Research Article

Postulation of a Union $X \subset \mathbb{P}^r$, $r \geq 4$, of a Given Zero-Dimensional Scheme and Several General Lines

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We study the Hilbert function of a general union of disjoint lines. We assume almost nothing on integers $t \geq 0$. Since $T \cap T = 0$, we have $Z \not\subset T$, and hence $h^0(T \cup Z, \mathcal{O}_{T \cup Z}(x)) = (x + 1)t + z$ for all $x \geq 0$. To study the Hilbert function of a general union of $Z$ and $T$ disjoint lines, we need the following relations:

$$z + (k + 1)e_{z,r,k} + f_{z,r,k} = \binom{k + r}{r}, \quad 0 \leq f_{z,r,k} \leq k. \quad (1)$$

The scheme $T \cup Z$ has maximal rank if and only if $h^0(\mathcal{I}_{T \cup Z}(k)) = 0$ for all integers $k$ with $t > e_{z,r,k}$ and $h^0(\mathcal{I}_{T \cup Z}(k)) = 0$ for the minimal integer $k \geq c$ such that either $e_{z,r,k} > t$ or $e_{z,r,k} + f_{z,r,k} = t$ and $f_{z,r,k} = 0$.

We first prove the following result, which says that if we may handle general unions of $Z$ and $T$ disjoint lines with respect to homogeneous polynomials in two consecutive degrees (not small with respect to the integer $z := \text{deg}(Z)$), then these information propagates to higher degree polynomials.

**Theorem 1.** Fix a zero-dimensional scheme $Z \subset \mathbb{P}^r$, $r \geq 4$, and an integer $c > 0$. Set $z := \text{deg}(Z)$. Assume $z \leq \binom{r+c}{c}$. If $r = 4$, assume $c \geq 8$. Let $\delta_1$ be any nonnegative integer such that $h^0(\mathcal{I}_{Z \cup X}(c)) \leq f_{z,r,c} + \delta_1$ for a general $X \in L(r,e_{z,r,c})$. Let $\delta_2$ be any nonnegative integer such that $h^0(\mathcal{I}_{Z \cup X}(c + 1)) \leq f_{z,r,c+1} + \delta_2$ for a general $X \in L(r,e_{z,r,c+1})$. Set $\delta := \max\{\delta_1,\delta_2\}$.

Fix an integer $y > e_{z,r,c+1}$, and let $Y$ be a general element of $L(r,y)$.

(a) For all integers $k \geq c$, either $h^0(\mathcal{I}_{Z \cup Y}(k)) \leq \delta$ or $h^0(\mathcal{I}_{Z \cup Y}(k)) \leq \delta$.

(b) Assume $f_{z,r,c} \leq f_{z,r,c+1}$. Then for all integers $k \geq c$, either $h^0(\mathcal{I}_{Z \cup Y}(k)) \leq \delta$ or $h^0(\mathcal{I}_{Z \cup Y}(k)) \leq \delta_1$.

Then we go to a case in which we only assume something about the integer $z = \text{deg}(Z)$ and prove the following result.
Theorem 2. Fix positive integers \( r, z, \) and \( c \) such that \( r \geq 4 \) and a zero-dimensional scheme \( Z \subset \mathbb{P}^r \) such that \( \deg(Z) = z \) and \( h^0(I(Z)) = 0 \). Set \( r' = (r^c - c)^{-1} - z \). If either \( r \geq 5 \) or \( r = 4 \) and \( c \geq 10 \), then set \( \tau' = \tau \). If \( r = 4 \) and \( c < 9 \), then set \( \tau' = r + 10 - c \). Fix any integer \( \tau \geq ((r^c - c + 1)^{-1} - z)/(c + \tau' + 1) \). Let \( X \subset \mathbb{P}^r \) be a general union of \( Z \) and \( t \) lines. Then \( X \) has maximal rank.

If \( z = (r^c) \), then we have the following result.

