{k-1} + 1/k$, for every $k = 1, 2, \ldots, p-1$, we get

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k \cdot H_{k,2}}{k} &= \sum_{k=1}^{p-1} \frac{1}{k}\left(H_{k-1} + \frac{1}{k}\right)\left(H_{k-1,2} + \frac{1}{k^2}\right) \\ &= \sum_{k=1}^{p-1} \frac{H_{k-1} \cdot H_{k-1,2}}{k} + \sum_{k=1}^{p-1} \frac{H_{k-1,2}}{k^2} \\ &\quad + \sum_{k=1}^{p-1} \frac{H_{k-1}}{k^3} + \sum_{k=1}^{p-1} \frac{1}{k^4}. \end{aligned} \tag{28}$$

Using particular congruences given in Lemma 5 with $m = 2$ and $m = 4$, we find that

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_{k-1,2}}{k^2} &= \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{i=1}^{k-1} \frac{1}{i^2} = \sum_{1 \leq i < k \leq p-1} \frac{1}{i^2k^2} \\ &= \frac{1}{2}\left(\sum_{k=1}^{p-1} \frac{1}{k^2}\right)^2 - \frac{1}{2}\sum_{k=1}^{p-1} \frac{1}{k^4} \equiv 0 \pmod{p}. \end{aligned} \tag{29}$$

Substituting the congruences (29), (13) of Lemma 6, and (11) with $m = 4$ of Lemma 5 into (28), we obtain

$$\sum_{k=1}^{p-1} \frac{H_k \cdot H_{k,2}}{k} \equiv \sum_{k=1}^{p-1} \frac{H_{k-1} \cdot H_{k-1,2}}{k} \pmod{p}. \tag{30}$$

The right hand side of (30) can be expressed as

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_{k-1} \cdot H_{k-1,2}}{k} &= \sum_{k=1}^{p-1} \frac{1}{k}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k-1}\right) \\ &\quad \times \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(k-1)^2}\right) = \sum_{k=1}^{p-1} \frac{1}{k} \\ &\quad \times \left(\sum_{1 \leq i < j \leq k-1} \frac{1}{ij^2} + \sum_{1 \leq i < j \leq k-1} \frac{1}{i^2j} + \sum_{i=1}^{k-1} \frac{1}{i^3}\right) \\ &= \sum_{1 \leq i < j < k \leq p-1} \frac{1}{ij^2k} + \sum_{1 \leq i < j < k \leq p-1} \frac{1}{i^2jk} \\ &\quad + \sum_{k=1}^{p-1} \frac{H_{k-1,3}}{k}. \end{aligned} \tag{31}$$

Taking (15) of Lemma 6 into (31) and comparing this with (30), we immediately obtain (27). □

Further, for the proof of Theorem 1 we will need two particular cases of the following identity due to Hernández [4].

**Lemma 8** (see [4]). *Let $n$ and $m$ be positive integers. Then*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m = k} \frac{1}{i_1 i_2 \cdots i_m} = \sum_{k=1}^{n} \frac{1}{k^m}. \tag{32}$$

**Lemma 9**. *Let $p \geq 7$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k \cdot H_{k,2}}{k} \equiv -\frac{3}{2} \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \pmod{p}. \tag{33}$$

*Proof.* The identity (32) of Lemma 8 with $m = 3$ and $n = p-1$ becomes

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \binom{p-1}{k} \sum_{1 \leq i \leq j \leq k} \frac{1}{ij} = \sum_{k=1}^{p-1} \frac{1}{k^3} = H_{p-1,3}. \tag{34}$$

For any fixed $k \le p-1$, we have the identity

$$\sum_{1\le i\le j\le k}\frac{1}{ij}=\frac{1}{2}\left(\left(\sum_{i=1}^{k}\frac{1}{i}\right)^2+\sum_{i=1}^{k}\frac{1}{i^2}\right)=\frac{1}{2}\left(H_k^2+H_{k,2}\right). \tag{35}$$

