Research Article

A Subclass of Harmonic Univalent Functions Associated with \( q \)-Analogue of Dziok-Srivastava Operator

Huda Aldweby and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia

Correspondence should be addressed to Maslina Darus; maslina@ukm.my

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We study a class of complex-valued harmonic univalent functions using a generalized operator involving basic hypergeometric function. Precisely, we give a necessary and sufficient coefficient condition for functions in this class. Distortion bounds, extreme points, and neighborhood of such functions are also considered.

1. Introduction

Let \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disc, and let \( S_{H} \) denote the class of functions which are complex valued, harmonic, univalent, and sense preserving in \( \mathbb{U} \) normalized by \( f(0) = f'(0) - 1 = 0 \). Each \( f \in S_{H} \) can be expressed as \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{U} \). We call \( h \) the analytic part and \( g \) the coanalytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense preserving in \( \mathbb{U} \) is that \( |h'(z)| > |g'(z)| \) in \( \mathbb{U} \) (see [1]). In [2], there is a more comprehensive study on harmonic univalent functions. Thus, for \( f = h + \overline{g} \in S_{H} \), we may write

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad (0 \leq b_1 < 1).
\]

Note that \( S_{H} \) reduces to \( S \), the class of normalized analytic univalent functions, if the coanalytic part of \( f = h + \overline{g} \) is identically zero.

The study of basic hypergeometric series (also called \( q \)-hypergeometric series) essentially started in 1748 when Euler considered the infinite product \( (q; q)_{\infty}^{-1} = \prod_{k=0}^{\infty} (1 - q^{k+1})^{-1} \). In the literature, we were told that the development of these functions was much slower until, in 1878, Heine converted a simple observation that \( \lim_{q \to 1} [(1 - q^s)/(1 - q)] = a \) which returns the theory of \( \phi_1 \) basic hypergeometric series to the famous theory of Gauss's \( _2F_1 \) hypergeometric series. The importance of basic hypergeometric functions is due to their application in deriving \( q \)-analogues of well-known functions, such as \( q \)-analogues of the exponential, gamma, and beta functions. In this paper, we define a class of starlike harmonic functions using basic hypergeometric functions and investigate its properties like coefficient condition, distortion theorem, and extreme points.

For complex parameters \( a_i, b_j, q \) \( (i = 1, \ldots, r, \ j = 1, \ldots, s, \ b_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, |q| < 1) \), we define the basic hypergeometric function, \( \Phi_r(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) \) by

\[
\Phi_r(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, q)_k \cdots (a_r, q)_k}{(q, q)_k (b_1, q)_k \cdots (b_s, q)_k} z^k, \quad (r = s + 1; \ r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \ z \in \mathbb{U})
\]

where \( \mathbb{N} \) denote the set of positive integers and \( (a, q)_k \) is the \( q \)-shifted factorial defined by

\[
(a, q)_k = \begin{cases} 1, & k = 0; \\ (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{k-1}), & k \in \mathbb{N}. \end{cases}
\]
We note that
\[
\lim_{q \to 1^-} \left[ r \Phi_s \left( q^a_1, \ldots, q^a_r; q^b_1, \ldots, q^b_s, q, (q-1)^{1+s-r} z \right) \right]
= r F_s \left( a_1, \ldots, a_r; b_1, \ldots, b_s, z \right),
\]
where, \( F_s \left( a_1, \ldots, a_r; b_1, \ldots, b_s, z \right) \) is the well-known generalized hypergeometric function. By the ratio test, one observes that for \( |q| < 1 \) and \( r = s + 1 \) the series defined in (2) converges absolutely in \( \mathbb{U} \) so that it represented an analytic function in \( \mathbb{U} \). For more mathematical background of basic hypergeometric functions, one may refer to [3, 4].

The \( q \)-derivative of a function \( h(x) \) is defined by
\[
D_q (h(x)) = \frac{h(qx) - h(x)}{(q-1)x}, \quad q \neq 1, \quad x \neq 0.
\] (5)

For a function \( h(z) = z^k \), we can observe that
\[
D_q (h(z)) = D_q (z^k) = \frac{1 - q^k}{1 - q} z^{k-1} = [k]_q z^{k-1}.
\] (6)

Then \( \lim_{q \to 1^-} D_q (h(z)) = \lim_{q \to 1} [k]_q z^{k-1} = k z^{k-1} = h'(z) \), where \( h'(z) \) is the ordinary derivative. For more properties of \( D_q \), see [4, 5].

