Research Article

Symmetry Reduction of the Two-Dimensional Ricci Flow Equation

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This paper is devoted to obtain the one-dimensional group invariant solutions of the two-dimensional Ricci flow ((2D) Rf) equation. By classifying the orbits of the adjoint representation of the symmetry group on its Lie algebra, the optimal system of one-dimensional subalgebras of the ((2D) Rf) equation is obtained. For each class, we will find the reduced equation by the method of similarity reduction. By solving these reduced equations, we will obtain new sets of group invariant solutions for the ((2D) Rf) equation.

1. Introduction

The Ricci flow was introduced by Hamilton in his seminal paper, "*Three-manifolds with positive Ricci curvature*" in 1982 [1]. Since then, Ricci flow has been a very useful tool for studying the special geometries which a manifold admits. Ricci flow is an evolution equation for a Riemannian metric which sometimes can be used in order to deform an arbitrary metric into a suitable metric that can specify the topology of the underlying manifold. If $(M, g(t))$ is a smooth Riemannian manifold, Ricci flow is defined by

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}, \tag{1}$$

where Ric denotes the Ricci tensor of the metric g. By using the concept of Ricci flow, Grisha Perelman completely proved the Poincaré conjecture around 2003 [2–4]. The Ricci flow also is used as an approximation to the renormalization group flow for the two-dimensional nonlinear σ-model, in quantum field theory; see [5] and references therein. The ricci flow equation is related to one of the models used in obtaining the quantum theory of gravity [6]. Because some difficulties appear when a quantum field theory is formulated, the studies focus on less dimensional models which are called mechanical models.

In this paper, we want to obtain new solutions of ((2D) Rf) equation by method of Lie symmetry group. As it is well known, Lie symmetry group method has an important role in the analysis of differential equations. The theory of Lie symmetry groups of differential equations was developed by Lie at the end of nineteenth century [7]. By this method, we can reduce the order of ODEs and investigate the invariant solutions. Also we can construct new solutions from known ones (for more details about the applications of Lie symmetries, see [8–10]). Lie's method led to an algorithmic approach to find special solution of differential equation by its symmetry group. These solutions are called group invariant solutions and obtained by solving the reduced system of differential equation having fewer independent variables than the original system. This fact that for some PDEs, the symmetry reductions are unobtainable by the Lie symmetry method, caused the creation of some generalizations of this method. These generalizations are called nonclassical symmetry method and was described in many references such as [11–14].

In this paper, we apply the Lie symmetry method to obtain the invariant solutions of ((2D) Rf) equation and classify them. This paper is organized as follows. In Section 2, by using the mechanical model of Ricci flow, Lie symmetries of ((2D) Rf) equation will be stated. Also we achieve some results from the structure of the Lie algebra of the Lie

symmetry group. In Section 3, we will construct an optimal system of one-dimensional subalgebras of the ((2D) Rf) equation which is useful for classifying of the group invariant solutions. In Section 4, the reduced equation for each element of optimal system is obtained. In Section 5, we will solve the reduced equations by method of Lie symmetry group and obtain the group invariant solutions of ((2D) Rf) equation.

2. Lie Symmetries of ((2D) Rf) Equation

As we know, transformations which map solutions of a differential equation to other solutions are called symmetries of the equation. The procedure of finding the Lie symmetry group of a PDE was described in many studies such as [8, 9, 15]. Before performing the Lie symmetries of Ricci flow, let us restate the mechanical model of Ricci flow that introduced by Cimpoiaus and Constantinescu [16].

