Research Article

New Subclasses of Biunivalent Functions Involving Dziok-Srivastava Operator

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We introduce two new subclasses of biunivalent functions which are defined by using the Dziok-Srivastava operator. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

1. Introduction

Let $A$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let $S$ denote the class of all functions in $A$ which are univalent in $U$.

Some of the important and well-investigated subclasses of the univalent function class $S$ include, for example, the class $S^*(\beta)$ of starlike functions of order $\beta$ in $U$ and the class $K(\beta)$ of convex functions of order $\beta$ in $U$. By definition, we have

$$S^*(\alpha) = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta, \right\},$$

$$K(\alpha) = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \right\}.$$

Ding et al. [1] introduced the following class $Q_\lambda(\beta)$ of analytic functions defined as follows:

$$Q_\lambda(\beta) = \left\{ f \in A : \text{Re} \left( (1 - \lambda) \frac{f(z)}{z} + \lambda f''(z) \right) > \beta, \right\},$$

$$0 \leq \beta < 1, \lambda \geq 0.$$

It is easy to see that $Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1$, $0 \leq \beta < 1$, $Q_\lambda(\beta) \subset Q_1(\beta) = \{ f \in A : \text{Re} f''(z) > \beta, 0 \leq \beta < 1 \}$ and hence $Q_\lambda(\beta)$ is univalent class (see [2–4]).

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$

$$f \left( f^{-1}(w) \right) = w \quad \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left( 2a_2^2 - a_3 \right) w^3$$

$$+ \left( 5a_3^2 - 5a_2a_3 + a_4 \right) w^4 + \cdots.$$
bi-univalent functions in $U$ given by (1). For a brief history and interesting examples in the class $\Sigma$ see [5].

Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class $\Sigma$ similar to the familiar subclasses $S^\ast (\beta)$ and $K(\beta)$ of starlike and convex functions of order $\beta$ ($0 \leq \beta < 1$), respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in A$ is in the class $S_\alpha^\ast (\alpha)$ of strongly bi-starlike functions of order $\alpha$ ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \left| \arg\left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, z \in U),$$

$$f \in \Sigma, \quad \left| \arg\left( \frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad (0 < \alpha \leq 1, z \in U),$$

where $g$ is the extension of $f^{-1}$ to $U$. The classes $S_\alpha^\ast (\alpha)$ and $K_\alpha^\ast (\alpha)$ of bi-starlike functions of order $\alpha$ and biconvex functions of order $\alpha$, corresponding, respectively, to the function classes $S^\ast (\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_\alpha^\ast (\alpha)$ and $K_\alpha^\ast (\alpha)$, they found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_1|$ and $|a_2|$ (for details, see [6, 7]).

For function $f$ given by (1) and $g$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) of $f$ and $g$ is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z).$$

For complex parameters $a_1, \ldots, a_q$ and $b_1, \ldots, b_l$ ($b_j \neq Z_0^n = \{0, -1, -2, \ldots\}; j = 1, \ldots, s$), the generalized hypergeometric function $_qF_s$ is defined by the following infinite series:

$$_qF_s \left( a_1, \ldots, a_q; b_1, \ldots, b_s; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n z^n}{(b_1)_n \cdots (b_s)_n n!} \quad (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, 3, \ldots\}; z \in U),$$

where $(\theta)_n$ is the Pochhammer symbol (or shift factorial) defined, in terms of the Gamma function $\Gamma$, by

$$\left( \theta \right)_n = \frac{\Gamma \left( \theta + n \right)}{\Gamma \left( \theta \right)} = \begin{cases} 1, & (n = 0) \\ \theta \left( \theta + 1 \right) \cdots \left( \theta + n - 1 \right), & (n \in \mathbb{N}). \end{cases}$$

Correspondingly a function $h(a_1, \ldots, a_q; b_1, \ldots, b_l; z)$ is defined by

$$h(a_1, \ldots, a_q; b_1, \ldots, b_l; z) = z \cdot _qF_s \left( a_1, \ldots, a_q; b_1, \ldots, b_s; z \right) \quad (z \in U).$$