Corresponding to the function \( \Phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s, q, z) \), consider
\[
\mathcal{S} \left( a_1, \ldots, a_r; b_1, \ldots, b_s; q, z \right) = z \Phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)
= z + \sum_{k=1}^{\infty} \left( a_1, \ldots, a_r; b_1, \ldots, b_s ; q \right) \frac{z^k}{k!}.
\] (7)

The authors [6] defined the linear operator \( H^r_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, f) \) by
\[
H^r_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q) f(z) = \mathcal{S} \left( a_1, \ldots, a_r; b_1, \ldots, b_s; q, f(z) \right)
= z + \sum_{k=2}^{\infty} \Gamma \left( a_1, q, k \right) q \frac{z^k}{k!},
\]
where (*) stands for convolution and
\[
\Gamma \left( a_1, q, k \right) = \frac{\left( a_1, q \right) \cdots \left( a_r, q \right) \frac{z^k}{k!}}{(q, q) \cdots (b_1, q) \cdots (b_s, q) k!}, \quad \left( a_1, q \right) = a_1 \left( a_1, q \right), \ldots, \left( a_r, q \right) = a_r \left( a_r, q \right).
\] (9)

To make the notation simple, we write
\[
H^r_s \left[ a_1, q, f(z) \right] = H^r_s \left( a_1, \ldots, a_r; b_1, \ldots, b_s; q, f(z) \right).
\] (10)

We define the operator (8) of harmonic function \( f = h + \overline{g} \) given by (1) as
\[
H^r_s \left[ a_1, q, f(z) \right] = H^r_s \left( a_1, q, h(z) \right) + H^r_s \left( a_1, q, \overline{g(z)} \right).
\] (11)

**Definition 1.** For \( 0 \leq \delta < 1 \), let \( S^r_H(a_1, q, \delta) \) denote the subfamily of starlike harmonic functions \( f \in S^r_H \) of the form (1) such that
\[
\frac{\partial}{\partial \theta} \left( \arg H^r_s \left[ a_1, q, f \right] \right) \geq \delta, \quad |z| = r < 1.
\] (12)

Following [7], a function \( f \) is said to be in the class \( V^r_H(a_1, \delta, q) = S^r_H(a_1, \delta, q) \cap V^r_H \) if \( f \) of the form (1) satisfies the condition that
\[
\arg \left( a_k \right) = \theta_k, \quad \arg \left( b_k \right) = \theta_k \quad (k \geq n + 1; \ n \in \mathbb{N})
\] (13)

and if there exists a real number \( \rho \) such that
\[
\theta_k + (k-1) \phi \equiv \pi \pmod{2\pi}, \quad \theta_k + (k-1) \phi \equiv 0, \quad (k \geq n + 1; \ n \in \mathbb{N}).
\] (14)

By specializing the parameters of \( H^r_s[a_1, q, f] \), we obtain different classes of starlike harmonic functions, for example,

(i) for \( r = s + 1, a_2 = b_1, \ldots, a_s = b_s, S^r_H(a_1, q, \delta) = SH(\delta) \) [8] is the class of sense-preserving harmonic univalent functions which are starlike of order \( \delta \) in \( \mathbb{U} \); that is, \( \partial / \partial \theta \left( \arg D^r f(z) \right) \geq \delta \);

(ii) for \( r = s + 1, a_2 = b_1, \ldots, a_r = b_s, a_1 = q^{s+1}, q \to 1, S^r_H(q^{s+1}, q, \delta) = R^s_H(n, \alpha) \) [9] is the class of starlike harmonic univalent functions with \( \partial / \partial \theta \left( \arg D^n f(z) \right) \geq \delta \), where \( D^r \) is the Ruscheweyh derivative (see [10]);

(iii) for \( r = \{1, \ldots, r\}, j = \{1, \ldots, s\}, r = s + 1, a_1 = q^n, \) and \( b_j = q^b, q \to 1, S^r_H(a_1, q, \delta) = S^r_H(a_1, \delta) \) [11] is the class of starlike harmonic univalent functions with \( \partial / \partial \theta \left( \arg H^r_s[a_1, f] \right) \geq \delta \), where \( H^r_s[a_1] \) is the Dziok-Srivastava operator (see [12]).

**2. Main Results**

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in \( S^r_H(a_1, \delta, q) \).