The metric tensor of the space, g_{ij}, can be written in the conformally flat frame

$$
\begin{aligned}
ds^2 = g_{ij} dx^i dx^j &= 2e^{\phi(x,y,t)} dx dy \\
&= \frac{1}{2} e^{\phi(X,Y,t)} \left(dX^2 + dY^2 \right)
\end{aligned}
\tag{2}
$$

using Cartesian coordinates X and Y or the complex variables $2x = Y + iX, 2y = Y - iX$ [6]. According to (1), the function $\phi(X, Y, t)$ must satisfy

$$
\frac{\partial}{\partial t} e^{\phi} = \Delta\phi,
\tag{3}
$$

where Δ is Laplacian. By introducing the field

$$
u(x, y, t) = e^{\phi},
\tag{4}
$$

Equation (3) takes the form $u_t = (\ln u)_{xy}$ or in the equivalent form:

$$
u^2 u_t + u_y u_x - u u_{xy} = 0.
\tag{5}
$$

Cimpoiaus and Constantinescu, also obtained the Lie symmetry group of this equation [16]. They proved that this equation admits a 6-parameter Lie group, G, with the following infinitesimal generators for its Lie algebra, $\mathfrak{g}$:

$$
X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = \partial_t,
$$

$$
X_4 = t\partial_t + u\partial_u, \qquad X_5 = x\partial_x - u\partial_u, \qquad X_6 = y\partial_y - u\partial_u.
\tag{6}
$$

The commutator table of Lie algebra for $\mathfrak{g}$ is given in Table 1, where the entry in the ith row and jth column is $[X_i, X_j] = X_i X_j - X_j X_i, i, j = 1, \ldots, 6$.

TABLE 1: The commutator table of $\mathfrak{g}$.

[,]	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	0	X_1	0
X_2	0	0	0	0	0	X_2
X_3	0	0	0	X_3	0	0
X_4	0	0	$-X_3$	0	0	0
X_5	$-X_1$	0	0	0	0	0
X_6	0	$-X_2$	0	0	0	0

Exponentiating the infinitesimal symmetries (6), we obtain the one-parameter groups $g_k(s)$ generated by $X_k, k = 1, \ldots, 6$ as follows:

$$
\begin{aligned}
g_1(s) &: (x, y, t, u) \longmapsto (x + s, y, t, u), \\
g_2(s) &: (x, y, t, u) \longmapsto (x, y + s, t, u), \\
g_3(s) &: (x, y, t, u) \longmapsto (x, y, t + s, u), \\
g_4(s) &: (x, y, t, u) \longmapsto (x, y, te^s, ue^s), \\
g_5(s) &: (x, y, t, u) \longmapsto (xe^s, y, t, ue^{-s}), \\
g_6(s) &: (x, y, t, u) \longmapsto (x, ye^s, t, ue^{-s}).
\end{aligned}
\tag{7}
$$

Consequently, we can state the following theorem.

Theorem 1. *If* $f = f(x, y, t)$ *is a solution of* (5), *so are functions*

$$
\begin{aligned}
g_1(s) \cdot f &= f(x - s, y, t), \\
g_2(s) \cdot f &= f(x, y - s, t), \\
g_3(s) \cdot f &= f(x, y, t - s), \\
g_4(s) \cdot f &= f(x, y, te^{-s}) e^s, \\
g_5(s) \cdot f &= f(xe^{-s}, y, t) e^{-s}, \\
g_6(s) \cdot f &= f(x, ye^{-s}, t) e^{-s}.
\end{aligned}
\tag{8}
$$

3. One-Dimensional Optimal System of Subalgebras for the ((2D) Rf) Equation

In this section, we obtain the one-dimensional optimal system of ((2D) Rf) equation by using symmetry group. Since every linear combination of infinitesimal symmetries is an infinitesimal symmetry, there is an infinite number of one-dimensional subgroups for G. Therefore, it is important to determine which subgroups give different types of solutions. For this, we must find invariant solutions which cannot be transformed to each other by symmetry transformations in the full symmetry group. This led to the concept of an optimal system of subalgebra. For one-dimensional subalgebras, this classification problem is the same as the problem of classifying the orbits of the adjoint representation [8]. Optimal set of subalgebras is obtained by selecting only one representative from each class of equivalent subalgebras. The problem of classifying the orbits is solved by taking a general element

Table 2: The adjoint representation table of the infinitesimal generators X_i.