Next the congruence (10) from Lemma 4 reduced modulo $p^2$ gives

$$(-1)^{k-1}\binom{p-1}{k}\equiv pH_k-1\left(\operatorname{mod} p^2\right) \tag{36}$$

for every $k=1,2,\dots,p-1$. Substituting (35), (36), and the congruence $H_{p-1,3}\equiv 0 \pmod{p^2}$ of Lemma 5 into (34), we immediately obtain

$$\sum_{k=1}^{p-1}\frac{1}{k}\left(pH_k-1\right)\left(H_k^2+H_{k,2}\right)\equiv 0\left(\operatorname{mod} p^2\right), \tag{37}$$

or equivalently,

$$p\left(\sum_{k=1}^{p-1}\frac{H_k^3}{k}+\sum_{k=1}^{p-1}\frac{H_k\cdot H_{k,2}}{k}\right)-\left(\sum_{k=1}^{p-1}\frac{H_k^2}{k}+\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}\right)\equiv 0\left(\operatorname{mod} p^2\right). \tag{38}$$

Further, (16) from Lemma 6 and the congruences $H_{p-1}\equiv H_{p-1,3}\equiv 0 \pmod{p^2}$ from Lemma 5 give

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k^2}{k}+\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}&\equiv\sum_{k=1}^{p-1}\frac{H_k}{k^2}+\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}\left(\operatorname{mod} p^2\right)\\&=\sum_{1\le i\le k\le p-1}\frac{1}{ik^2}+\sum_{1\le i\le k\le p-1}\frac{1}{i^2k}\\&=\left(\sum_{i=1}^{p-1}\frac{1}{i}\right)\left(\sum_{k=1}^{p-1}\frac{1}{k^2}\right)+\sum_{k=1}^{p-1}\frac{1}{i^3}\\&=H_{p-1}\cdot H_{p-1,2}+H_{p-1,3}\equiv 0\left(\operatorname{mod} p^2\right).\end{aligned} \tag{39}$$

Substituting (39) into (38), we find that

$$\sum_{k=1}^{p-1}\frac{H_k^3}{k}+\sum_{k=1}^{p-1}\frac{H_k\cdot H_{k,2}}{k}\equiv 0\left(\operatorname{mod} p\right). \tag{40}$$

Taking (14) of Lemma 6 into (40) yields (33). This concludes the proof. □

**Lemma 10**. *Let $p\ge 7$ be a prime. Then*

$$\begin{aligned}\sum_{1\le i<j<k\le p-1}\frac{1}{i^2jk}&\equiv\sum_{1\le i<j<k\le p-1}\frac{1}{ijk^2}\\&\equiv-\frac{1}{2}\sum_{1\le i<j<k\le p-1}\frac{1}{ij^2k}\left(\operatorname{mod} p\right).\end{aligned} \tag{41}$$

*Proof.* For simplicity, we denote

$$\begin{aligned}A&:=\sum_{1\le i<j<k\le p-1}\frac{1}{i^2jk},\qquad B:=\sum_{1\le i<j<k\le p-1}\frac{1}{ij^2k},\\C&:=\sum_{1\le i<j<k\le p-1}\frac{1}{ijk^2}.\end{aligned} \tag{42}$$

Obviously, the following identity holds

$$\begin{aligned}&\left(\sum_{1\le i<j<k\le p-1}\frac{1}{ijk}\right)\left(\sum_{l=1}^{p-1}\frac{1}{l}\right)\\&\quad=A+B+C+4\sum_{1\le i<j<k<l\le p-1}\frac{1}{ijkl}.\end{aligned} \tag{43}$$

The well-known Newton's identities (see, e.g., [13]) imply

$$\begin{aligned}4\sum_{1\le i<j<k<l\le p-1}\frac{1}{ijkl}&=\sum_{1\le i<j<k\le p-1}\frac{1}{ijk}H_{p-1}-\sum_{1\le i<j\le p-1}\frac{1}{ij}H_{p-1,2}\\&\quad+\sum_{i=1}^{p-1}\frac{1}{i}H_{p-1,3}-H_{p-1,4},\end{aligned} \tag{44}$$

whence since all the sums $H_{p-1}, H_{p-1,2}, H_{p-1,3}$, and $H_{p-1,4}$ are divisible by a prime $p\ge 7$, we obtain (cf. [5, Theorem 1.5] or [14])

$$\sum_{1\le i<j<k<l\le p-1}\frac{1}{ijkl}\equiv 0\left(\operatorname{mod} p\right). \tag{45}$$