**Theorem 2.** Let \( f = h + \overline{g} \) be given by (1). If
\[
\sum_{k=2}^{\infty} \left( \left| k \right|_q - \delta \left| a_k \right| + \left| k \right|_q + \delta \left| b_k \right| \right) \Gamma \left( a_1, q, k \right)
\leq 1 - \frac{1 + \delta}{1 - \delta} \left| b_1 \right|,
\] (15)

where \( a_1 = 1, \ 0 \leq \delta < 1, \) and \( \Gamma(a_1, q, k) \) is given by (9), then \( f \in S^r_H(a_1, \delta, q) \).
Proof. To prove that \( f \in S_H^*(a_1, \delta, q) \), we only need to show that if (15) holds, then the required condition (12) is satisfied. For (12), we can write

\[
\frac{\partial}{\partial \theta} \left( \text{arg} H'_s[a_1, q] f(z) \right) = \Re \left\{ \frac{zD_q (H'_s[a_1, q] h(z))}{H'_s[a_1, q] h(z) + H'_s[a_1, q] g(z)} \right\} - \frac{\sum_{k=1}^{\infty} [k]_q + \delta}{1 - \delta} \Gamma (a_1, q, k) \|b_k\| \]

\[
= 2 (1 - \delta) |z| \left\{ 1 - \frac{1 + \delta}{1 - \delta} |b_1| \right\}
\]

\[
- \left[ \sum_{k=2}^{\infty} \left( \frac{[k]_q - \delta}{1 - \delta} |a_k| + \frac{[k]_q + \delta}{1 - \delta} |b_k| \right) \Gamma (a_1, q, k) \right]\]

\[
x \Gamma (a_1, q, k) \right\},
\]

\[
(16)
\]

The last expression is nonnegative by (15), and so, \( f \in S_H^*(a_1, \delta, q) \).

Now, we obtain the necessary and sufficient conditions for \( f = h + g \) given by (14).

Theorem 3. Let \( f = h + g \) be given by (11). Then, \( f \in V_{TH}(a_1, \delta, q) \) if and only if

\[
\sum_{k=2}^{\infty} \left( \frac{[k]_q - \delta}{1 - \delta} |a_k| + \frac{[k]_q + \delta}{1 - \delta} |b_k| \right) \Gamma (a_1, q, k) \leq 1 - \frac{1 + \delta}{1 - \delta} |b_1|,
\]

(19)

where \( a_1 = 1, 0 \leq \delta < 1 \), and \( \Gamma (a_1, q, k) \) is given by (9).

Proof. Since \( V_{TH}(a_1, \delta, q) \subset S_H^*(a_1, \delta, q) \), we only to prove the only if part of the theorem. So that for functions \( f \in V_{TH}(a_1, \delta, q) \), we notice that the condition \( (\partial/\partial \theta) (\text{arg} H'_s[a_1, q] f(z)) \geq \delta \) is equivalent to

\[
\frac{\partial}{\partial \theta} \left( \text{arg} H'_s[a_1, q] f(z) \right) = \Re \left\{ \frac{zD_q (H'_s[a_1, q] h(z))}{H'_s[a_1, q] h(z) + H'_s[a_1, q] g(z)} \right\}
\]

\[
- \frac{\sum_{k=1}^{\infty} [k]_q + \delta}{1 - \delta} \Gamma (a_1, q, k) \|b_k\| \]

\[
= \Re \left\{ \frac{zD_q (H'_s[a_1, q] h(z))}{H'_s[a_1, q] h(z) + H'_s[a_1, q] g(z)} \right\} - \frac{\sum_{k=1}^{\infty} [k]_q + \delta}{1 - \delta} \Gamma (a_1, q, k) \|b_k\| \]

\[
\geq 0.
\]

(20)

That is,

\[
\Re \left\{ \frac{(1 - \delta) z + \sum_{k=2}^{\infty} \left( \frac{[k]_q - \delta}{1 - \delta} \Gamma (a_1, q, k) |a_k| z^k - \sum_{k=1}^{\infty} \left( \frac{[k]_q + \delta}{1 - \delta} \Gamma (a_1, q, k) |b_k| z^k \right) \right)}{z + \sum_{k=2}^{\infty} \Gamma (a_1, q, k) |a_k| z^k + \sum_{k=1}^{\infty} \Gamma (a_1, q, k) |b_k| z^k} \right\} \geq 0.
\]

(21)
The previous condition must hold for all values of \( z \) in \( U \). Upon choosing \( \phi \) according to (14), we must have

\[
\frac{(1 - \delta) - (1 + \delta)|b_1|}{1 + |b_1| + \sum_{k=2}^{\infty} (|a_k| + |b_k|) \Gamma(a_1, q, k) r^{k-1}} - \frac{\sum_{k=2}^{\infty} ((|k|_q - \delta)|a_k| + (|k|_q + \delta)|b_k|) \Gamma(a_1, q, k) r^{k-1}}{1 + |b_1| + \sum_{k=2}^{\infty} (|a_k| + |b_k|) \Gamma(a_1, q, k) r^{k-1}} \geq 0.
\]

(22)

If condition (19) does not hold, then the numerator in (22) is negative for \( r \) sufficiently close to 1. Hence, there exist \( z_0 = r_0 \) in (0,1) for which the quotient of (22) is negative. This contradicts the fact that \( f \in \text{clco} V_H(a_1, \delta, q) \), and this completes the proof.