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	X_4	$X_5 - sX_1$	X_6
X_2	X_1	X_2	X_3	X_4	X_5	$X_6 - sX_2$
X_3	X_1	X_2	X_3	$X_4 - sX_3$	X_5	X_6
X_4	X_1	X_2	$e^s X_3$	X_4	X_5	X_6
X_5	$e^s X_1$	X_2	X_3	X_4	X_5	X_6
X_6	X_1	$e^s X_2$	X_3	X_4	X_5	X_6

in the Lie algebra and simplifying it as much as possible by imposing various adjoint transformation on it [15, 17]. Adjoint representation of each X_i, $i = 1, \ldots, 6$ is defined by Lie series

$$\mathrm{Ad}\left(\exp\left(s \cdot X_i\right) \cdot X_j\right) = X_j - s \cdot \left[X_i, X_j\right]$$
$$+ \frac{s^2}{2} \cdot \left[X_i, \left[X_i, X_j\right]\right] - \cdots, \tag{9}$$

where s is a parameter and $[X_i, X_j]$ is the commutator of the Lie algebra for $i, j = 1, \ldots, 6$ [8]. It is important to note that following the convention of [8], we used the right invariant vector fields to define the Lie algebra in this paper. As a consequence a minus sign is present in Lie series.

Taking into account the table of commutator, we can compute all the adjoint representations corresponding to the Lie group of the ((2D) Rf) equation. They are presented in Table 2. Note that, the (i, j) entry indicate $\mathrm{Ad}(\exp(s \cdot X_i) \cdot X_j)$.

Now we can state the following theorem.

Theorem 2. *A one-dimensional optimal system for Lie algebra of ((2D) Rf) equation is given by*

$$\begin{aligned}
&(1)\ X_1 + aX_2 + bX_3, \\
&(2)\ X_1 \pm X_2 + cX_4, \\
&(3)\ X_1 \pm X_3 + cX_6, \\
&(4)\ X_1 + cX_4 + dX_6, \\
&(5)\ X_2 \pm X_3 + cX_5, \\
&(6)\ X_2 + cX_4 + dX_5, \\
&(7)\ X_3 + cX_5 + dX_6, \\
&(8)\ X_4 + cX_5 + dX_6,
\end{aligned} \tag{10}$$

where $a, b, c, d \in \mathbb{R}$ *and* $a \neq 0, b \neq 0$.

Proof. Let $F_i^s : \mathfrak{g} \to \mathfrak{g}$ be the adjoint transformation defined by $X \mapsto \mathrm{Ad}(\exp(sX_i) \cdot X)$, for $i = 1, \ldots, 6$. The matrix of F_i^s, $i = 1, \ldots, 6$, with respect to basis $\{X_1, \ldots, X_6\}$ is

$$M_1^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -s & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -s & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_3^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_4^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_5^s = \begin{bmatrix} e^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_6^s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^s & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

respectively. If $X = \sum_{i=1}^{6} a_i X_i$, then we have

$$F_6^{s_6} \circ F_5^{s_5} \circ \cdots \circ F_1^{s_1} : X$$
$$\longmapsto e^{s_5} a_1 X_1 + e^{s_6} a_2 X_2 + e^{s_4} a_3 X_3 + \left(a_4 - e^{s_4} s_3 a_3\right) X_4$$
$$+ \left(a_5 - e^{s_5} s_1 a_1\right) X_5 + \left(a_6 - e^{s_6} s_2 a_2\right) X_6. \tag{12}$$

Now, we try to vanish the coefficients of X by acting the adjoint representations $M_i^{s_i}$ on X, and choosing suitable parameters s_i in each step. Therefore, we can simplify X as follows.

If $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$, then we can make the coefficients of X_5, X_6, and X_4 vanish by $F_1^{S_1}$, $F_2^{S_2}$, and $F_3^{S_3}$; by setting $s_1 = a_5/a_1$, $s_2 = a_6/a_2$ and $s_3 = a_4/a_3$, respectively. Scaling X if necessary, we can assume that $a_1 = 1$. So X is reduced to the case (1).

