A Remark on the Global Attractors of the Nonlinear Evolution Equations

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We study the existence of global attractor of the nonlinear elastic rod oscillation equation when the forcing term belongs only to $H^{-1}(\Omega)$; furthermore, we prove that the fractal dimension of global attractor is finite.

1. Introduction

Let $\Omega$ be an open bounded set of $\mathbb{R}^3$ with smooth boundary $\partial \Omega$. We consider the following equation:

$$
\begin{align*}
&u_{tt} - \Delta u - \Delta u_t - \omega \Delta u_{tt} + f(u) = g(x), \\
&(x, t) \in \Omega \times \mathbb{R}^+,
\end{align*}
$$

(1)

where $\omega > 0$ and $g \in H^{-1}(\Omega)$. The nonlinear term $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, and satisfies the following:

$$
\liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, 
$$

(2)

$$
|f'(s)| \leq C(1 + |s|^4), \quad \forall s \in \mathbb{R},
$$

(3)

where $\lambda_1$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ and $C$ is a positive constant.

In line with the Galerkin methods introduced in [1], we know that (1) has a unique solution $u \in C([0, T]; H^1_0(\Omega))$, $u_t \in C([0, T]; H^1_0(\Omega))$, for $g \in H^{-1}(\Omega)$. The proof has no essential difference between $g(x) \in L^2(\Omega)$ and $g(x) \in H^{-1}(\Omega)$, so we omit it; see [2].

Equation (1), which appears as a class of nonlinear evolution equations, like the strain solitary wave equation and dispersive-dissipative wave equation, is used to represent the propagation problems of a lengthwise wave in nonlinear elastic rods and ion-sonic of space transformation by weak nonlinear effect; see [3–6]. For (1), when $g(x) \in L^2(\Omega)$, in [2], the author has discussed the existence of global strong solutions in $(H^2(\Omega) \cap H^1_0(\Omega)) \times (H^2(\Omega) \cap H^1_0(\Omega))$; in [7, 8], the authors have obtained the existence of global attractors in the weak topological space and the strong topology space, respectively. Recently, existence of the uniform compact attractors has been proved about the nonautonomous case of (1); that is, $g(x) = g(x, t)$. In this paper, we prove existence of global attractor and its fractal dimension for (1) under the condition that $g(x)$ only satisfies the lower regularity.

2. The Main Results

Without loss of generality, we denote $H = L^2(\Omega), V = H^1_0(\Omega)$, and $H^*, V^*$ is, respectively, the dual space of $H, V$. Write $\mathscr{H} = H^1_0(\Omega) \times H^1_0(\Omega)$. Let $A = -\Delta$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$; we define $D(A^{(s/2)}); s \in \mathbb{R}$ is Hilbert space family, and its inner product and norm are

$$
\langle \cdot, \cdot \rangle_{D(A^{(s/2)})} = \left( A^{(s/2)} \cdot, A^{(s/2)} \cdot \right), \quad \| \cdot \|_{D(A^{(s/2)})} = \| A^{(s/2)} \cdot \|.
$$

(4)

The following results will be used later.

Lemma 1 (see [8]). Assume that $f$ satisfies (2) and (3), $g \in H^{-1}(\Omega)$; then, the solution semigroup $\{S(t)\}_{t \geq 0}$ has a bounded
absorbing set \( B_0 \) in \( \mathcal{H} \); that is, for any bounded subset \( B \subset \mathcal{H} \), there exists \( T = T(B) \) such that
\[
S(t)B \subset B_0, \quad \forall t \geq T. \tag{5}
\]

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary, and one assumes that \( f \) satisfies (2) and (3), \( g \in H^{-1}(\Omega) \); then, the semigroup \( \{S(t)\}_{t \geq 0} \) possesses a global attractor \( \mathcal{A}_0 \) on \( \mathcal{H} \).