Dziok and Srivastava [9] (see also [10]) considered a linear operator

$$H \left( a_1, \ldots, a_q; b_1, \ldots, b_l \right) : A \rightarrow A,$$

defined by the following Hadamard product:

$$H \left( a_1, \ldots, a_q; b_1, \ldots, b_l \right) f(z) = h(a_1, \ldots, a_q; b_1, \ldots, b_l; z) \ast f(z), \quad (q \leq s + 1; q, s \in \mathbb{N}_0; z \in U).$$

If $f \in A$ is given by (1), then we have

$$H \left( a_1, \ldots, a_q; b_1, \ldots, b_l \right) f(z) = z + \sum_{n=0}^{\infty} \Gamma_n \left[ a_1; b_1 \right] a_n z^n \quad (z \in U),$$

where

$$\left[ a_1; b_1 \right] = \left( \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n n!} \right) \quad (n \in \mathbb{N}).$$

To make the notation simple, we write

$$H_{q,s} \left[ a_1; b_1 \right] = H \left( a_1, \ldots, a_q; b_1, \ldots, b_l \right) f(z).$$

It easily follows from (14) that

$$z \left( H_{q,s} \left[ a_1; b_1 \right] \right)' = a_1 H_{q,s} \left[ a_1 + 1; b_1 \right] - (a_1 - 1) H_{q,s} \left[ a_1; b_1 \right].$$

The linear operator $H_{q,s} \left[ a_1; b_1 \right]$ is a generalization of many other linear operators considered earlier.

The object of the present paper is to introduce two new subclasses of the bi-univalent functions which are defined by using the Dziok-Srivastava operator and find estimates on the coefficients $|a_1|$ and $|a_2|$ for functions in these new subclasses of the function class $\Sigma$ employing the techniques used earlier by Srivastava et al. [5] (see also [11]).

In order to derive our main results, we have to recall here the following lemma [12].

**Lemma 1.** If $h \in P$, then $|c_k| \leq 2$ for each $k$, where $P$ is the family of all functions $h$ analytic in $U$ for which $\text{Re} h(z) > 0$ holds.

Unless otherwise mentioned, we assume throughout this paper that $a_1, b_j \in \mathbb{C} \setminus \mathbb{Z}_0^i, i = 1, \ldots, s, j = 1, \ldots, q, s \leq$ $s + 1; q, s \in \mathbb{N}_0, 0 < \alpha \leq 1, \lambda \geq 1, z \in U, \Gamma_n \left[ a_1; b_1 \right]$ is given by (15) and all powers are understood as principle values.
2. Coefficient Bounds of the Function Class

Definition 2. One says that a function \( f(z) \) given by (1) is said to be in the class \( T_{q,s}^\alpha[a_1; b_1, \alpha, \lambda] \) if it satisfies the following condition:

\[
f \in \Sigma \quad \left| \arg \left( \frac{(1 - \lambda) H_{q,s} [a_1; b_1; z]}{z} + \lambda \left( H_{q,s} [a_1; b_1; z] \right)' \right) \right| < \frac{\alpha \pi}{2}, \quad (18)
\]

where the function \( g \) is given by

\[
g(w) = H_{q,s}^{-1} [a_1; b_1; z]
= w - \Gamma_3 [a_1; b_1] a_2 w^2 + \left( 2(\Gamma_2 [a_1; b_1])^2 a_2^2 - \Gamma_3 [a_1; b_1] a_3 \right) w^3 - \left( 5(\Gamma_2 [a_1; b_1])^3 a_2^3 - 5\Gamma_2 [a_1; b_1] \right)
\times \left( \Gamma_3 [a_1; b_1] a_2 a_3 + \frac{\Gamma_4 [a_1; b_1]}{a_2} \right) w^4 + \cdots.
\]

Remark 3. (i) For \( q = 2, s = 1, \alpha = a_2 = b_1 = 1, \) we have \( T_{q,s}^\alpha [1, 1; 2; \alpha, \lambda] = B_2(\alpha, \lambda), \) where the class \( B_2(\alpha, \lambda) \) was introduced and studied by Frasin and Aouf [11].