If $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 = 0$, then we can make the coefficients of X_5 and X_6 vanish by $F_1^{S_1}$ and $F_2^{S_2}$; by setting $s_1 = a_5/a_1$ and $s_2 = a_6/a_2$, respectively. Also we can make the coefficient of X_2, ± 1 by $F_6^{S_6}$; by setting $s_6 = -\ln|a_2|$. Scaling X if necessary, we can assume that $a_1 = 1$. So X is reduced to the case (2).

If $a_1 \neq 0$, $a_3 \neq 0$ and $a_2 = 0$, then we can make the coefficients of X_5 and X_4 vanish by $F_1^{S_1}$ and $F_3^{S_3}$; by setting $s_1 = a_5/a_1$ and $s_3 = a_4/a_3$, respectively. Also we can make the coefficient of X_3, ± 1 by $F_4^{S_4}$; by setting $s_4 = -\ln|a_3|$. Scaling X if necessary, we can assume that $a_1 = 1$. So X is reduced to the case (3).

If $a_1 \neq 0$ and $a_2 = a_3 = 0$, then we can make the coefficient of X_5 vanish by $F_1^{S_1}$; by setting $s_3 = a_5/a_1$.

TABLE 3: Lie invariants, similarity solutions, and reduced equations.

i	$\mathfrak{h}_i$	$\{z_i, w_i, v_i\}$	u_i	Similarity reduced equations
1	$X_1 + X_6$	$\{ye^{-x}, t, uy\}$	$\dfrac{f(z,w)}{y}$	$f^2 f_w - z^2 f_z^2 + zff_z + z^2 ff_{zz} = 0$
2	$X_2 + X_4$	$\{x, te^{-y}, ue^{-y}\}$	$f(z,w)e^y$	$f^2 f_w - wf_z f_w + wff_{zw} = 0$
3	$X_3 + X_5 + dX_6$	$\{\dfrac{y}{x^d}, \ln\dfrac{e^t}{x}, ux^{d+1}\}$	$\dfrac{f(z,w)}{x^{d+1}}$	$f_w(f^2 - f_z) - dzf_z^2 + f(df_z + dzf_{zz} + f_{zw}) = 0$
4	$X_2 + X_3 + X_5$	$\{\ln\dfrac{e^y}{x}, \ln\dfrac{e^t}{x}, ux\}$	$\dfrac{f(z,w)}{x}$	$f^2 f_w - f_z^2 - f_w f_z + ff_{zz} + ff_{zw} = 0$
5	$X_2 + X_5$	$\{\ln\dfrac{e^y}{x}, t, ux\}$	$\dfrac{f(z,w)}{x}$	$f^2 f_w - f_z^2 + ff_{zz} = 0$
6	$X_3 + X_6$	$\{x, t - \ln y, uy\}$	$\dfrac{f(z,w)}{y}$	$f^2 f_w - f_z f_w + ff_{zw} = 0$
7	$X_1 + X_2$	$\{y - x, t, u\}$	$f(z,w)$	$f^2 f_w - f_z^2 + ff_{zz} = 0$
8	$X_2 + X_3$	$\{x, t - y, u\}$	$f(z,w)$	$f^2 f_w - f_z f_w + ff_{zw} = 0$

Scaling X if necessary, we can assume that $a_1 = 1$. So X is reduced to the case (4).

If $a_2 \neq 0$, $a_3 \neq 0$ and $a_1 = 0$, then we can make the coefficients of X_6 and X_4 vanish by $F_2^{S_2}$ and $F_3^{S_3}$; by setting $s_2 = a_6/a_2$ and $s_3 = a_4/a_3$, respectively. Also we can make the coefficient of X_3, ± 1 by $F_4^{S_4}$; by setting $s_4 = -\ln|a_3|$. Scaling X if necessary, we can assume that $a_2 = 1$. So X is reduced to the case (5).