Proposition 3. Fix integers \( r \geq 4 \) and \( c > 0 \). Let \( Z \subset \mathbb{P}^r \) be a zero-dimensional scheme such that \( \deg(Z) = (r^c) \) and \( h^0(I(Z)) = 0 \). Then for each integer \( t \geq 0 \), a general union of \( Z \) and \( t \) lines has maximal rank.

In the case \( Z = (c + 1)P \), we get the case \( r \geq 4 \) of [4]. We omit the proof of Proposition 3, because the proof of [4] works verbatim (e.g., quote [4, Lemma 5.1] instead of Lemmas 10, 12, and 13).

2. The Proofs

For all integers \( z \geq 0, r \geq 4, c \geq 0, x \in Z, \) and \( k \geq 0 \), define the integers \( e_{z,r,k}, x \) and \( f_{z,r,k}, x \) by the relations

\[
 z + (k + 1)e_{z,r,k} + f_{z,r,k} = \left( \frac{r + k}{r} \right) + x, \tag{2}
\]

\[
 0 \leq f_{z,r,k} \leq k. \tag{3}
\]

Notice that \( e_{z,r,k} = e_{z,k,0} \) and \( f_{z,r,k} = f_{z,r,k,0} \). For any \( t \geq 0, c > 0 \), the critical value of the triple \( (z, r, t) \) (resp., \( (z, r, c) \)) is the minimal integer \( k > 0 \) (resp., \( k > c \)) such that \( z + (k + 1)t \leq (r^c) \), that is, the minimal integer \( k > 0 \) (resp., \( k > c \)) such that \( t \leq e_{z,r,k} \).

Taking the equation in (2) minus the same equation for the integer \( k := k - 1 \), we get

\[
e_{z,r,k-1,k} + (k + 1)(e_{z,r,k} - e_{z,r,k-1}) + f_{z,r,k} - f_{z,r,k-1} = (k + r - 1). \tag{3}
\]

Take \( Z, z, c, \delta, \) and \( \delta_1 \) as in Theorem 1. For each integer \( k \geq 0 \) consider the following assertion \( A_k \):

\( A_k \): we have \( h^0(I(X)) \leq f_{z,r,k} + \delta \) for a general \( A \in L(r, e_{z,r,k}) \).

Take any \( A \in L(r, e_{z,r,k}) \) such that \( Z \cap A = \emptyset \). Since \( h^0(I(Z) \cup A) = \deg(Z) + (k + 1)\deg(A) \), we have \( h^0(I(Z)) \geq h^0(I(X)) + f_{z,r,k} \). Hence \( A_k \) is true if and only if \( h^0(I(Z)) \leq \delta \) for a general \( A \in L(r, e_{z,r,k}) \). Call \( A_k \) the same assertion with \( \delta_1 \) instead of \( \delta \).

Remark 4. By the definition of the integers \( \delta_1 \) and \( \delta \), the assertions \( A_0, A_{c+1}, \) and \( A_{c+1} \) are true.

For any hyperplane \( H \subset \mathbb{P}^r \), and any closed subscheme \( W' \subset \mathbb{P}^r \), let \( \text{Res}_{H}(W') \) be the closed subscheme of \( \mathbb{P}^r \) with \( \mathcal{I}_{W'} : \mathcal{I}_{H} \) as its ideal sheaf. If \( W = Z \cup Y \) with \( Z \cap H = \emptyset \) and \( Y \) is a reduced scheme, then \( \text{Res}_{H}(W) = Z \cup Y' \), where \( Y' \) is the union of the irreducible components of \( Y \) not contained in \( H \).

Lemma 5. Fix a hyperplane \( H \subset \mathbb{P}^n, \) an integer \( k > 0 \) and a closed subscheme \( W \subset \mathbb{P}^r \). If \( h^0(I(W)) > h^0(I(W)) \) \((k-1)) \), then \( h^0(I(W)) \) \((k-1)) \) \( -1 \) for a general \( P \in H \).