(ii) For \( q = 2, s = 1, \alpha = a_2 = b_1 = \lambda = 1, \) we have \( T_{q,s}^\alpha [1, 1; 2; \alpha, 1] = H_2(\alpha, \lambda), \) where the class \( H_2(\alpha, \lambda) \) was introduced and studied by Srivastava et al. [5].

Theorem 4. Letting \( f(z) \) given by (1) be in the class \( T_{q,s}^\alpha [a_1; b_1, \alpha, \lambda], \) then

\[
|a_2| = \frac{2\alpha}{\Gamma_2 [a_1; b_1] \sqrt{(\lambda + 1)^2 + \alpha (1 + 2\lambda - \lambda^2)}}, \quad (20)
\]

\[
|a_3| = \frac{4\alpha^2}{\Gamma_2 [a_1; b_1] (\lambda + 1)^2} + \frac{2\alpha}{\Gamma_3 [a_1; b_1] (\lambda + 1)}, \quad (21)
\]

Proof. It follows from (18) that

\[
(1 - \lambda) H_{q,s} [a_1; b_1; z] z + \lambda (H_{q,s} [a_1; b_1; z])' = \left( p(z) \right)^2,
\]

\[
(1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) = \left( q(w) \right)^2,
\]

where \( p(z) \) and \( q(w) \) in \( P \) have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad (23)
\]

\[
q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \quad (24)
\]

Now, equating the coefficients in (22), we get

\[
(\lambda + 1) \Gamma_2 [a_1; b_1] a_2 = \alpha p_1, \quad (25)
\]

\[
(2\lambda + 1) \Gamma_3 [a_1; b_1] a_3 = \alpha p_2 + \frac{\alpha (\alpha - 1)}{2} p_1^2, \quad (26)
\]

\[
- (\lambda + 1) \Gamma_3 [a_1; b_1] a_3 = \alpha q_1, \quad (27)
\]

\[
(2\lambda + 1) \left( 2(\Gamma_2 [a_1; b_1])^2 a_2^2 - \Gamma_3 [a_1; b_1] a_3 \right) = \frac{\alpha q_2}{2} + \frac{\alpha (\alpha - 1)}{2} q_1^2.
\]

From (25) and (27), we get

\[
p_1 = -q_1, \quad (29)
\]

\[
2(\lambda + 1)^2 (\Gamma_2 [a_1; b_1])^2 a_2^2 = \alpha^2 \left( p_1^2 + q_1^2 \right). \quad (30)
\]

Now from (26), (28), and (30), we obtain

\[
2 (2\lambda + 1) (\Gamma_2 [a_1; b_1])^2 a_2^2
= \alpha (p_2 + q_2) + \frac{\alpha (\alpha - 1)}{2} (p_1^2 + q_1^2)
= \alpha (p_2 + q_2) + \frac{\alpha (\alpha - 1)}{2} 2(\lambda + 1)^2 (\Gamma_2 [a_1; b_1])^2 a_2^2.
\]

Therefore, we have

\[
a_2^2 = \frac{\alpha^2 \left( p_2 + q_2 \right)}{(\Gamma_2 [a_1; b_1])^2 \left( (\lambda + 1)^2 + \alpha (1 + 2\lambda - \lambda^2) \right)}. \quad (32)
\]

Applying Lemma 1 for the coefficients \( p_2 \) and \( q_2, \) we immediately have

\[
|a_2| \leq \frac{2\alpha}{\Gamma_2 [a_1; b_1] \sqrt{(\lambda + 1)^2 + \alpha (1 + 2\lambda - \lambda^2)}}. \quad (33)
\]

This gives the bound on \( |a_2| \) as asserted in (20).