If $a_2 \neq 0$ and $a_1 = a_3 = 0$, then we can make the coefficient of X_6 vanish by $F_2^{S_2}$; by setting $s_2 = a_6/a_2$. Scaling X if necessary, we can assume that $a_2 = 1$. So X is reduced to the case (6).

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, then we can make the coefficient of X_4 vanish by $F_3^{S_3}$; by setting $s_3 = a_4/a_3$. Scaling X if necessary, we can assume that $a_3 = 1$. So X is reduced to the case (7).

If $a_1 = a_2 = a_3 = 0$, then X is reduced to the case (8).

There is not any more possible case for investigating and the proof is complete. $\qquad\square$

4. Similarity Reduction of ((2D) Rf) Equation

In this section, the two-dimensional Ricci flow equation will be reduced by expressing it in the new coordinates. The ((2D) Rf) equation is expressed in the coordinates (x, y, t, u), we must search for this equation's form in the suitable coordinates for reducing it. These new coordinates will be obtained by looking for independent invariants (z, w, f) corresponding to the generators of the symmetry group. Hence, by using the new coordinates and applying the chain rule, we obtain the reduced equation. We express this procedure for one of the infinitesimal generators in the optimal system (10) and list the result for some other cases.

For example, consider the case (4) in Theorem 2 when $c = 0$ and $d = 1$; therefore, we have $X := X_1 + X_6$. For determining independent invariants I, we ought to solve the PDEs $X(I) = 0$, that is

$$(X_1 + X_6)I = \left(\partial_x + y\partial_y - u\partial_u\right)I$$
$$= \frac{\partial I}{\partial x} + y\frac{\partial I}{\partial y} + 0\frac{\partial I}{\partial t} - u\frac{\partial I}{\partial u} = 0. \tag{13}$$

For solving this PDE, the following associated characteristic ODE must be solved:

$$\frac{dx}{1} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{-u}. \tag{14}$$

Hence, three functionally independent invariants $z = ye^{-x}$, $w = t$, and $f = uy$ are obtained. If we treat f as a function of z and w, we can compute formulae for the derivatives of u with respect to x, y, and t in terms of z, w, f and the derivatives of f with respect to z and w. By using the chain rule and the fact that $u = f(z,w)y^{-1}$, we have

$$u_t = \left(f_z z_t + f_w w_t\right)y^{-1} = f_w y^{-1},$$
$$u_x = -f_z e^{-x},$$
$$u_y = f_z e^{-x} y^{-1} - fy^{-2}, \tag{15}$$
$$u_{xy} = -e^{-2x}f_{zz}.$$

After substituting the above relations into (5), we obtain

$$u^2 u_t + u_y u_x - uu_{xy}$$
$$= y^{-3}\left(f^2 f_w - f_z^2 z^2 + ff_z z + ff_{zz}z^2\right) = 0. \tag{16}$$

So the reduced equation is

$$f^2 f_w - z^2 f_z^2 + zff_z + z^2 ff_{zz} = 0. \tag{17}$$

This equation has two independent variables z and w and one dependent variable f. In a similar way, we can compute all of the similarity reduction equations corresponding to the infinitesimal symmetries that mentioned in Theorem 2. Some of them are listed in Table 3.

TABLE 4: ODEs obtained from the reduced equations of Table 3.

