Research Article

Some Properties for Certain Class of Analytic Functions with Varying Arguments

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We introduce a new class $V(k, \lambda, A, B)$ of analytic functions with varying arguments in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ defined by the Salagean operator. The object of the present paper is to determine coefficient estimates, extreme points, and distortion theorems for functions $f(z)$ belonging to the class $V(k, \lambda, A, B)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in U , we say that f is subordinate to g , written $f < g$ if there exists a Schwarz function w , which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function g is univalent in U , then we have the following equivalence (cf, e.g., [1, 2]):

$$f(z) < g(z) \iff f(0) = g(0), f(U) \subset g(U). \quad (2)$$

For function $f(z) \in \mathcal{A}$, Salagean [3] defined the differential operator D^k as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^k f(z) &= D(D^{k-1}f(z)), \\ (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}). \end{aligned} \quad (3)$$

For $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $k \in \mathbb{N}_0$, and for all $z \in U$, let $S(k, \lambda, A, B)$ denotes the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criterion:

$$(1 - \lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' < \frac{1 + Az}{1 + Bz}. \quad (4)$$

It is noticed that for suitable choices of k , λ , A , and B , we obtain the following subclasses studied by various authors:

- (1) $S(k, \lambda, A, B) = G_k(\lambda, 1, A, B) = G_k(\lambda, A, B)$ ($\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $k \in \mathbb{N}_0$) (see Sivasubramanian et al. [4], with $b = 1$);
- (2) $S(k, \lambda, 2\alpha - 1, 1) = S\mathfrak{S}(k, \lambda, \alpha)$ ($k \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $\lambda \geq 0$) (see Aouf et al. [5], with $b = 1$);
- (3) $S(0, \lambda, 2\alpha - 1, 1) = B(\lambda, \alpha)$ ($0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$) (see Chunyi and Owa [6]);
- (4) $S(0, 0, 2\alpha - 1, 1) = B(\alpha)$ ($0 \leq \alpha < 1$) (see Chen [7, 8] and Goel [9]);
- (5) $S(1, 0, 2\alpha - 1, 1) = C(\alpha)$ ($0 \leq \alpha < 1$) (see Srivastava and Owa [10]).

Also we note the following.

- (1) Putting $k = 0$ in (4), the class $S(0, \lambda, A, B)$ reduces to the class $S(\lambda, A, B) = \{(1 - \lambda)(f(z)/z) + \lambda f'(z) < (1 + Az)/(1 + Bz)\}$, $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

Definition 1 (see [11]). A function $f(z)$ defined by (1) is said to be in the class $V(\theta_n)$ if $f(z) \in \mathcal{A}$ and $\arg(a_n) = \theta_n$ for all $n \geq 2$. If, furthermore, there exists a real number β such that

$$\theta_n + (n-1)\beta \equiv \pi \pmod{2\pi}, \quad (5)$$

then $f(z)$ is said to be in the class $V(\theta_n; \beta)$. The union of $V(\theta_n; \beta)$ taken over all possible sequences $\{\theta_n\}$ and all possible real numbers β is denoted by V .

Let $V(k, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z)$ in $S(k, \lambda, A, B)$.

We note the following:

- (1) $V(0, \lambda, 2\alpha - 1, 1) = V_\lambda(\alpha)$ ($0 \leq \alpha < 1$, $\lambda \geq 0$) (see Aouf et al. [12]);
- (2) $V(0, 0, 2\alpha - 1, 1) = B_\alpha$ ($0 \leq \alpha < 1$) (see Srivastava and Owa [10]);
- (3) $V(1, 0, 2\alpha - 1, 1) = C_\alpha$ ($0 \leq \alpha < 1$) (see Srivastava and Owa [10]).

Also we note that

- (i) $V(k, \lambda, 2\alpha - 1, 1) = V\mathfrak{F}(k, \lambda, \alpha)$ ($k \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $\lambda \geq 0$) denotes the subclass of V , consisting of functions $f(z)$ belonging to the class $S\mathfrak{F}(k, \lambda, \alpha)$;
- (ii) $V(0, \lambda, A, B) = V(\lambda, A, B)$ ($\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$) denotes the subclass of V , consisting of functions $f(z)$ belonging to the class $S(\lambda, A, B)$.

2. Coefficient Estimates

Unless otherwise mentioned, we will assume in the reminder of this paper that $\lambda \geq 0$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $k \in \mathbb{N}_0$.

Theorem 2. Let the function $f(z) \in \mathcal{A}$ be of the form (1), if

$$\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] (1+B) |a_n| \leq (B-A), \quad (6)$$

then $f(z) \in S(k, \lambda, A, B)$.