The following theorem gives the distortion bounds for functions in \( V_H(a_1, \delta, q) \) which yield a covering result for this class.

**Theorem 4.** If \( f \in V_H(a_1, \delta, q) \), then

\[
|f(z)| \leq (1 + |b_1|) r + \frac{1 - \delta}{\Gamma(a_1, q, 2) (2l_q - \delta)} \sum_{k=2}^{\infty} \left( |a_k| + |b_k| \right) r^{k-1} \\
\times \Gamma(a_1, q, 2)^2 \left( 1 + \frac{1 - \delta}{1 - \delta} |b_1| \right)^2 \\
\leq (1 + |b_1|) r + \frac{1 - \delta}{\Gamma(a_1, q, 2) ((q + 1) - \delta)} \\
\times \left[ 1 - \frac{1 - \delta}{1 - \delta} |b_1| \right]^2 r^2.
\]

(26)

That is,

\[
|f(z)| \leq (1 + |b_1|) r + \frac{1 - \delta}{\Gamma(a_1, q, 2) (2l_q - \delta)} \sum_{k=2}^{\infty} \left( |a_k| + |b_k| \right) r^{k-1} \\
\times \Gamma(a_1, q, 2)^2 \left( 1 + \frac{1 - \delta}{1 - \delta} |b_1| \right)^2 \\
\leq (1 + |b_1|) r + \frac{1 - \delta}{\Gamma(a_1, q, 2) ((q + 1) - \delta)} \\
\times \left[ 1 - \frac{1 - \delta}{1 - \delta} |b_1| \right]^2 r^2.
\]

(27)

**Corollary 5.** Let \( f \) be of the form (1) so that \( f \in V_H(a_1, \delta, q) \). Then,

\[
\left\{ w : |w| < \frac{2 \Gamma(a_1, q, 2) - 1 - (\Gamma(a_1, q, 2) - 1) \delta}{(q + 1) - \delta} \Gamma(a_1, q, 2) \\
- \frac{2 \Gamma(a_1, q, 2) - 1 - (\Gamma(a_1, q, 2) - 1) \delta}{((q + 1) + \delta) \Gamma(a_1, q, 2)} |b_1| \right\} \subset f(U).
\]

(28)

Next, one determines the extreme points of closed convex hull of \( V_H(a_1, \delta, q) \) denoted by \( \text{clco} V_H(a_1, \delta, q) \).

**Theorem 6.** Set

\[
\lambda_k = \frac{1 - \delta}{(|k|_q - \delta) \Gamma(a_1, q, k)},
\]

(29)

\[
\mu_k = \frac{1 - \delta}{(|k|_q + \delta) \Gamma(a_1, q, k)}.
\]

For \( b_1 \) fixed, the extreme points for \( \text{clco} V_H(a_1, \delta, q) \) are

\[
\left\{ z + \lambda_k x z^k + b_1 z \right\} \cup \left\{ z + b_1 z + \mu_k x z^k \right\},
\]

(30)

where \( k \geq 2 \) and \( |x| = 1 - |b_1| \).

**Proof.** Any function \( f \in \text{clco} V_H(a_1, \delta, q) \) may be expressed as

\[
f(z) = z + \sum_{k=2}^{\infty} |a_k| e^{a_k z^k} + b_1 z + \sum_{k=2}^{\infty} |b_k| e^{b_k z^k}.
\]
where the coefficients satisfy the inequality (15). Set \( h_1(z) = z, g_1(z) = b_1z, h_k(z) = \ldots \), for \( k = 2, 3, \ldots \). Writing \( X_k = |a_k|/|\lambda_k|, Y_k = |b_k|/|\mu_k|, k = 2, 3, \ldots \) and \( X_1 = 1 - \sum_{k=2}^{\infty} X_k; Y_1 = 1 - \sum_{k=2}^{\infty} Y_k \), we have

\[
f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)).
\]

