Research Article

Applications of Soft Sets in BE-Algebras

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The notion of intersectional soft subalgebras of a BE-algebra is introduced, and related properties are investigated. Characterization of an intersectional soft subalgebra is discussed. The problem of classifying intersectional soft subalgebras by their inclusive subalgebras will be solved.

1. Introduction

In 1966, Imai and Iséki [1] and Iséki [2] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Ma et al. studied (∈, ∈ ∨ q)-tyle (interval-valued) fuzzy ideals in BCI-algebras and soft R₀ -algebras (see [3–5]). As a generalization of a BCK-algebra, H. S. Kim and Y. H. Kim [6] introduced the notion of a BE-algebra and investigated several properties. In [7], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

Various problems in system identification involve characteristics which are essentially nonprobabilistic in nature [9]. In response to this situation, Zadeh [10] introduced fuzzy set theory as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [11]. To solve complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [12]. Maji et al. [13] and Molodtsov [12] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [13] described the application of soft set theory to a decision making problem. Maji et al. [14] also studied several operations on the theory of soft sets. Chen et al. [15] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çağman et al. [16] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory and provided an example which shows that the method can be successfully applied to the problems that
contain uncertainties. Feng [17] considered the application of soft rough approximations in multicriteria group decision making problems. Aktaş and Çağman [18] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. After that, many algebraic properties of soft sets are studied (see [19–29]).

In this paper, we introduce the notion of int-soft subalgebras of a BE-algebra and investigate their properties. We consider characterization of an int-soft subalgebra, and solve the problem of classifying int-soft subalgebras by their inclusive subalgebras.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a BE-algebra we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold (see [6]):

\[
\begin{align*}
\forall x \in X \left( x * x = 1 \right), \\
\forall x \in X \left( x * 1 = 1 \right), \\
\forall x \in X \left( 1 * x = x \right), \\
\forall x, y, z \in X \left( x * (y * z) = y * (x * z) \right). \\
\end{align*}
\]

(1) (2) (3) (4)

A relation “≤” on a BE-algebra $X$ is defined by

\[
\forall x, y \in X \left( x \leq y \iff x * y = 1 \right).
\]

(5)

A BE-algebra $(X; *, 1)$ is said to be transitive (see [7]) if it satisfies

\[
\forall x, y, z \in X \left( y * z \leq (x * y) * (x * z) \right).
\]

(6)

A BE-algebra $(X; *, 1)$ is said to be self distributive (see [6]) if it satisfies

\[
\forall x, y, z \in X \left( x * (y * z) = (x * y) * (x * z) \right).
\]

(7)

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [7]).

A mapping $\mu : X \to Y$ of BE-algebras is called a homomorphism if $\mu(x * y) = \mu(x) * \mu(y)$ for all $x, y \in X$.

A soft set theory is introduced by Molodtsov [12], and Çağman and Enginoğlu [30] provided new definitions and various results on soft set theory.

In what follows, let $U$ be an initial universe set, and let $E$ be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of $U$ and $A, B, C, \ldots \subseteq E$.

Definition 1 (see [12, 13]). A soft set $(f, A)$ over $U$ is defined to be the set of ordered pairs

\[
(f, A) := \{(x, f(x)) : x \in E, f(x) \in \mathcal{P}(U)\},
\]

where $f : E \to \mathcal{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

The function $f$ is called an approximate function of the soft set $(f, A)$.

In what follows, denote by $S(U)$ the set of all soft sets over $U$ by Çağman and Enginoğlu [30].

For any soft sets $(f, X)$ and $(g, X)$ over $U$, we call $(f, X)$ a soft subset of $(g, X)$, denoted by $(f, X) \subseteq (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The soft $\{\cup, \cap\}$ of $(f, X)$ and $(g, X)$ is defined to be a soft set

\[
\left\{ \left( f(x) \cup g(x) \right) : x \in X \right\},
\]

where

\[
\left\{ \left( f(x) \cap g(x) \right) : x \in X \right\}.
\]

for all $x \in X$.

Definition 2 (see [31, 32]). Assume that $E$ has a binary operation $\rightarrow$. For any nonempty subset $A$ of $E$, a soft set $(f, A)$ over $U$ is said to be intersectional over $U$ if it satisfies

\[
\forall x, y \in A \left( x \rightarrow y \in A \implies f(x) \cap f(y) \subseteq f(x \rightarrow y) \right).
\]

(9)

For a soft set $(f, A)$ over $U$ and a subset $\gamma$ of $U$, the $\gamma$-inclusive set of $(f, A)$, denoted by $i_\gamma(f, A)$, is defined to be the set

\[
i_\gamma(f, A) := \{x \in A | \gamma \subseteq f(x)\}.
\]

(10)

3. Intersectional Soft Subalgebras

In what follows, we take $E = X$ as a set of parameters, which is a BE-algebra under the operation $\ast$ unless otherwise specified.