Next, in order to find the bound on \( |a_3|, \) by subtracting (28) from (26) and using (29), we get

\[
2 (2\lambda + 1) \Gamma_3 [a_1; b_1] a_3 - 2(2\lambda + 1) (\Gamma_2 [a_1; b_1])^2 a_2^2
= \alpha p_2 + \frac{\alpha (\alpha - 1)}{2} p_1 - \left( \alpha q_2 + \frac{\alpha (\alpha - 1)}{2} q_1 \right)
= \alpha (p_2 - q_2).
\]

It follows from (30) and (34) that

\[
2(2\lambda + 1) \Gamma_3 [a_1; b_1] a_3
= \alpha^2 \left( (\lambda + 1)^2 + q_1^2 \right) \left( (\lambda + 1)^2 \right)^2 + \alpha (p_2 - q_2).
\]

And, then,

\[
a_3 = \frac{\alpha^2 \left( p_1^2 + q_1^2 \right)}{2(\lambda + 1)^2 \Gamma_3 [a_1; b_1]} + \frac{\alpha (p_2 - q_2)}{2(\lambda + 1)^2 \Gamma_3 [a_1; b_1]}. \quad (36)
\]
Applying Lemma 1 once again for the coefficients $p_1$, $p_2$, $q_1$, and $q_2$, we readily get

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + 1)^2} |\Gamma_3[a_1; b_1]| + \frac{2\alpha}{(2\lambda + 1)} |\Gamma_3[a_1; b_1]|. \quad (37)$$

This completes the proof of Theorem 4. □

Remark 5. (i) Taking $q = 2, s = 1$, and $a_1 = a_2 = b_1 = 1$ in Theorem 4, we obtain the result obtained by Frasin and Aouf [11, Theorem 2.2].

(ii) Taking $q = 2, s = 1$, and $a_1 = a_2 = b_1 = \lambda = 1$ in Theorem 4, we obtain the result obtained by Srivastava et al. [5, Theorem 1].

3. Coefficient Bounds of the Function Class

$T_{q,s}^{\Sigma_1}[a_1; b_1, \beta; \lambda]$

Definition 6. One says that a function $f(z)$ given by (1) is said to be in the class $T_{q,s}^{\Sigma_1}[a_1; b_1, \beta; \lambda]$ if it satisfies the following condition:

$$f \in \Sigma, \quad \Re \left\{ (1 - \lambda) \frac{H_{q,s} [a_1; b_1; z]}{z} + \lambda \left( H_{q,s} [a_1; b_1; z] \right)' \right\} > \beta,$$

$$\Re \left\{ (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) \right\} > \beta,$$

where the function $g$ is defined by (19).

Remark 7. (i) For $q = 2, s = 1$, and $a_1 = a_2 = b_1 = 1$, we have $T_{q,s}^{\Sigma_1}[1, 1; 2; \beta; \lambda] = \mathcal{B}_2(\beta, \lambda)$, where the class $\mathcal{B}_2(\beta, \lambda)$ was introduced and studied by Frasin and Aouf [11].

(ii) For $q = 2, s = 1$, and $a_1 = a_2 = b_1 = \lambda = 1$, we have $T_{q,s}^{\Sigma_1}[1, 1; 1; 1; \beta; \lambda] = \mathcal{H}_2(\beta, \lambda)$, where the class $\mathcal{H}_2(\beta, \lambda)$ was introduced and studied by Srivastava et al. [5].