Inserting (45) and $H_{p-1}=\sum_{i=1}^{p-1}1/i\equiv 0 \pmod{p}$ into (43), we get

$$A+B+C\equiv 0\left(\operatorname{mod} p\right). \tag{46}$$

Further, by the substitution trick $i,j,k\to p-i,p-j,p-k$,

$$\begin{aligned}A&=\sum_{1\le i<j<k\le p-1}\frac{1}{i^2jk}\\&=\sum_{1\le p-i<p-j<p-k\le p-1}\frac{1}{(p-i)^2(p-j)(p-k)}\\&\equiv\sum_{1\le k<j<i\le p-1}\frac{1}{kji^2}=C\left(\operatorname{mod} p\right).\end{aligned} \tag{47}$$

From (47) we see that $C\equiv A \pmod{p}$, which substituting into (46) gives

$$2A+B\equiv 0\left(\operatorname{mod} p\right). \tag{48}$$

Finally, (47) and (48) yield $C\equiv A\equiv -B/2 \pmod{p}$, as desired. □

**Lemma 11**. *Let $p\ge 7$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv-\sum_{1\le i<j<k\le p-1}\frac{1}{ij^2k}\left(\operatorname{mod} p\right). \tag{49}$$

*Proof.* We follow proof of the congruence (3) in Theorem 1.1 of [15]. By Lemma 5, $H_{p-1} := \sum_{j=1}^{p-1} 1/j \equiv 0 \pmod{p}$, or equivalently, for each $j = 1, 2, \ldots, p-2$ holds

$$\frac{1}{j+1} + \frac{1}{j+2} + \cdots + \frac{1}{p-1} \equiv -\left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1} + \frac{1}{j}\right) \pmod{p}. \tag{50}$$

Applying the congruence (50), we find that

$$\begin{aligned}
\sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} &= \sum_{j=2}^{p-1} \frac{1}{j^2} \sum_{i=1}^{j-1} \frac{1}{i} \sum_{k=j+1}^{p-1} \frac{1}{k} \\
&= \sum_{j=2}^{p-1} \frac{1}{j^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1}\right) \\
&\quad \times \left(\frac{1}{j+1} + \frac{1}{j+2} + \cdots + \frac{1}{p-1}\right) \\
&\equiv \sum_{j=2}^{p-1} \frac{1}{j^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j-1}\right) \\
&\quad \times \left(-\left(1 + \frac{1}{2} + \cdots + \frac{1}{j}\right)\right) \pmod{p} \\
&= \sum_{j=1}^{p-1} \frac{1}{j^2} \left(H_j - \frac{1}{j}\right)\left(-H_j\right) \\
&= -\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} + \sum_{j=1}^{p-1} \frac{H_j}{j^3} \\
&= -\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} + \sum_{j=1}^{p-1} \frac{H_{j-1} + (1/j)}{j^3} \\
&= -\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} + \sum_{1\le i<j\le p-1} \frac{1}{ij^3} + \sum_{j=1}^{p-1} \frac{1}{j^4} \\
&= -\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} + \sum_{j=1}^{p-1} \frac{H_{j-1}}{j^3} + H_{p-1,4} \pmod{p},
\end{aligned} \tag{51}$$

whence it follows that

$$\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} \equiv \sum_{j=1}^{p-1} \frac{H_{j-1}}{j^3} + H_{p-1,4} - \sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} \pmod{p}. \tag{52}$$

Since by the first part of (13) from Lemma 6 and by (11) of Lemma 5 with $m = 4$,

$$\sum_{j=1}^{p-1} \frac{H_{j-1}}{j^3} \equiv H_{p-1,4} \equiv 0 \pmod{p}, \tag{53}$$

substituting this into (52), we obtain (49). □

The first congruence of the following result was recently established by Sun [6, Theorem 1.1, congruence (1.5)].