Definition 3. A soft set $(f, X)$ over $U$ is called an intersectional soft subalgebra (briefly, int-soft subalgebra) over $U$ if it satisfies

\[
\forall x, y \in X \left( f(x \rightarrow y) \supseteq f(x) \cap f(y) \right).
\]

(11)

Example 4. Let $E = X$ be the set of parameters where $X = \{1, a, b, c, d\}$ is a BE-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
* & 1 & a & b & c \\
\hline
1 & 1 & a & b & c \\
a & a & 1 & b & c \\
b & b & a & 1 & c \\
c & c & b & 1 & 1 \\
d & d & 1 & 1 & 1 \\
\end{array}
\]

(12)

Let $(f, X)$ be a soft set over $U$ defined as follows:

\[
f : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_3 & \text{if } x = 1, \\
\gamma_1 & \text{if } x \in \{a, c, d\}, \\
\gamma_2 & \text{if } x = b, \end{cases}
\]

(13)

where $\gamma_1, \gamma_2,$ and $\gamma_3$ are subsets of $U$ with $\gamma_1 \subseteq \gamma_2 \subseteq \gamma_3$. It is easy to check that $(f, X)$ is an int-soft subalgebra over $U$. 
Example 5. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set, where $X = \{1, a, b, c, d, 0\}$ is a BE-algebra [7] with the following Cayley table:

$$
\begin{array}{c|cccccc}
* & 1 & a & b & c & d & 0 \\
\hline
1 & 1 & a & b & c & d & 0 \\
a & 1 & a & b & c & d & 0 \\
b & 1 & 1 & c & c & c & c \\
c & 1 & a & b & a & b & a \\
d & 1 & 1 & 1 & a & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

Let $\{\gamma_n \mid n = 1, 2, 3, 4\}$ be a class of subsets of $U$ which is a poset under the following Hasse diagram:

$$
\begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\end{array}
$$

Let $(f, X)$ be a soft set over $U$ defined as follows:

$$
f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\gamma_4 & \text{if } x = 1, \\
\gamma_1 & \text{if } x \in \{a, c, d\}, \\
\gamma_2 & \text{if } x = b, \\
\gamma_3 & \text{if } x = 0.
\end{cases}
$$

It is easy to check that $(f, X)$ is an int-soft subalgebra over $U$.

Theorem 6. A soft set $(f, X)$ over $U$ is an int-soft subalgebra over $U$ if and only if $i_X(f; y)$ is a subalgebra of $X$ for all $y \in \mathcal{P}(U)$.

Proof. Assume that $(f, X)$ is an int-soft subalgebra over $U$. Let $y \in \mathcal{P}(U)$ and $x, y \in i_X(f; y)$. Then, $y \subseteq f(x)$ and $y \subseteq f(y)$. It follows from (II) that

$$
f (x \ast y) \supseteq f (x) \cap f (y) \supseteq y,
$$

that is, $x \ast y \in i_X(f; y)$. Thus, $i_X(f; y)$ is a subalgebra of $X$.

Conversely, suppose that $i_X(f; y)$ is a subalgebra of $X$ for all $y \in \mathcal{P}(U)$. Let $x, y \in X$ be such that $f(x) = \gamma_x$ and $f(y) = \gamma_y$. Take $y = \gamma_x \cap \gamma_y$. Then, $x, y \in i_X(f; y)$, and so $x \ast y \in i_X(f; y)$ by assumption. Hence,

$$
f (x \ast y) \supseteq \gamma_x \cap \gamma_y = f (x) \cap f (y).
$$

Therefore, $(f, X)$ is an int-soft subalgebra over $U$. \qed

Lemma 7. Every int-soft subalgebra $(f, X)$ over $U$ satisfies the following inclusion:

$$
(\forall x \in X) \ (f (x) \subseteq f (1)).
$$

Proof. Using (I) and (II), we have

$$
f (1) = f (x \ast x) \supseteq f (x) \cap f (x) = f (x)
$$

for all $x \in X$. \qed

Proposition 8. For any int-soft subalgebra $(f, X)$ over $U$, if a fixed element $x \in X$ satisfies $f(x) = f(1)$, then

$$
(\forall y \in X) \ (f(y) \subseteq f(x \ast y)).
$$

Proof. Assume that a fixed element $x \in X$ satisfies $f(x) = f(1)$. Then,

$$
f (y) = f (1) \cap f (y) = f (x) \cap f (y) \subseteq f (x \ast y)
$$

for all $y \in X$. \qed

Proposition 9. Let $(f, X)$ be an int-soft subalgebra over $U$. If a fixed element $x \in X$ satisfies the following condition:

$$
(\forall y \in X) \ (f (y) \subseteq f (y \ast x)),
$$

then $f(x) = f(1)$.