**Proof.** Since \( L^2(\Omega) \to H^{-1}(\Omega) \) is dense, for any \( \eta > 0 \), there exists \( g(x) \in L^2(\Omega) \) such that
\[
\|g(x) - g^\theta(x)\|_{H^{-1}} \leq \eta, \quad \text{for any } g \in H^{-1}(\Omega). \tag{6}
\]
The remained proof of Lemma 2 is similar to that of [7], so we omit it. \( \square \)

**Lemma 3** (see [9]). Let \( B \) be a bounded subset in Hilbert space \( X \), the mapping \( V : B \to X \), such that \( B \subset V(B) \), and satisfy
\[
\begin{align*}
&\|V(v) - V(\tilde{v})\|_X \leq \|v - \tilde{v}\|_X, \quad \forall v, \tilde{v} \in B, \\
&\|Q_NV(v) - Q_NV(\tilde{v})\|_X \leq \delta \|v - \tilde{v}\|_X, \quad (0 < \delta < 1),
\end{align*}
\]
where \( Q_N : X \to X_N^+ \) is orthogonal mapping and \( X_N^+ \) is spanned subspace by the former \( N \)th eigenvector of \( X \); then, the fractal dimension of \( B \) satisfies
\[
d_F(B) \leq \frac{N \ln \left( 8k^2 \delta^2 / (1 - \delta^2) \right)}{\ln (2/(1 - \delta^2))}, \tag{8}
\]
where \( k \) is the Gaussian constant.

**Our main result is as follows.**

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary; one assumes that \( f \) satisfies (2) and (3), \( g \in H^{-1}(\Omega) \); then, the fractal dimension of the global attractor \( \mathcal{A}_0 \) of the semigroup \( \{S(t)\}_{t \geq 0} \) is finite.

**Proof.** According to Lemma 2, provided
\[
u(x,0) = \nu_0(x), \quad \nu(x,0) = \nu_0(x) \in \mathcal{A}_0, \tag{9}
\]
then,
\[
u(x,t) = S(t)\nu_0, \quad \nu(x,t) = S(t)\nu_0 \in \mathcal{A}_0. \tag{10}
\]
Let \( w = u - \nu \) satisfy the following equation:
\[
w_t - \Delta w - \Delta \nu_t - \omega \Delta \nu_0 + f(u) - f(\nu) = 0. \tag{11}
\]
Taking the scalar product of (11) in \( L^2(\Omega) \) with \( w_1 \), we obtain that
\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \left( \|w_1\|^2 + \|A^{1/2}w_1\|^2 + \omega \|A^{1/2}w_1\|^2 \right) \\
&\quad + \|A^{1/2}w_1\|^2 + (f(u) - f(\nu), w_1) = 0.
\end{align*}
\]
As the global attractor \( \mathcal{A}_0 \) is bounded in \( \mathcal{H}(\Omega) \), so there exists \( M_0 > 0 \) such that
\[
\max_{x \in \Omega} \|u\|, \quad \max_{x \in \Omega} \|v\|, \quad \|A^{1/2}u\|, \quad \|A^{1/2}v\| \leq M_0, \tag{13}
\]
Therefore, using (3) and (13), it follows that
\[
(f(u) - f(\nu), w_1) = \int_0^t \int_\Omega f'(\theta u + (1 - \theta) v) \cdot \theta \cdot w_1 \cdot dx.
\]
Let \( C_i \) \( (i = 1, 2, 3) \) be constant independent of \( \omega \); by Gronwall's inequality, we get
\[
\|w_1\|^2 + \|A^{1/2}w_1\|^2 + \omega \|A^{1/2}w_1\|^2 \\
\leq \left( \|w_1(0)\|^2 + \|A^{1/2}w_1(0)\|^2 + \|A^{1/2}w_1(0)\|^2 \right) e^{2C_1t}. \tag{16}
\]
For some \( t_1 > 0 \), define \( l = e^{2C_1t_1} \); hence, we prove that the first inequality in Lemma 3 holds true.

Taking the inner product of (11) in \( L^2(\Omega) \) with \( Q_Nw_1(x,t) \), we commute the operator \( A \) with the projection \( Q_N \) to get
\[
\int_\Omega |\left( f(u) - f(\nu) \right) \cdot Q_Nw_1 | \cdot dx \\
\leq \int_\Omega \left| f'(\theta u + (1 - \theta) v) \cdot |w_1| \cdot |Q_Nw_1| \right| \cdot dx \leq C_4 \|w_1\| \cdot \|Q_Nw_1\|. 
\]
\[ \begin{align*}
&\leq C_4 \| w \|_2 \cdot \left\| \mathcal{A} \left( A \right)^{1/2} Q_N w \right\|_{L^2_{t,N+1}} \\
&\leq \frac{C_4}{2} \lambda^{-1/2}_{N+1} \| w \|^2 + \frac{C_4}{2} \lambda^{-1/2}_{N+1} \left\| Q_N \mathcal{A} \left( A \right)^{1/2} w \right\|^2 \\
&\leq \frac{C_4}{2\lambda_1} \lambda^{-1/2}_{N+1} \left\| A \right\|^{1/2} w \right\|^2 + \frac{C_4}{2} \lambda^{-1/2}_{N+1} \left\| Q_N \mathcal{A} \left( A \right)^{1/2} w \right\|^2 .
\end{align*} \]