In particular, set

\[
f_1(z) = z + b_1 z, \quad f_k(z) = z + \lambda_k x z^k + b_1 z + \mu_k y z^k, \quad (k \geq 2, |x| + |y| = 1 - |b_1|).
\]

Therefore, the extreme points of \( \text{clco} V_{\mathcal{F}}(a_1, \delta, q) \) are contained in \( \{ f_k(z) \} \). To see that \( f_1 \) is not an extreme point, note that \( f_1 \) may be written as a convex linear combination of functions in \( \text{clco} V_{\mathcal{F}}(a_1, \delta, q) \) as follows:

\[
f_1(z) = \frac{1}{2} \left\{ f_1(z) + \lambda z (1 - |b_1|) z^2 \right\} + \frac{1}{2} \left\{ f_1(z) - \lambda z (1 - |b_1|) z^2 \right\}.
\]

If both \( |x| \neq 0 \) and \( |y| \neq 0 \), we will show that it can also be expressed as a convex linear combination of functions in \( \text{clco} V_{\mathcal{F}}(a_1, \delta, q) \). In our case, let us define the generalized \( q \cdot \gamma \)-neighborhood of a function \( f \in S_{\mathcal{F}} \) as follows:

\[
N^q_\gamma(f) = \left\{ F = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k : \sum_{k=2}^{\infty} [k]_q (|a_k - A_k| + |b_k - B_k|) \right. 
\]

\[
+ |b_1 - B_1| \leq \gamma \left. \right\}.
\]

In our case, let us define the generalized \( q \cdot \gamma \)-neighborhood of \( f \) to be the set

\[
N^q_\gamma(f) = \left\{ F = \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[ (|k|_q - \delta)|a_k - A_k| \right. 
\]

\[
+ \left. (|k|_q + \delta)|b_k - B_k| \right] \right. 
\]

\[
+ (1 + \delta)|b_1 - B_1| \leq (1 - \delta)\gamma \left. \right\}.
\]

Theorem 7. Let \( f \) be given by (1). If \( f \) satisfies the conditions

\[
\sum_{k=2}^{\infty} [k]_q (|k|_q - \delta)|a_k| \Gamma(a_1, q, k)
\]

\[
+ \sum_{k=2}^{\infty} [k]_q (|k|_q + \delta)|b_k| \Gamma(a_1, q, k) \leq 1 - \delta,
\]

\[
0 \leq \delta < 1,
\]

\[
\gamma \leq \frac{1 - \delta}{q + 1 - \delta} \left( 1 - \frac{1 + \delta}{1 - \delta}|b_1| \right),
\]

then \( N^q_\gamma(f) \subset S^*_H(a_1, \delta, q) \).

Proof. Let \( f \) satisfy (39) and let

\[
F(z) = z + B_1 z + \sum_{k=2}^{\infty} \left( A_k z^k + B_k z^k \right)
\]

belong to \( N^q_\gamma(f) \). We have

\[
(1 + \delta)|B_1| + \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[ (|k|_q - \delta)|A_k| \right. 
\]

\[
+ \left. (|k|_q + \delta)|B_k| \right] \leq (1 + \delta)|B_1 - b_1| + (1 + \delta)|b_1|
\]
\[
+ \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[ \left( \lfloor k \rfloor^q_1 - \delta \right) |A_k - a_k| + \left( \lfloor k \rfloor^q_1 + \delta \right) |B_k - b_k| \right] \\
+ \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[ \left( \lfloor k \rfloor^q_1 - \delta \right) |A_k| + \left( \lfloor k \rfloor^q_1 + \delta \right) |B_k| \right] \\
\leq (1 - \delta) \gamma + (1 + \delta) |b_1| \\
+ \frac{1}{(q + 1) - \delta} \sum_{k=2}^{\infty} \Gamma(a_1, q, k) \left[ \lfloor k \rfloor^q_1 \left( \lfloor k \rfloor^q_1 - \delta \right) |A_k| + \lfloor k \rfloor^q_1 \left( \lfloor k \rfloor^q_1 + \delta \right) |B_k| \right] \\
\leq (1 - \delta) \gamma + (1 + \delta) |b_1| + \frac{1}{(q + 1) - \delta} \left[ (1 - \delta) - (1 + \delta) |b_1| \right] \leq 1 - \delta.
\]

Hence,
\[ \gamma' \leq \frac{1 - \delta}{(q + 1) - \delta} \left( 1 - \frac{1 + \delta}{1 - \delta} |b_1| \right), \quad F \in S_H^* (a_1, \delta, q). \]

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