Proof. Taking $y = 1$ in (22) implies that $f(1) \subseteq f(1 \ast x) = f(x)$ by (3). It follows from Lemma 7 that $f(x) = f(1)$. \qed

For any BE-algebras $X$ and $Y$, let $\mu : X \rightarrow Y$ be a function and $(f, X)$, and let $(g, Y)$ be soft sets over $U$.

(1) The soft set

$$
\mu^{-1}(g, Y) = \{(x, \mu^{-1}(g)(x)) : x \in X, \mu^{-1}(g)(x) \in \mathcal{P}(U)\},
$$

where $\mu^{-1}(g)(x) = g(\mu(x))$, is called the soft preimage of $(g, Y)$ under $\mu$.

(2) The soft set

$$
\mu (f, X) = \{(y, \mu (f)(y)) : y \in Y, \mu (f)(y) \in \mathcal{P}(U)\},
$$

where

$$
\mu (f)(y) = \begin{cases} 
\bigcup_{x \in \mu^{-1}(y)} f (x) & \text{if } \mu^{-1}(y) \neq \emptyset, \\
\emptyset & \text{otherwise},
\end{cases}
$$

is called the soft image of $(f, X)$ under $\mu$.

Proposition 10. For any BE-algebras $X$ and $Y$, let $\mu : X \rightarrow Y$ be a function. Then,

$$
(\forall (f, X) \in S(U)) \left( (f, X) \subseteq \mu^{-1}(\mu (f, X)) \right).
$$

Proof. Note that $\mu^{-1}(\mu(x)) \neq \emptyset$ for all $x \in X$. Hence,

$$
f (x) \subseteq \bigcup_{a \in \mu^{-1}(\mu(x))} f (a) = \mu (f)(\mu (x)) = \mu^{-1}(\mu (f)(x))
$$

for all $x \in X$, and therefore (26) is valid. \qed
Theorem 11. Let $\mu : X \to Y$ be a homomorphism of BE-algebras and $(g, Y) \in S(U)$. If $(g, Y)$ is an int-soft subalgebra over $U$, then the soft preimage $\mu^{-1}(g, Y)$ of $(g, Y)$ under $\mu$ is also an int-soft subalgebra over $U$.

Proof. For any $x_1, x_2 \in X$, we have
\[
\mu^{-1}(g)(x_1) \cap \mu^{-1}(g)(x_2) = g(\mu(x_1)) \cap g(\mu(x_2)) \\
\subseteq g(\mu(x_1) \ast \mu(x_2)) \\
= g(\mu(x_1 \ast x_2)) \\
= \mu^{-1}(g)(x_1 \ast x_2).
\]
Hence, $\mu^{-1}(g, Y)$ is also an int-soft subalgebra over $U$.

Theorem 12. Let $\mu : X \to Y$ be a homomorphism of BE-algebras and $(f, X) \in S(U)$. If $(f, X)$ is an int-soft subalgebra over $U$ and $\mu$ is injective, then the soft image $\mu(f, X)$ of $(f, X)$ under $\mu$ is also an int-soft subalgebra over $U$.

Proof. Let $y_1, y_2 \in Y$. If at least one of $\mu^{-1}(y_1)$ and $\mu^{-1}(y_2)$ is empty, then the inclusion
\[
\mu(f)(y_1) \cap \mu(f)(y_2) \subseteq \mu(f)(y_1 \ast y_2)
\]
is clear. Assume that $\mu^{-1}(y_1) \neq \emptyset$ and $\mu^{-1}(y_2) \neq \emptyset$. Since $\mu$ is injective, we have
\[
\mu(f)(y_1) \cap \mu(f)(y_2) \\
= \left( \bigcup_{x \in \mu^{-1}(y_1)} f(x) \right) \cap \left( \bigcup_{x \in \mu^{-1}(y_2)} f(x) \right) \\
\subseteq \bigcup_{x \in \mu^{-1}(y_1) \cap \mu^{-1}(y_2)} f(x) \\
= \mu(f)(y_1 \ast y_2).
\]
Therefore, $\mu(f, X)$ is an int-soft subalgebra over $U$.

Theorem 13. Let $(f, X) \in S(U)$ and define a soft set $(f^*, X)$ over $U$ by
\[
f^* : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} f(x) & \text{if } x \in i_X(f; y), \\ \delta & \text{otherwise,} \end{cases}
\]
where $y$ is any subset of $U$ and $\delta$ is a subset of $U$ satisfying $\delta \subseteq \bigcap_{x \in i_X(f; y)} f(x)$. If $(f, X)$ is an int-soft subalgebra over $U$, then so is $(f^*, X)$.