Proof. The assumption implies that \( H \) is not contained in the base scheme of \( I(W) \). The long cohomology exact sequence associated to the following exact sequence

\[
 0 \to I_{W}(W) \to I_{W}(k-1) \to I_{W}(k) \to I_{W}(H) \to 0 \tag{4}
\]

gives the following inequalities:

\[
 (1) \quad h^0(I(W)) \leq h^0(I(W)) + h^0(H), \tag{1}
\]

\[
 (2) \quad h^0(I(W)) \leq h^0(I(W)) + h^0(H), \tag{2}
\]

We have \( \text{Res}_{H}(W \cup \{P\}) = \text{Res}_{H}(W) \) and \( h^0(H, I(W) \cup \{P\}) = h^0(H, I(W)) - 1 \), because a general \( P \in H \) is not in the base locus of \( I(W) \). Apply (4) first to \( W \) and then to \( W \cup \{P\} \).

As in [2–4], we will call Castelnuovo’s sequence for any of the inequalities in the proof of Lemma 5.

Lemma 6. Fix integers \( n \geq 3, k > 0, \) and \( f \geq 0 \) such that \( (k + 1)(e + 2f) \leq (\frac{nk}{n-1}) \). Let \( X \subset \mathbb{P}^n \) be a general union of \( \ell \) lines and \( f \) reducible conics. Then \( h^1(I(X)) = 0 \).

Proof. For each \( P \in \text{Sing}(X) \), let \( C_P \subset X \) be the connected component of \( X \) containing \( P \). Write \( X = Y \cup \bigcup_{P \in \text{Sing}(X)} C_P \). For each \( P \in \text{Sing}(X) \), let \( N_P \subset \mathbb{P}^n \) be a general 3-dimensional linear space containing \( C_P \). Let \( E_P \subset N_P \) be the sundial with \( C_P \) as its support. Set \( X' := Y \cup \bigcup_{P \in \text{Sing}(X)} E_P \). Since \( X \) and each \( N_P \) are general, \( X' \) is a general union of \( \ell \) lines and \( f \) sundials. Hence \( X' \) has maximal rank [3]. Since \( (k + 1)(e + 2f) \leq (\frac{nk}{n-1}) \), we get \( h^1(I(X')) = 0 \). Fix \( P \in \text{Sing}(X) \). Let \( \eta \) be the nilpotent sheaf of \( E_P \). We have an exact sequence

\[
 0 \to \eta \to \mathcal{O}_X(E_P) \to \mathcal{O}_{C_P} \to 0. \tag{5}
\]

Since \( \eta \) is supported by a single point, \( P \), we have \( h^1(E_P, \eta) = 0 \). Hence (5) gives the surjectivity of the restriction map \( h^0(E_P, \mathcal{O}_{E_P}(k)) \to h^0(C_P, \mathcal{O}_{C_P}(k)) \). Looking at all the connected components of \( X \) and \( X' \), we get the surjectivity of the restriction map \( h^1(X', \mathcal{O}_{X'}(k)) \to h^1(X, \mathcal{O}_X(k)) \). Hence \( h^1(I(X)) \leq h^1(I(X')) = 0 \).

Lemma 7. \( A_k \) is true for all integers \( k \geq c \).

Proof. By Remark 4 we may assume \( k \geq c + 2 \). We use induction on \( k \). Since \( A_{c+1} \) is true, we may assume that \( A_{k-1} \) is
true. Fix a hyperplane $H \subset \mathbb{P}^r$ such that $H \cap Z = \emptyset$. Fix a general $Y \in L(r, e_{z,r,k-1})$. We have $Y \cap Z = 0$, $H \cap Y$ is a general union of $e_{z,r,k-1}$ points of $H$, and $h^0(\mathcal{I}_{Z,Y}(k)) \leq f_{z,r,k-1} + \delta$.