Theorem 8. Letting $f(z)$ given by (1) be in the class $T_{q,s}^{\Sigma_1}[a_1; b_1, \beta; \lambda]$, $0 \leq \beta < 1$ and $\lambda \geq 1$, then

$$|a_3| = \sqrt{2(1 - \beta)} \frac{|\Gamma_2[a_1; b_1]|}{\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)}.$$

$$|a_3| = \sqrt{2(1 - \beta)} \frac{2(1 - \beta)^2}{\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)}. \quad (40)$$

Proof. It follows from (38) that

$$H_{q,s} [a_1; b_1; z] = \beta + (1 - \beta) \frac{g(z)}{z} + \lambda \left( H_{q,s} [a_1; b_1; z] \right)',$$

$$\frac{g(w)}{w} + \lambda g'(w) = \beta + (1 - \beta) q(w), \quad \text{or, equivalently,}$$

$$a_3 = \frac{(\lambda+1)^2}{\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)} + \frac{2(1 - \beta)^2}{\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)}.$$

Applying Lemma 1 once again for the coefficients $p_1, p_2, q_1,$ and $q_2$, we readily get

$$|a_3| \leq \frac{4(1 - \beta)^2}{\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)} \quad (54)$$

where $p(z)$ and $q(w)$ have the forms (23) and (24), respectively.

As in the proof of Theorem 4, by suitably comparing coefficients in (41), we get

$$(\lambda + 1) \Gamma_3[a_1; b_1] a_2 = (1 - \beta) p_1, \quad (42)$$

$$(2\lambda + 1) \Gamma_3[a_1; b_1] a_3 = (1 - \beta) p_2, \quad (43)$$

and

$$(\lambda + 1) \Gamma_2[a_1; b_1] a_2 = (1 - \beta) q_1, \quad (44)$$

$$2(\lambda + 1) \left( 2(\Gamma_2[a_1; b_1])^2 a_2^2 - \Gamma_3[a_1; b_1] a_3 \right) = (1 - \beta) q_2. \quad (45)$$

From (42) and (44), we get

$$p_1 = -q_1, \quad (46)$$

$$2(\lambda + 1)^2(\Gamma_2[a_1; b_1])^2 a_2^2 = (1 - \beta)^2 \left( p_1^2 + q_1^2 \right). \quad (47)$$

Also, from (43) and (45), we find that

$$2(\lambda + 1) \left( 2(\Gamma_2[a_1; b_1])^2 a_2^2 = (1 - \beta)^2 (p_2 + q_2). \quad (48)$$

Therefore, we have

$$|a_3| \leq \frac{(1 - \beta)}{(\Gamma_2[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1) + 1)(p_2 + q_2)}. \quad (49)$$

Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have

$$|a_3| \leq \frac{\sqrt{2(1 - \beta)}}{|\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)}. \quad (50)$$

This gives the bound on $|a_3|$ as asserted in (39).

As in the proof of Theorem 4, by suitably comparing coefficients in (41), we get

$$(\lambda + 1) \Gamma_3[a_1; b_1] a_2 = (1 - \beta) p_1, \quad (42)$$

$$(2\lambda + 1) \Gamma_3[a_1; b_1] a_3 = (1 - \beta) p_2, \quad (43)$$

$$2(\lambda + 1)^2(\Gamma_2[a_1; b_1])^2 a_2^2 = (1 - \beta)^2 \left( p_1^2 + q_1^2 \right). \quad (47)$$

Also, from (43) and (45), we find that

$$2(\lambda + 1) \left( 2(\Gamma_2[a_1; b_1])^2 a_2^2 = (1 - \beta)^2 (p_2 + q_2). \quad (48)$$

Therefore, we have

$$|a_3| \leq \frac{(1 - \beta)}{(\Gamma_2[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1) + 1)(p_2 + q_2)}. \quad (49)$$

Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have

$$|a_3| \leq \frac{\sqrt{2(1 - \beta)}}{|\Gamma_3[a_1; b_1]|(\lambda + 1)^2 + 2(\lambda + 1)}. \quad (50)$$

This completes the proof of Theorem 8. □
Remark 9. (i) Taking $q = 2, s = 1$, and $a_1 = a_2 = b_1 = 1$, in Theorem 8, we obtain the result obtained by Frasin and Aouf [11, Theorem 3.2].

(ii) Taking $q = 2, s = 1$, and $a_1 = a_2 = b_1 = \lambda = 1$, in Theorem 8, we obtain the result obtained by Srivastava et al. [5, Theorem 2].

References


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