Proof. If $(f, X)$ is an int-soft subalgebra over $U$, then $i_X(f; y)$ is a subalgebra of $X$ for all $y \subseteq U$ by Theorem 6. Let $x, y \in X$. If $x, y \in i_X(f; y)$, then $x \ast y \in i_X(f; y)$. Hence,
\[
f^*(x \ast y) = f(x \ast y) = f(x) \cap f(y) = f^*(x) \cap f^*(y). \tag{32}
\]
If $x \notin i_X(f; y)$ or $y \notin i_X(f; y)$, then $f^*(x) = \delta$ or $f^*(y) = \delta$.

\[
f^*(x \ast y) \supseteq \delta = f^*(x) \cap f^*(y). \tag{33}
\]
Therefore, $(f^*, X)$ is an int-soft subalgebra over $U$.

Theorem 14. If $(f, X)$ and $(g, X)$ are int-soft subalgebras over $U$, then the soft intersection $(f \cap g)(X)$ of $(f, X)$ and $(g, X)$ is an int-soft subalgebra over $U$.

Proof. Let $x, y \in X$. Then,
\[
(f \cap g)(x \ast y) = f(x \ast y) \cap g(x \ast y) \\
\supseteq (f(x) \cap g(x)) \cap (f(y) \cap g(y)) \\
= (f(x) \cap g(x)) \cap (f(y) \cap g(y)) \\
= (f \cap g)(x) \cap (f \cap g)(y).
\]
Hence, $(f, X) \cap (g, X)$ is an int-soft subalgebra over $U$.

The following example shows that the soft union of int-soft subalgebras over $U$ may not be an int-soft subalgebra over $U$.

Example 15. Let $E = X$ be the set of parameters where $X = \{1, a, b, c\}$ is a BE-algebra [7] with the following Cayley table:
\[
\begin{array}{c|ccc}
\ast & 1 & a & b & c \\
\hline
1 & 1 & a & b & c \\
a & 1 & a & b & c \\
b & 1 & 1 & a & a \\
c & 1 & 1 & a & 1 \\
\end{array}
\]
Let $(f, X)$ and $(g, X)$ be soft sets over $U$ defined, respectively, as follows:
\[
f : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} y_3 & \text{if } x = 1, \\ y_2 & \text{if } x = a, \\ y_1 & \text{if } x = b, \\ \delta & \text{if } x = c, \end{cases}
\]
\[
g : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} y_4 & \text{if } x = 1, \\ y_2 & \text{if } x = a, \\ y_3 & \text{if } x = b, \\ y_1 & \text{if } x = c, \end{cases}
\]
where $y_1, y_2, y_3, y_4$, and $y_5$ are subsets of $U$ with $y_1 \subseteq y_2 \subseteq y_3 \subseteq y_4 \subseteq y_5$. It is easy to check that $(f, X)$ and $(g, X)$ are int-soft subalgebras over $U$. But $(f \cup g)(c \ast b) = (f \cup g)(a) = y_2 \not\supseteq y_3 = (f \cup g)(c) \cap (f \cup g)(b)$.
Theorem 16. Let \((f, X)\) be an int-soft subalgebra over \(U\). Let \(γ_1\) and \(γ_2\) be subsets of \(U\) such that \(γ_1 ⊊ γ_2\). If the \(γ_1\)-inclusive set of \((f, X)\) is equal to the \(γ_2\)-inclusive set of \((f, X)\), then there is no \(x \in X\) such that \(γ_1 ⊊ f(x) ⊊ γ_2\).

Proof. Straightforward.

The converse of Theorem 16 is not true in general as seen in the following example.

Example 17. Let \(E = X = \{a, b, c, d\}\) be the initial universe set where \(X = \{1, a, b, c, d\}\) is a BE-algebra as in Example 4. Consider a soft set \((f, X)\) which is given by

\[
\begin{align*}
\text{if } x = 1, & \quad \{1, a\} \\
\text{if } x \in \{a, c, d\}, & \quad \{1, a, c\} \\
\text{if } x = b, & \quad \emptyset
\end{align*}
\]

Then, \((f, X)\) is an int-soft subalgebra over \(U\). The \(γ\)-inclusive sets of \((f, X)\) are described as follows:

\[
i_X(f; γ) = \begin{cases} X & \text{if } γ \in \{\{0\}, \{1\}, \{1, a\}\}, \\
\{1, b\} & \text{if } γ \in \{\{1\}, \{a, c\}, \{1, a, c\}\}, \\
\{c\} & \text{if } γ \in \{\{a\}, \{a, c\}, \{1, a, c\}\}, \\
\emptyset & \text{otherwise}
\end{cases}
\]

If we take \(γ_1 = \{1, c\}\) and \(γ_2 = \{1, b, c\}\), then \(γ_1 \not\subseteq γ_2\) and there is no \(x \in X\) such that \(γ_1 \subseteq f(x) \not\subseteq γ_2\). But \(i_X(f; γ)= \{1, b\} \not\subseteq \{1\}\).