**Lemma 12**. *Let $p \ge 7$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}, \tag{54}$$

$$\sum_{k=1}^{p-1} \frac{H_k \cdot H_{k,2}}{k} \equiv 0 \pmod{p}, \tag{55}$$

$$\sum_{k=1}^{p-1} \frac{H_k^3}{k} \equiv 0 \pmod{p}. \tag{56}$$

*Proof.* Comparing the congruences (27) of Lemma 7 and (33) of Lemma 9, we have

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv -\frac{2}{3}\left(\sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} + \sum_{1\le i<j<k\le p-1} \frac{1}{i^2jk}\right) \pmod{p}. \tag{57}$$

Since by (41) of Lemma 10,

$$\sum_{1\le i<j<k\le p-1} \frac{1}{i^2jk} \equiv -\frac{1}{2} \sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} \pmod{p}, \tag{58}$$

then substituting this into (57), we obtain

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv -\frac{1}{3} \sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} \pmod{p}. \tag{59}$$

Finally, as by (49) of Lemma 11,

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv -\sum_{1\le i<j<k\le p-1} \frac{1}{ij^2k} \pmod{p}, \tag{60}$$

then comparing this with (59) implies

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}, \tag{61}$$

which coincides with (54).

Finally, (54) and (33) of Lemma 9 yield (55), while (54) and (14) of Lemma 6 yield (56). □

*Remarks 13.* Applying a standard technique expressing sum of powers in terms of Bernoulli numbers, Sun in [6, proof of (1.5) of Theorem 1.1] showed that

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv -\sum_{j=0}^{p-3} B_j B_{p-3-j} \pmod{p}. \tag{62}$$

The above congruence and (54) yield the following curious congruence for a prime $p \ge 7$ established by Zhao [5, congruence (3.19) of Corollary 3.6]:

$$\sum_{j=0}^{p-3} B_j B_{p-3-j} \equiv 0 \pmod{p}. \tag{63}$$

As noticed in [6, proof of Lemma 2.8], the above congruence immediately follows from an identity of Matiyasevich (cf. [16, equation (1.3)]).

Furthermore, the congruences (54), (59), and (41) from Lemma 10 immediately give

$$\begin{aligned}\sum_{1\le i<j<k\le p-1}\frac{1}{i^2jk} &\equiv \sum_{1\le i<j<k\le p-1}\frac{1}{ijk^2}\\ &\equiv \sum_{1\le i<j<k\le p-1}\frac{1}{ij^2k}\equiv 0\pmod{p}.\end{aligned}\tag{64}$$

Notice that the congruences (64) were proved by Zhao [5, Corollary 3.6, congruence (3.20)] applying a technique expressing sum of powers in terms of Bernoulli numbers.

The following result is contained in [17, Lemma 2.4].

**Lemma 14**. *Let $p\ge 7$ be a prime. Then*

$$2\sum_{k=1}^{p-1}\frac{1}{k}\equiv -p\sum_{k=1}^{p-1}\frac{1}{k^2}\pmod{p^4}.\tag{65}$$

*Proof.* Multiplying the identity

$$1+\frac{p}{k}+\frac{p^2}{k^2}=\frac{p^3-k^3}{k^2(p-k)}\tag{66}$$

by $-p/k^2$ $(1\le k\le p-1)$, we obtain

$$-\frac{p}{k^2}\left(1+\frac{p}{k}+\frac{p^2}{k^2}\right)=\frac{-p^4+pk^3}{k^4(p-k)}\equiv\frac{p}{k(p-k)}\pmod{p^4},\tag{67}$$

which can be written as

$$\frac{1}{k}+\frac{1}{p-k}\equiv-\left(\frac{p}{k^2}+\frac{p^2}{k^3}+\frac{p^3}{k^4}\right)\pmod{p^4}.\tag{68}$$

After summation of the above congruence over $k=1,\dots,p-1$ we immediately obtain

$$2H_{p-1}\equiv -pH_{p-1,2}-p^2H_{p-1,3}-p^3H_{p-1,4}\pmod{p^4}.\tag{69}$$

Taking the congruences $H_{p-1,3}\equiv 0\pmod{p^2}$ and $H_{p-1,4}\equiv 0\pmod{p}$ of Lemma 5 into (69), it becomes (65). □

*Proof of Theorem 1.* The first congruence of (5) is in fact the congruence (16) of Lemma 6.