i	Symmetry group generators	Optimal system	Invariants $\{s, g\}$	Reduced equation
1	$V_1 = \dfrac{1}{2}z \ln z \partial_z + w\partial_w$ $V_2 = \partial_w$ $V_3 = -\dfrac{1}{2}z \ln z \partial_z + f\partial_f$ $V_4 = z\partial_z$	$\mathscr{A}_1^1 : V_2$ $\mathscr{A}_1^2 : V_3$ $\mathscr{A}_1^3 : V_1 + V_3$ $\mathscr{A}_1^4 : V_2 + V_4$	$\{z, f\}$ $\{w, f(\ln z)^2\}$ $\{z, \dfrac{f}{w}\}$ $\{w - \ln z, f\}$	$sg'^2 - gg' - sgg'' = 0$ $g^2(g' + 2) = 0$ $g^3 - s^2 g'^2 + sgg' + s^2 gg'' = 0$ $g^2 g' - g'^2 + gg'' = 0$
2	$V_1 = w\partial_w + f\partial_f$ $V_2 = z\partial_z - f\partial_f$ $V_3 = \partial_z$	$\mathscr{A}_2^1 : V_2$ $\mathscr{A}_2^2 : V_3$	$\{w, fz\}$ $\{w, f\}$	$g^2 g' = 0$ $g^2 g' = 0$
3	$V_1 = z \ln z \partial_z + w\partial_w - f(1 + \ln z)\partial_f$ $V_2 = \partial_w$ $V_3 = z\partial_z - f\partial_f$	$\mathscr{A}_3^1 : V_1$ $\mathscr{A}_3^2 : V_2$ $\mathscr{A}_3^3 : V_3$	$\left\{\dfrac{w}{\ln z}, fz \ln z\right\}$ $\{z, f\}$ $\{w, fz\}$	$gg'(g + 2ds - 1) + dg^2 + s(ds - 1)(gg'' - g'^2) = 0$ $sg'^2 - gg' - sgg'' = 0$ $g^2 g' = 0$
4	$V_1 = z\partial_z + w\partial_w - f\partial_f$ $V_2 = \partial_w$ $V_3 = \partial_z$	$\mathscr{A}_4^1 : V_1$ $\mathscr{A}_4^2 : V_2$ $\mathscr{A}_4^3 : V_2 - V_3$	$\left\{\dfrac{w}{z}, fz\right\}$ $\{z, f\}$ $\{z + w, f\}$	$gg'(1 - g - 2s) - g^2 + s(s - 1)(g'^2 - gg'') = 0$ $-g'^2 + gg'' = 0$ $g^2 g' - 2g'^2 + 2sgg'' = 0$
5	$V_1 = \dfrac{z}{2}\partial_z + w\partial_w$ $V_2 = \partial_w$ $V_3 = -\dfrac{z}{2}\partial_z + f\partial_f$ $V_4 = \partial_z$	$\mathscr{A}_5^1 : V_3$ $\mathscr{A}_5^2 : V_1 + V_3$ $\mathscr{A}_5^3 : V_2 + V_4$	$\{w, fz^2\}$ $\left\{z, \dfrac{f}{w}\right\}$ $\{w - z, f\}$	$g' + 2 = 0$ $g^3 - g'^2 + gg'' = 0$ $g^2 g' - g'^2 + gg'' = 0$
6	$V_1 = w\partial_w$ $V_2 = \partial_w$ $V_3 = z\partial_z - f\partial_f$ $V_4 = \partial_z$	$\mathscr{A}_6^1 : V_4$ $\mathscr{A}_6^2 : V_2 + V_3$	$\{w, f\}$ $\{w - \ln z, zf\}$	$g' = 0$ $g^2 g' + g'^2 - gg'' = 0$
7	$V_1 = \dfrac{z}{2}\partial_z + w\partial_w$ $V_2 = \partial_w$ $V_3 = -\dfrac{z}{2}\partial_z + f\partial_f$ $V_4 = \partial_z$	$\mathscr{A}_7^1 : V_2$ $\mathscr{A}_7^2 : V_3$ $\mathscr{A}_7^3 : V_2 + V_4$ $\mathscr{A}_7^4 : V_1 - V_3$	$\{z, f\}$ $\{w, fz^2\}$ $\{w - z, f\}$ $\left\{\dfrac{w}{z}, fz\right\}$	$g'^2 - gg'' = 0$ $g^2(g' + 2) = 0$ $g^2 g' - g'^2 + gg'' = 0$ $g^2 g' - s^2 g'^2 + 2sgg' + g^2 + s^2 gg'' = 0$
8	$V_1 = w\partial_w$ $V_3 = z\partial_z - f\partial_f$ $V_2 = \partial_w$ $V_4 = \partial_z$	$\mathscr{A}_8^1 : V_2 + V_4$ $\mathscr{A}_8^2 : V_2 + V_3$	$\{w - z, f\}$ $\{w - \ln z, zf\}$	$g^2 g' + g'^2 - gg'' = 0$ $g^2 g' + g'^2 - gg'' = 0$

5. Group Invariant Solutions of ((2D) Rf) Equation

In this section, we reduce the equations obtained in last section to ODEs and solve them.