Theorem 18. Let \((f, X)\) be an int-soft subalgebra over \(U\). Let \(γ_1\) and \(γ_2\) be subsets of \(U\) such that \(γ_1 \subseteq γ_2\) and \(\{γ_1, γ_2\}, f(x)\) are totally ordered by set inclusion for all \(x \in X\). If there is no \(x \in X\) such that \(γ_1 \subseteq f(x) \not\subseteq γ_2\), then the \(γ_1\)-inclusive set of \((f, X)\) is equal to the \(γ_2\)-inclusive set of \((f, X)\).

Proof. Since \(γ_1 \subseteq γ_2\), we have \(i_X(f; γ_2) \subseteq i_X(f; γ_1)\). If \(x \in i_X(f; γ_1)\), then \(γ_1 \subseteq f(x)\) since \(\{γ_1, γ_2\}, f(x)\) is totally ordered by inclusion and there is no \(x \in X\) such that \(γ_1 \subseteq f(x) \not\subseteq γ_2\), it follows that \(γ_2 \subseteq f(x)\), that is, \(x \in i_X(f; γ_2)\). Therefore, the \(γ_1\)-inclusive set of \((f, X)\) is equal to the \(γ_2\)-inclusive set of \((f, X)\).

Theorem 19. Let \((f, X)\) be a soft set over \(U\) in which \(\text{Im}(f)\) is totally ordered by set inclusion. For each subset \(γ\) of \(\text{Im}(f)\), if the \(γ\)-inclusive set of \((f, X)\) is a subalgebra of \(X\), then \((f, X)\) is an int-soft subalgebra over \(U\).

Proof. Let \(x, y \in X\) be such that \(f(x) = γ_1\) and \(f(y) = γ_2\). Then, either \(γ_1 \subseteq γ_2\) or \(γ_2 \subseteq γ_1\). We may assume that \(γ_1 \subseteq γ_2\) without loss of generality. Then, \(x \in i_X(f; γ_1)\), \(y \in i_X(f; γ_2)\), and \(i_X(f; γ_1) \subseteq i_X(f; γ_2)\). Since \(i_X(f; γ_1)\) is a subalgebra of \(X\), it follows that \(x \ast y \in i_X(f; γ_1)\) so that

\[
f(x \ast y) = γ_1 \cap γ_2 = f(x) \cap f(y).
\]

Therefore, \((f, X)\) is an int-soft subalgebra over \(U\).
Proof. Let $\gamma_1$ and $\gamma_2$ be subsets of $U$ such that $i_X(f;\gamma_1) = i_X(f;\gamma_2)$. Assume that there exists $x \in X$ such that $\gamma_1 \not\subseteq f(x) \subseteq \gamma_2$. Then, $i_X(f;\gamma_2)$ is a proper subset of $i_X(f;\gamma_1)$, which contradicts the hypothesis.

Conversely, suppose that there is no $x \in X$ such that $\gamma_1 \not\subseteq f(x) \subseteq \gamma_2$. Obviously, $i_X(f;\gamma_1) \subseteq i_X(f;\gamma_2)$. If $x \in i_X(f;\gamma_1)$, then $\gamma_1 \subseteq f(x)$. It follows from the assumption that $\gamma_2 \not\subseteq f(x)$, that is, $x \in i_X(f;\gamma_2)$. Therefore, $i_X(f;\gamma_1) = i_X(f;\gamma_2)$. \hfill $\Box$

Remark 23. As a consequence of Theorem 22, if $E = X$ is a finite BE-algebra, then the $\gamma$-inclusive sets of an int-soft subalgebra $(f, X)$ over $U$ form a chain. But $f(x) \subseteq f(1)$ for all $x \in X$. Therefore, $i_X(f;\gamma_0)$, where $\gamma_0 = f(1)$, is the smallest inclusive subalgebra but not always $i_X(f;\gamma_0) = \{1\}$ as seen in the following example, and so we have the chain

$$i_X(f;\gamma_0) \subseteq i_X(f;\gamma_1) \subseteq \cdots \subseteq i_X(f;\gamma_n) = X,$$

where $\gamma_0 \not\subset \gamma_1 \not\subset \cdots \not\subset \gamma_n$.