The identity (32) of Lemma 8 with $m=2$ and $n=p-1$ becomes

$$\sum_{1\le i\le k\le p-1}(-1)^{k-1}\binom{p-1}{k}\frac{1}{ik}=\sum_{k=1}^{p-1}\frac{1}{k^2}=H_{p-1,2}.\tag{70}$$

Substituting the congruence (10) of Lemma 4 into the left hand side of equality (70), we find that

$$\begin{aligned}\sum_{1\le i\le k\le p-1}\frac{(-1)^{k-1}}{ik}\binom{p-1}{k}\equiv&-\sum_{1\le i\le k\le p-1}\frac{1}{ik}+p\sum_{1\le i\le k\le p-1}\frac{H_k}{ik}\\&-\frac{p^2}{2}\sum_{1\le i\le k\le p-1}\frac{H_k^2}{ik}+\frac{p^2}{2}\\&\times\sum_{1\le i\le k\le p-1}\frac{H_{k,2}}{ik}\pmod{p^3}.\end{aligned}\tag{71}$$

By Wolstenholme's theorem,

$$\sum_{1\le i\le k\le p-1}\frac{1}{ik}=\frac{1}{2}\left(H_{p-1}^2+H_{p-1,2}\right)\equiv\frac{1}{2}H_{p-1,2}\pmod{p^2}.\tag{72}$$

Next we have

$$\sum_{1\le i\le k\le p-1}\frac{H_k}{ik}=\sum_{k=1}^{p-1}\frac{H_k}{k}\sum_{i=1}^{k}\frac{1}{i}=\sum_{k=1}^{p-1}\frac{H_k^2}{k}.\tag{73}$$

Further, using (56) of Lemma 12,

$$\sum_{1\le i\le k\le p-1}\frac{H_k^2}{ik}=\sum_{k=1}^{p-1}\frac{H_k^2}{k}\sum_{i=1}^{k}\frac{1}{i}=\sum_{k=1}^{p-1}\frac{H_k^3}{k}\equiv 0\pmod{p}.\tag{74}$$

Similarly, using (55) of Lemma 12,

$$\sum_{1\le i\le k\le p-1}\frac{H_{k,2}}{ik}=\sum_{k=1}^{p-1}\frac{H_{k,2}}{k}\sum_{i=1}^{k}\frac{1}{i}=\sum_{k=1}^{p-1}\frac{H_{k,2}\cdot H_k}{k}\equiv 0\pmod{p}.\tag{75}$$

Now inserting (72)–(75) into (71), we obtain

$$\sum_{1\le i\le k\le p-1}\frac{(-1)^{k-1}}{ik}\binom{p-1}{k}\equiv-\frac{1}{2}H_{p-1,2}+p\sum_{k=1}^{p-1}\frac{H_k^2}{k}.\tag{76}$$

The equality (70) and the congruence (76) give

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k}\equiv\frac{3}{2p}\sum_{k=1}^{p-1}\frac{1}{k^2}\pmod{p^2}.\tag{77}$$

The congruences (77) and (65) of Lemma 14 yield

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k}\equiv-\frac{3}{p^2}\sum_{k=1}^{p-1}\frac{1}{k}\pmod{p^2}.\tag{78}$$

Finally, the congruences (77) and (78) complete proof of Theorem 1. □

*Remark 15.* From the identity

$$H_{k-1}^3 = \left(H_k - \frac{1}{k}\right)^3 = H_k^3 - 3\frac{H_k^2}{k} + 3\frac{H_k}{k^2} - \frac{1}{k^3} \tag{79}$$

immediately follows that

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} = \frac{1}{3}\left(H_{p-1}^3 - H_{p-1,3}\right). \tag{80}$$

Inserting in the right hand side of the above identity the congruences $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1,3} \equiv -(6p^2 B_{p-5}/5) \pmod{p^3}$ from [8, Theorem 5.1(a) with $k = 3$], we find that for a prime $p \geq 7$,

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} - \sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv \frac{2p^2 B_{p-5}}{5} \left(\bmod p^3\right). \tag{81}$$

However, the determination of $\sum_{k=1}^{p-1}(H_k^2/k) \pmod{p^3}$ seems to be a difficult problem.

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