(i) First assume $f_{z,r,k-1} < f_{z,r,k}$. Let $E \subset H$ be a general union of $e_{z,r,k} - e_{z,r,k-1}$ lines; this is possible, because $e_{z,r,k} - e_{z,r,k-1} \geq 0$ (Lemma 10). By (3), we have $e_{z,r,k} - e_{z,r,k-1} = (k + 1) - 1$. Since $E$ has maximal rank in $H$ if and only if $H \cap E$ is a general union of lines and sundials, the Castelnuovo's semicontinuity theorem for cohomology ([5, Theorem III.12.8]) implies that it is sufficient to prove that $h^0(\mathcal{I}_{Z,Y,U}(k)) \leq f_{z,r,k} + \delta$.

Lemma 8. Assume $f_{z,r,c} \leq f_{z,r,c+1}$. Then $A_k^c$ is true for all $k \geq c$.

Proof. $A_k^c$ is true (Remark 4). Now assume $k = c + 1$. To copy step (i) of the proof of Lemma 7, it is sufficient to have $e_{z,r,c+1} \geq e_{z,r,c}$. Use Lemma 12. The case $k \geq c + 2$ is done as in the proof of Lemma 7. 

Lemma 9. Fix $Z$ as in Theorem 1 and integers $k > c$ and $t > e_{z,r,k}$. Let $Y$ be a general element of $L(r,t)$. Then $h^0(\mathcal{I}_{Z,Y}(k)) \leq \delta$.

Proof. It is sufficient to do the case $t = e_{z,r,k} + 1$. Fix a hyperplane $H \subset \mathbb{P}^r$. First assume $k \geq c + 2$. Take a general $A \subseteq L(r,e_{z,r,k-1})$. We have $A \cap Z = \emptyset$, $A_{k-1}$ gives $h^0(\mathcal{I}_{A\cup Z}(k-1)) = f_{z,r,k-1}$. Let $F \subset \mathbb{P}^r$ be a general union of $e_{z,r,k} + 1 - 2f_{z,r,k-1}$ lines of $H$ and $f_{z,r,k-1}$ sundials whose support in contained in $H$. We conclude as in step (ii) of the proof of Lemma 7.

Proof of Theorem 1. First assume $z + (k + 1) y \leq (t + k)$ (we allow the equality). By the semicontinuity theorem for cohomology ([5, Theorem III.12.8]), it is sufficient to find $A \subseteq L(r,y)$ such that $A \cap Z = \emptyset$ and $h^0(\mathcal{I}_{Z,Y}(k)) \leq \delta$. By (2) and the assumption $y > e_{z,r,c}$, we have $k \geq c + 2$ and $y \leq e_{z,r,k}$. Fix a general $W \subset L(r,e_{z,r,k})$. Let $U \subset W$ be the union of the lines of $W$. By $A_k$, we have $h^0(\mathcal{I}_{Z,U}(k)) \leq \delta$. Since $Z \cup U$ is a union of some of the connected components of $Z \cup W$, we have $h^0(\mathcal{I}_{Z,Z'}(k)) \leq \delta$.

Now we check part (b). Since $f_{z,r,c} \leq f_{z,r,c+1}$, Lemma 8 gives $A_k^c$ for all $k \geq c$.

Lemma 10. Assume $f_{z,r,c} \leq f_{z,r,c+1}$. Then $A_k^c$ is true for all $k \geq c$. Then we get Lemma 9 with $\delta_i$ instead of $\delta$. Then we continue as in the proof of part (a).

Proof of Theorem 2. Fix a hyperplane $H \subset \mathbb{P}^r$ such that $H \cap Z = \emptyset$. If either $r \geq 5$ or $r = 4$ and $c \geq 10$ for each integer $k \geq c$, set $x_k = \max(0, -r + k - c - 1)$. If $r = 4$ and $c \leq 9$, set $x_k := -r$. Hence $x_k = x_{k+1} = -r$. If $r = 4$ and $c \leq 9$, we have $x_k = -r$ for all $k \leq 10$. Consider the following statement $A_k^r$, $k \geq c$:

$a_k^r$: we have $h^0(\mathcal{I}_{Z,U}(k)) = f_{z,r,k+1}$ for a general $A \subseteq L(r,e_{z,r,k+1})$.