(18)

Then,
\[ \frac{1}{2} \frac{d}{dt} \left( \| Q_N w \|^2 + \left\| A \left( A \right)^{1/2} Q_N w \right\|^2 + \omega \left\| A \left( A \right)^{1/2} Q_N w \right\|^2 \right) \]
\[ + \left( 1 - C_4 \lambda^{-1/2}_{N+1} \right) \left\| A \left( A \right)^{1/2} Q_N w \right\|^2 \]
\[ \leq \frac{C_4}{\lambda_1} \lambda^{-1/2}_{N+1} + \lambda^{-1}_{N+1} \left\| A \left( A \right)^{1/2} w \right\|^2 .
\]

(20)

We let
\[ y(t) = \| Q_N w \|^2 + \left\| A \left( A \right)^{1/2} Q_N w \right\|^2 + \omega \left\| A \left( A \right)^{1/2} Q_N w \right\|^2 .
\]

(21)

Choosing \( N \) large enough that \( 1 - C_4 \lambda^{-1/2}_{N+1} > 0 \) and setting \( \alpha = \min\{\lambda^{-1}_{N+1}, \lambda^{-1/2}_{N+1}, 1 - C_4 \lambda^{-1/2}_{N+1}\} \), integrating with (16) and (20), we get
\[ y'(t) + \alpha y(t) \leq \frac{C_4}{\lambda_1} \lambda^{-1/2}_{N+1} \\
\times \left( \left\| w_1(0) \right\|^2 + \left\| A \left( A \right)^{1/2} w(0) \right\|^2 + \left\| A \left( A \right)^{1/2} w_1(0) \right\|^2 \right) e^{2C_1t} .
\]

(22)

Gronwall's inequality implies that
\[ y(t) \leq y(0) e^{-\alpha t} + \frac{1}{2C_3 + \alpha} \left\{ \frac{C_4}{\lambda_1} \lambda^{-1/2}_{N+1} + \lambda^{-1}_{N+1} \right\} \\
\times \left( \left\| w_1(0) \right\|^2 + \left\| A \left( A \right)^{1/2} w(0) \right\|^2 + \left\| A \left( A \right)^{1/2} w_1(0) \right\|^2 \right) e^{2C_1t} .
\]

(23)

So, we have
\[ \left\| Q_N w \right\|^2 \leq y(t) \leq \left\| w_0 \right\|^2 \\
\times \left( e^{-\alpha t} + \frac{1}{2C_3 + \alpha} \left\{ \frac{C_4}{\lambda_1} \lambda^{-1/2}_{N+1} + \lambda^{-1}_{N+1} \right\} e^{2C_1t} \right) .
\]

(24)

Take a proper \( t_1 > 0 \) and \( N \) large enough such that
\[ e^{-\alpha t_1} + \frac{1}{2C_3 + \alpha} \left\{ \frac{C_4}{\lambda_1} \lambda^{-1/2}_{N+1} + \lambda^{-1}_{N+1} \right\} e^{2C_1t_1} \leq \delta < 1 .
\]

(25)

Therefore, for \( t = t_1 \), \( S(t_1) \) satisfies the condition of Lemma 3; then, the fractal dimension of the global attractor \( \mathcal{A}_0 \) satisfies
\[ d_F(B) \leq \frac{N \ln \left( 8k^2 l^2 / \left( 1 - \delta^2 \right) \right)}{\ln \left( 2/(1 - \delta^2) \right)} .
\]

(26)

This implies that the global attractor for semigroup \( \{ S(t) \}_{t \geq 0} \) generated by the problem (1) has a finite fractal dimension. \( \square \)

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**References**