For example, (17) admits a 4-parameter family of Lie operators with following infinitesimal generators:

$$V_1 = \frac{1}{2}z \ln z \partial_z + w\partial_w,$$

$$V_2 = \partial_w,$$

$$V_3 = -\frac{1}{2}z \ln z \partial_z + f\partial_f, \tag{18}$$

$$V_4 = z\partial_z.$$

The invariants associated to the infinitesimal generator V_2, are $s = z$ and $g = f$. By substituting these invariants into (17) and using chain rule, the reduced equation is obtained as follows:

$$sg'^2 - gg' - sgg'' = 0 \tag{19}$$

the solution of this equation is $g(s) = c_2 s^{c_1} = c_2 z^{c_1}$, where c_1 and c_2 are arbitrary constants. therefore we have $f(z) = c_2 z^{c_1} = c_2 (ye^{-x})^{c_1} = c_2 y^{c_1} e^{-c_1 x}$. So $u = fy^{-1} = c_2 y^{c_1-1} e^{-c_1 x}$ is a solution of (5).

By similar arguments, we can obtain other invariant solutions of (17). Also by reducing other equations in Table 3, we can find other solutions of ((2D) Rf) equation. Some of the similarity reduced equations are listed in Table 4.

Table 5: Group invariant solutions of the ((2D) Rf) equation.

$\mathscr{A}_j^i$	Invariant solution
$\mathscr{A}_1^2$	$-2s + c_1$
$\mathscr{A}_1^3$	$\dfrac{1}{2c_1^2}\left(1 - \tanh\left(\dfrac{\ln s - c_2}{2c_1}\right)^2\right)$
$\mathscr{A}_2^1$	c_1
$\mathscr{A}_3^2$	$c_2 s^{c_1}$
$\mathscr{A}_3^1$	$\dfrac{c_1(1+c_1)s^{c_1}}{-s^{c_1}(1+c_1-ds)+dc_1c_2(1+c_1)(ds-1)^{c_1+1}}$
$\mathscr{A}_4^3$	$\dfrac{2c_1 e^{c_1(s+c_2)}}{-1 + e^{c_1(s+c_2)}}$
$\mathscr{A}_5^2$	$\dfrac{1}{2c_1^2}\left(1 - \tanh\left(\dfrac{s+c_2}{2c_1}\right)^2\right)$
$\mathscr{A}_7^1$	$c_2 e^{c_1 s}$
$\mathscr{A}_7^4$	$\dfrac{-c_1^2 e^{c_1/s}}{e^{c_1/s}(c_1 - s) - sc_1^2 c_2}$
$\mathscr{A}_8^1$	$\dfrac{c_1 e^{c_1(s+c_2)}}{1 - e^{c_1(s+c_2)}}$

In Table 5, we obtain the invariant solutions of ((2D) Rf) equation corresponding to some of the similarity-reduced equations.

6. Conclusion

In this paper, by using the adjoint representation of the symmetry group on its Lie algebra, we have constructed an optimal system of one-dimensional subalgebras for a well-known partial differential equation in mathematical physics called: two-dimensional Ricci flow equation. Moreover, we have obtained the similarity-reduced equations for each element of optimal system as well as some group invariant solutions of two-dimensional Ricci flow equation.

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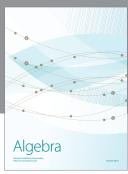
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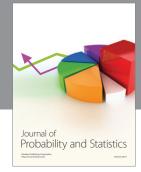
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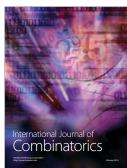Differential Equations

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