Take any $A \subseteq L(r,e_{z,r,k+1})$ such that $Z \cap A = \emptyset$. We have $h^0(\mathcal{I}_{Z,U}(k)) = h^0(\mathcal{I}_{Y,Z}(k)) = f_{z,r,k+1} + \delta$. Hence $A_k^r$ is true if and only if $h^0(\mathcal{I}_{Z,U}(k)) = 0$ for a general $A \subseteq L(r,e_{z,r,k+1})$.

(a) In this step we prove $A_k^r$ and $A_{k+1}^r$. $A_c^r$ is true, because we assumed that $h^0(\mathcal{I}_{Z,Y}(c)) = 0$. To check $A_{c+1}^c$, we first notice that $x_k = x_{k+1}$, and hence we may apply (3) for the integer $x := x_c$. Since $f_{z,r,c+1} = 0$, we
have $f_{x,x+1} \geq f_{z,x,r}$. Since $e_{z,x,r} = 0$, Lemma 15 gives $e_{z,x+1,r} \geq e_{z,x,r}$. Hence the construction in step (i) of the proof of Lemma 7 works verbatim.

(b) In this step we prove $A''_k$ for all $k \geq c$. Since the cases $k = c, c+1$ are true by step (a), we may assume that $k \geq c + 2$ and that $A''_{k-1}$ is true. Taking the difference of (2) with $k' := k - 1$ and $x := x'$ and integers $k, x, x'$, we get

$$e_{z,k-1,x} + (k + 1)\left(e_{z,x,k} - e_{z,k-1,x'}\right) + f_{z,r,k,x} - f_{z,r,k-1,x'} + x - x' = \left(k + r - 1\right).$$

(7)

We apply (7) with $x' = x_{k-1}$ and $x = x_k$. Hence either $x = x_k$ or $x = x_{k-1}$. In both cases, we have $0 \leq x - x' \leq 1$. If $x = x_k$, then we may apply the numerical Lemmas 10, 12, and 13 used in the proof of Theorem 1. We need different numerical lemmas if $x \neq x'$, that is, if $x_{k-1} < 0$ and $x_k = x_{k-1} + 1$.

Fix a general $Y \in L(r, e_{z,k-1,1})$. We have $Z \cap Y = \emptyset$, hence $h^1(\mathcal{J}_{Z,Y}(k-1)) = 0$. Since $h^0(\mathcal{J}_{Z,Y}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1}$ and $Y \cap H$ is a general union of $e_{z,k-1,1}$ points of $H$. Since $h^1(\mathcal{J}_{Z,Y}(k-1)) = 0$, $h^0(\mathcal{J}_{Z,Y}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1}$, we have $0 = h^0(\mathcal{J}_{Z,Y}(k-1)) = f_{z,r,k-1,x_{k-1}} - x_{k-1} - 1$ for a general $P \in P^0$ (and even a general $P \in H$ by Lemma 5 if $f_{z,r,k-1,x_{k-1}} - x_{k-1} - 1 \geq 0$). This is always the case if $x_{k-1} < 0$, that is, if $x_{k-1} \neq x_{k-1}$.

(b1) First assume $f_{z,r,k-1,x_{k-1}} \leq f_{z,r,k,x_k}$. Since $0 \leq x_k - x_{k-1} - 1 \leq 1$, Lemma 15 gives $e_{z,r,x_k} \geq e_{z,r,k-1,x_{k-1}} + 2$. Let $E_i \subset H$ be a general union of $e_{z,r,k-1,x_{k-1}} - 2$ lines. Take a general reducible conic $T \subset H$. Let $T' \subset P^0$ be a general sundial of $P^0$, and let $T'' \subset H$ be a general sundial of $H$ with $T$ as its support. Set $F := E_1 \cup T'$. Since $Z \cup Y \cup E_1 \cup T'$ is a specialization of a union of $Z$ and deg$(Y \cup E_1 \cup T')$ lines, it is sufficient to prove that $h^0(\mathcal{J}_{Z,Y\cup E_1}(k)) = 0$. Castelnuovo’s sequence gives $h^1(\mathcal{J}_{Z,Y\cup E_1}(k)) = 0$. Lemma 14 gives $e_{z,r,k-1} \geq 0$. By (7) to apply Lemma 5, we need that $Y \cap H$ has at least one point; this is the reason why this proof does not work if $k < c = 1$.