Proof. A function $f(z)$ of the form (1) belongs to the class $S(k, \lambda, A, B)$ if and only if there exists a function w , $|w(z)| \leq |z|$, such that

$$(1-\lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (7)$$

or, equivalently,

$$\left| \frac{(1-\lambda) (D^k f(z)/z) + \lambda (D^k f(z))' - 1}{B [(1-\lambda) (D^k f(z)/z) + \lambda (D^k f(z))'] - A} \right| < 1. \quad (8)$$

Thus, it is sufficient to prove that

$$\left| (1-\lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' - 1 \right| - \left| B \left[(1-\lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' \right] - A \right| < 0. \quad (9)$$

Indeed, letting $|z| = r$ ($0 \leq r < 1$) we have

$$\begin{aligned} & \left| (1-\lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' - 1 \right| \\ & - \left| B \left[(1-\lambda) \frac{D^k f(z)}{z} + \lambda (D^k f(z))' \right] - A \right| \\ & = \left| \sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] a_n z^{n-1} \right| \\ & - \left| (B-A) + \sum_{n=2}^{\infty} B n^k [1 + \lambda(n-1)] a_n z^{n-1} \right| \quad (10) \\ & \leq \left(\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] |a_n| r^{n-1} \right. \\ & \quad \left. - (B-A) + \sum_{n=2}^{\infty} B n^k [1 + \lambda(n-1)] |a_n| r^{n-1} \right) \\ & < \sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] (1+B) |a_n| - (B-A). \end{aligned}$$

In view of (6), the last inequality is less than zero; hence $f(z) \in S(k, \lambda, A, B)$. This completes the proof of Theorem 2. $\square$

Theorem 3. Let the function $f(z) \in \mathcal{A}$ be of the form (1), then $f(z)$ is in the class $V(k, \lambda, A, B)$ if and only if

$$\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] (1+B) |a_n| \leq (B-A). \quad (11)$$

Proof. In view of Theorem 2, we need only to show that each function $f(z)$ from the class $V(k, \lambda, A, B)$ satisfies the coefficient inequality (6). Let $f(z) \in V(k, \lambda, A, B)$. Then, by (8) and (1), we have

$$\left| \frac{\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] a_n z^{n-1}}{(B-A) + \sum_{n=2}^{\infty} B n^k [1 + \lambda(n-1)] a_n z^{n-1}} \right| < 1. \quad (12)$$

Since $f(z) \in V$, $f(z)$ lies in the class $V(\theta_n, \beta)$ for some sequence $\{\theta_n\}$ and a real number β such that $\theta_n + (n-1)\beta \equiv \pi \pmod{2\pi}$ ($n \geq 2$), set $z = re^{i\beta}$ in the above inequality and since $\operatorname{Re}\{w(z)\} < |w(z)| < 1$, then

$$\frac{\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] |a_n| r^{n-1}}{(B-A) - \sum_{n=2}^{\infty} B n^k [1 + \lambda(n-1)] |a_n| r^{n-1}} < 1. \quad (13)$$

It is clear that the denominator of the left-hand said cannot vanish for $r \in [0, 1)$. Moreover, it is positive for $r = 0$ and in consequence for $r \in [0, 1)$. Thus, we have

$$\sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)] (1+B) |a_n| r^{n-1} \leq (B-A), \quad (14)$$

which, upon letting $r \rightarrow 1^-$, readily yields the assertion (11). This completes the proof of Theorem 3. $\square$

Corollary 4. Let the function $f(z) \in \mathcal{A}$ defined by (1) be in the class $V(k, \lambda, A, B)$, then

$$|a_n| \leq \frac{(B-A)}{n^k [1 + \lambda(n-1)](1+B)} \quad (n \geq 2). \quad (15)$$

The result is sharp for the function

$$f(z) = z + \frac{(B-A)}{n^k [1 + \lambda(n-1)](1+B)} e^{i\theta_n} z^n \quad (n \geq 2). \quad (16)$$

3. Distortion Theorems

Theorem 5. Let the function $f(z) \in \mathcal{A}$ defined by (1) be in the class $V(k, \lambda, A, B)$. Then

$$\begin{aligned} |z| - \frac{(B-A)}{2^k(1+B)(1+\lambda)} |z|^2 \\ \leq |f(z)| \leq |z| + \frac{(B-A)}{2^k(1+B)(1+\lambda)} |z|^2. \end{aligned} \quad (17)$$

The result is sharp.

Proof. Since

$$\Psi(n) = n^k [1 + \lambda(n-1)](1+B) \quad (18)$$

is an increasing function of n ($n \geq 2$), from Theorem 2, we have

$$\begin{aligned} 2^k(1+B)(1+\lambda) \sum_{n=2}^{\infty} |a_n| \\ \leq \sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)](1+B) |a_n| \leq (B-A), \end{aligned} \quad (19)$$

that is,

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(B-A)}{2^k(1+B)(1+\lambda)}. \quad (20)$$

Thus,

$$\begin{aligned} |f(z)| &= \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^n \\ &\leq |z| + \frac{(B-A)}{2^k(1+B)(1+\lambda)} |z|^2. \end{aligned} \quad (21)$$

Similarly, we get

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^n \geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^2 \\ &\geq |z| - \frac{(B-A)}{2^k(1+B)(1+\lambda)} |z|^2. \end{aligned} \quad (22)$$

Finally, the result is sharp for the function

$$f(z) = z + \frac{(B-A)}{2^k(1+B)(1+\lambda)} e^{i\theta_2} z^2 \quad (23)$$

at $z = \pm |z| e^{-i\theta_2}$.