Example 24. Let $A$ be a subalgebra of a BE-algebra $(f, X)$ such that $A \neq \{0, 1\}$. Let $(f; X)$ be the int-soft subalgebra over $U$ which is given in the proof of Theorem 21. Then, $\operatorname{Im}(f) = \{\emptyset, \gamma\}$. Further, the $\gamma$-inclusive sets of $(f; X)$ are $i_X(f;\theta) = X$ and $i_X(f;\gamma) = A$. Thus, we have $f(1) = \gamma$ but $i_X(f;\gamma) = A \neq \{0, 1\}$.

Corollary 25. Let $E = X$ be a finite BE-algebra, and let $(f; X)$ be an int-soft subalgebra over $U$. If $\operatorname{Im}(f) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, then the family of $\gamma_i$-inclusive sets $i_X(f;\gamma_i)$, $1 \leq i \leq n$, constitutes all the $\gamma$-inclusive sets of $(f; X)$.

Proof. Let $\gamma \subseteq U$ and $\gamma \not\subseteq \gamma_i$, where $\gamma_i, \gamma_j \in \operatorname{Im}(f)$, then $i_X(f;\gamma_i) = i_X(f;\gamma_j) = i_X(f;\gamma)$ by Theorem 22. If $\gamma \not\subseteq \gamma_i$, where $\gamma_j$ is the least element (under the set inclusion) of $\operatorname{Im}(f)$, then $i_X(f;\gamma_i) = X = i_X(f;\gamma)$. Assume that $\gamma \not\subset \gamma_j$, where $\gamma_i$ is the greatest element (under the set inclusion) of $\operatorname{Im}(f)$. If there is $x \in X$ such that $f(x) = \gamma \not\subseteq U$ and $\gamma \not\subseteq \gamma_i$, then $\gamma_i \in \operatorname{Im}(f)$. It is a contradiction. From Theorem 22 that $i_X(f;\gamma_i) = i_X(f;\gamma)$. Thus, for any $\gamma \subseteq U$, the inclusive subalgebra is one of $\{i_X(f;\gamma) \mid \gamma_i \in \operatorname{Im}(f)\}$. \hfill $\Box$

The following example shows that two int-soft subalgebras over $U$ may have an identical family of $\gamma$-inclusive sets but the int-soft subalgebras over $U$ may not be equal.

Example 26. Let $E = X$ be the set of parameters, and let $U = X$ be the initial universe set where $X = \{1, a, b, c\}$ is a BE-algebra as in Example 15. Consider a soft set $(f; X)$ over $U$ which is given by

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\gamma_1 & \text{if } x = 1, \\
\gamma_2 & \text{if } x = a, \\
\gamma_3 & \text{if } x \in \{b, c\},
\end{cases}$$

where $\gamma_1 \not\subseteq \gamma_2 \not\subseteq \gamma_3$ are subsets of $U$. It is easy to verify that $(f; X)$ is an int-soft subalgebra over $U$. The $\gamma$-inclusive sets of $(f; X)$ are $i_X(f;\gamma_1) = \{1\}$, $i_X(f;\gamma_2) = \{1, a\}$ and $i_X(f;\gamma_3) = X$. Now let $\delta_1$, $\delta_2$, and $\delta_3$ be subsets of $U$ such that $\delta_1 \not\subseteq \delta_2 \not\subseteq \delta_3$ and $\delta_i \neq \gamma_j$ for $i = 1, 2, 3$ and $j = 1, 2, 3$. Define a soft set $(g; X)$ over $U$ as follows:

$$g : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 
\delta_1 & \text{if } x = 1, \\
\delta_2 & \text{if } x = a, \\
\delta_3 & \text{if } x \in \{b, c\}.
\end{cases}$$

Then, $(g, X)$ is an int-soft subalgebra over $U$, and the $\gamma$-inclusive sets of $(g, X)$ are $i_X(g;\delta_1) = \{1\}$, $i_X(g;\delta_2) = \{1, a\}$ and $i_X(g;\delta_3) = X$. Hence, the two int-soft subalgebras $(f, X)$ and $(g, X)$ over $U$ have an identical family of $\gamma$-inclusive sets. However, it is clear that $(f, X)$ is not equal to $(g, X)$.

Lemma 27. Let $E = X$ be a finite BE-algebra, and let $(f; X)$ be an int-soft subalgebra over $U$. If $\gamma_i$ and $\gamma_j$ are elements of $\operatorname{Im}(f)$ such that $i_X(f;\gamma_i) = i_X(f;\gamma_j)$, then $\gamma_i = \gamma_j$.

Proof. Straightforward. \hfill $\Box$

Theorem 28. Let $E = X$ be a finite BE-algebra, and let $(f; X)$ and $(g; X)$ be two int-soft subalgebras over $U$ having the identical family of $\gamma$-inclusive sets. If $\operatorname{Im}(f) = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ and $\operatorname{Im}(g) = \{\delta_0, \delta_1, \ldots, \delta_k\}$, where

$$\gamma_0 \not\subset \gamma_1 \not\subset \cdots \not\subset \gamma_n,$$

then we have

1. $r = k$,
2. $i_X(f;\gamma_i) = i_X(g;\delta_i)$, $0 \leq i \leq r$,
3. $(\forall x \in X)(f(x) = \gamma_i \Rightarrow g(x) = \delta_i, 0 \leq i \leq r)$.

Proof. Corollary 25 implies that the only $\gamma$-inclusive sets of $(f, X)$ and $(g, X)$ are the two families $i_X(f;\gamma_i)$ and $i_X(g;\delta_i)$. Since $(f, X)$ and $(g, X)$ have the same family of $\gamma$-inclusive sets, we have $r = k$ which proves (1).

2. Using (1) and Remark 23, we have two chains of $\gamma$-inclusive sets:

$$i_X(f;\gamma_0) \subseteq i_X(f;\gamma_1) \subseteq \cdots \subseteq i_X(f;\gamma_n) = X,$$

$$i_X(g;\delta_0) \subseteq i_X(g;\delta_1) \subseteq \cdots \subseteq i_X(g;\delta_k) = X.$$

Clearly, we have

$$\left(\forall \gamma_i, \gamma_j \in \operatorname{Im}(f)\right) (\gamma_i \not\subset \gamma_j \Rightarrow i_X(f;\gamma_i) \subset i_X(f;\gamma_j)),$$

(49)

$$\left(\forall \delta_i, \delta_j \in \operatorname{Im}(g)\right) (\delta_i \not\subset \delta_j \Rightarrow i_X(g;\delta_i) \subset i_X(g;\delta_j)).$$

(50)

Since two families of $\gamma$-inclusive sets are identical, it is clear that $i_X(f;\gamma_i) = i_X(g;\delta_i)$. By hypothesis, $i_X(f;\gamma_i) = i_X(g;\delta_i)$ for some $j > 0$. Assume that $i_X(f;\gamma_i) \not\subseteq i_X(g;\delta_i)$. Then, $i_X(f;\gamma_i) = i_X(g;\delta_i)$ for some $j > 1$, and $i_X(g;\delta_k) = i_X(f;\gamma_i)$ for some $j > 1$. Thus, by (49) and (50), we have $i_X(f;\gamma_i) = i_X(f;\gamma_j)$ and $i_X(f;\gamma_i) = i_X(g;\delta_k)$. This is a contradiction, and so $i_X(f;\gamma_i) = i_X(g;\delta_i)$. By mathematical induction on $i$, $0 \leq i \leq r$, we finally obtain $i_X(f;\gamma_i) = i_X(g;\delta_i)$, $0 \leq i \leq r$.

3. Let $x \in X$ be such that $f(x) = \gamma_i$ and $g(x) = \delta_j$, where $0 \leq i \leq r$ and $0 \leq j \leq r$. It is sufficient to show that
Let $E = X$ be a BE-algebra. Given any chain of subalgebras

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = X,$$

(51)

there exists an int-soft subalgebra over $U$ whose $\gamma$-inclusive sets are exactly the subalgebras of this chain.

**Proof.** Consider a class $\{y_i \mid i = 0, 1, 2, \ldots, r\}$ of subsets of $U$ such that

$$y_0 \supseteq y_1 \supseteq \cdots \supseteq y_r.$$  

(52)

Define a soft set $(f, X) : E \rightarrow \mathcal{P}(U)$ by $f(A_i) = y_0$ and $f(A_i - A_{i-1}) = y_i$, $0 < i \leq r$. We will prove that $(f, X)$ is an intersectional soft BE-algebra over $U$. Note that if $x \in A_i$, then $y_i \subseteq f(x)$. If $x \in A_j$, then either $x \in A_i - A_j$ or $x \in A_j$ for $i > j$. Thus, if $x \in A_j - A_i$, then $f(x) = y_i$. If $x \in A_i$, then $y_i \subseteq y_j \subseteq f(x)$. Let $x, y \in X$. We distinguish two cases as follows:

**Case 1.** Let $x, y \in A_i - A_{i-1}$. Then, $f(x) = y_i = f(y)$. Since $A_i$ is a subalgebra, we have $x \ast y \in A_i$, and so either $x \ast y \in A_i - A_{i-1}$ or $x \ast y \in A_{i-1}$. In any case we know that

$$f(x) \cap f(y) = y_i \subseteq f(x \ast y).$$

(53)