This completes the proof of Theorem 5. $\square$

Corollary 6. Under the hypotheses of Theorem 5, $f(z)$ is included in a disc with center at the origin and radius r_1 given by

$$r_1 = 1 + \frac{(B-A)}{2^k(1+B)(1+\lambda)}. \quad (24)$$

Theorem 7. Let the function $f(z) \in \mathcal{A}$ defined by (1) belong to the class $V(k, \lambda, A, B)$ ($k \geq 1$). Then

$$\begin{aligned} 1 - \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)} |z| \\ \leq |f'(z)| \leq 1 + \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)} |z|. \end{aligned} \quad (25)$$

The result is sharp for the function $f(z)$ given by (23) at $z = \pm |z| e^{-i\theta_2}$.

Proof. Since $\{n\Psi(n)\}$, where $\Psi(n)$ given by (18) is increasing function of n ($n \geq 2$), in view of Theorem 3, we have

$$\begin{aligned} \frac{2^k(1+B)(1+\lambda)}{2} \sum_{n=2}^{\infty} n |a_n| \\ \leq \sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)](1+B) |a_n| \leq (B-A), \end{aligned} \quad (26)$$

that is,

$$\sum_{n=2}^{\infty} n |a_n| \leq \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)}. \quad (27)$$

Thus,

$$\begin{aligned} |f'(z)| &= \left| 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq 1 + \sum_{n=2}^{\infty} n |a_n| |z| \\ &\leq 1 + \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)} |z|. \end{aligned} \quad (28)$$

Similarly, we get

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} \geq 1 - \sum_{n=2}^{\infty} n |a_n| |z| \\ &\geq 1 - \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)} |z|. \end{aligned} \quad (29)$$

Finally, the result is sharp for the function $f(z)$ given by (23). This completes the proof of Theorem 7. $\square$

Corollary 8. Let the function $f(z) \in \mathcal{A}$ defined by (1) be in the class $V(k, \lambda, A, B)$. Then $f'(z)$ is included in a disc with center at the origin and radius r_2 given by

$$r_2 = 1 + \frac{(B-A)}{2^{k-1}(1+B)(1+\lambda)}. \quad (30)$$

4. Extreme Points

Theorem 9. Let the function $f(z) \in \mathcal{A}$ defined by (1) be in the class $V(k, \lambda, A, B)$, with $\arg(a_n) = \theta_n$ where $[\theta_n + (n-1)\beta] \equiv \pi \pmod{2\pi}$. Define $f_1(z) = z$ and

$$f_n(z) = z + \frac{(B-A)}{n^k [1 + \lambda(n-1)](1+B)} e^{i\theta_n} z^n. \quad (31)$$

Then $f(z)$ is in the class $V(k, \lambda, A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z), \quad (32)$$

where $\mu_n \geq 0$ ($n \geq 1$) and $\sum_{n=1}^{\infty} \mu_n = 1$.

Proof. If $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$ with $\sum_{n=1}^{\infty} \mu_n = 1$ and $\mu_n \geq 0$, then

$$\begin{aligned} & \sum_{n=2}^{\infty} n^k [1 + \lambda(n-1)](1+B) \frac{(B-A)}{n^k [1 + \lambda(n-1)](1+B)} \mu_n \\ &= \sum_{n=2}^{\infty} (B-A) \mu_n = (B-A)(1 - \mu_1) \leq (B-A). \end{aligned} \quad (33)$$

Hence, $f(z) \in V(k, \lambda, A, B)$.

Conversely, let the function $f(z)$ defined by (1) belong to the class $V(k, \lambda, A, B)$, and define

$$\begin{aligned} \mu_n &= \frac{n^k [1 + \lambda(n-1)](1+B)}{(B-A)} |a_n|, \\ \mu_1 &= 1 - \sum_{n=2}^{\infty} \mu_n. \end{aligned} \quad (34)$$

From Theorem 3, $\sum_{n=1}^{\infty} \mu_n \leq 1$ and so $\mu_n \geq 0$. Since $\mu_n f_n(z) = \mu_n z + a_n z^n$, then

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z + \sum_{n=2}^{\infty} a_n z^n = f(z). \quad (35)$$

This completes the proof of Theorem 9. $\square$

Remark 10. Specializing k, λ, A , and B in the above results, we obtain the corresponding results for the corresponding classes $V\mathfrak{S}(k, \lambda, \alpha)$ and $V(\lambda, A, B)$ defined in Section 1.

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