**Case 2.** For $i > j$, let $x \in A_i - A_{i-1}$ and $y \in A_j - A_{j-1}$. Then, $f(x) = y_i$, $f(y) = y_j$ and $x \ast y \in A_i$. It follows that

$$f(x) \cap f(y) = y_i \cap y_j = y_i \subseteq f(x \ast y).$$

(54)

Hence, we conclude that $(f, X)$ is an intersectional soft BE-algebra over $U$. From the definition of $(f, X)$, we have $\mathrm{Im}(f) = \{y_0, y_1, \ldots, y_r\}$. Thus, the $\gamma$-inclusive sets of $X$ are given by the chain of subalgebras

$$i_X(f; y_0) \subseteq i_X(f; y_1) \subseteq \cdots \subseteq i_X(f; y_r) = X.$$  

(55)

Now, $i_X(f; y_0) = \{x \in X \mid y_0 \subseteq f(x)\} = A_0$. Finally, we prove that $i_X(f; y_i) = A_i$ for $0 < i \leq r$. Clearly $A_i \subseteq i_X(f; y_i)$. If $x \in i_X(f; y_i)$, then $y_i \subseteq f(x)$, and so $x \notin A_j$ for $j > i$. Hence, $f(x) \in \{y_1, y_2, \ldots, y_i\}$, and thus $x \in A_k$ for some $k \leq i$. Since $A_k \subseteq A_i$, we have $x \in A_i$, and so $i_X(f; y_i) = A_i$ for $0 \leq i \leq r$. This completes the proof.

**Theorem 29** is illustrated as an example.

**Example 30.** Let $U = \mathbb{Z}$ be the initial universe set, and let $E = X$ be the set of parameters where $X = \{1, a, b, c, d, 0\}$ is a BE-algebra as in Example 5. Consider subalgebras $A_1 = \{1\}$,

$$A_2 = \{1, a\}, A_3 = \{1, a, c, d\}, A_4 = X.$$

Then, $A_0 \nsubseteq A_2 \nsubseteq A_3 \nsubseteq A_4$. Define a soft set $(f, X)$ over $U$ by

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} Z & \text{if } x = 1, \\ 2Z & \text{if } x = a, \\ 4Z & \text{if } x \in \{c, d\}, \\ 8Z & \text{if } x \in \{b, 0\}. \end{cases}$$

(56)

Then, $(f, X)$ is an int-soft subalgebra over $U$ with $i_X(f; 2Z) = \{1\} = A_1$, $i_X(f; 4Z) = \{1, a, c, d\} = A_2$, $i_X(f; 8Z) = X = A_4$. and $i_X(f; 4Z) = A_3$, and $i_X(f; 8Z) = X = A_4$.

**Theorem 31.** Let $(f, X)$ be a soft set over $U$, and let $\gamma$ be a subset of $U$. Define a soft set $(f^*, X)$ over $U$ by

$$f^* : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} f(x) & \text{if } x \in i_X(f; \gamma), \\ \emptyset & \text{otherwise}. \end{cases}$$

(57)

If $(f, X)$ is an int-soft subalgebra over $U$, then so is $(f^*, X)$.

**Proof.** Let $x, y \in X$. If $x, y \in i_X(f; \gamma)$, then $x \ast y \in i_X(f; \gamma)$, and so

$$f^*(x \ast y) = f(x) \cap f(y) = f^*(x) \cap f^*(y).$$

(58)

If $x \notin i_X(f; \gamma)$ or $y \notin i_X(f; \gamma)$, then $f^*(x) = \emptyset$ or $f^*(y) = \emptyset$. Hence,

$$f^*(x) \cap f^*(y) = \emptyset \subseteq f^*(x \ast y).$$

(59)

Therefore, $(f^*, X)$ is an int-soft subalgebra over $U$.

### 4. Conclusion

Using the notion of int-soft sets, we have introduced the concept of int-soft subalgebras in BE-algebras and investigated related properties. We have considered characterization of an int-soft subalgebra and solved the problem of classifying int-soft subalgebras by their inclusive subalgebras. We have shown that

- (1) every soft image of an int-soft subalgebra is also an int-soft subalgebra;
- (2) the soft intersection of int-soft subalgebras is an int-soft subalgebra.

We have made a new int-soft subalgebra from the old one. Work is ongoing. Some important issues for future work are as follows:

- (1) to develop strategies for obtaining more valuable results,
- (2) to apply these notions and results for studying related notions in other (soft) algebraic structures,
- (3) to study the soft set application in ideal and filter theory of BE-algebras.
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