(b2) Now assume $f_{z,r,k-1,x_{k-1}} > f_{z,r,k,x_k}$. Since $0 \leq x_k - x_{k-1} - 1 \leq 1$, Lemma 16 gives $e_{z,r,x_k} - e_{z,r,k-1,x_{k-1}} \geq 2(f_{z,r,k-1,x_{k-1}} - f_{z,r,k-1,x_{k-1}}) + 2$. We repeat step (i) of the proof of Lemma 7 taking inside $F$ of a general union $F$ of $e_{z,r,x_k} - e_{z,r,k-1,x_{k-1}} - 2(f_{z,r,k-1,x_{k-1}} - f_{z,r,k-1,x_{k-1}} + 1)$ lines, $f_{z,r,k-1,x_{k-1}} - f_{z,r,k-1,x_{k-1}} + 1$ reducible conics. Then we take the general sundials (in $H$ and in $P^0$) with these conics as their supports. To apply Lemma 6, we need $e_{z,r,k-1,x_{k-1}} \geq f_{z,r,k-1,x_{k-1}} - f_{z,r,k-1,x_{k-1}} + 1$. This inequality is true by Lemma 15.

(c) Fix an integer $t \geq \left(\frac{(r+c+1)}{r} - z\right)/(c + r + 1)$. Let $k$ be the critical value for the triple $(r, c, t)$, i.e. the minimal positive integer such that $z + (k + 1)t \leq (k + 1)r$. Since $t \geq \left(\frac{(r+c+1)}{r} - z\right)/(c + r + 1)$, either $k \geq c + r + 1$ or $k = c + r'$ and $f_{z,r,k} = 0$. To prove Theorem 2 for the integer $t$ it is sufficient to prove $h^1(\mathcal{J}_{Z,Y}(k)) = 0$ and $h^0(\mathcal{J}_{Z,Y}(k - 1)) = 0$ for a general $Y \in L(r, t)$. Since $L(r, t)$ is irreducible, the semicontinuity theorem for cohomology says that to prove Theorem 2 for the integer $t$ it is sufficient to prove the existence of $A \in L(r, t)$ and $B \in L(r, t)$ such that $A \cap Z = B \cap Z = 0$, $h^0(\mathcal{J}_{Z,Y}(k)) = 0$ and $h^1(\mathcal{J}_{Z,Y}(k - 1)) = 0$. Since $t \geq \left(\frac{(r+c+1)}{r} - z\right)/(c + r + 1)$, we have $k \geq r' + 1$. Hence $x_k = x_{k-1} = 0$ since $x_k = 0$ the existence of $A$ is proved as.

3. Numerical Lemmas

Lemma 10. Assume $r \geq 4$. Fix an integer $c > 0$. One has $e_{z,r,k} \geq e_{z,r,k-1} + k$ for all $k \geq c + 2$.

Proof. Assume $e_{z,r,k} \leq e_{z,r,k-1} + k - 1$. From (3), we get

$$e_{z,r,k} + (k + 1)(k + 1) - f_{z,r,k} \geq f_{z,r,k-1} \geq \left(\frac{r + k - 1}{r - 1}\right)^2.$$ (8)

From (2) for the integer $k' := k - 1$, we get

$$e_{z,r,k-1} = \left(\frac{(r+1)}{r} - z - f_{z,r,k-1}\right).$$ (9)

Since $f_{z,r,k} \leq k$ and $k' = (r+1)(r+k)$, from (11) and (12), we get

$$k(k^2 + 2k + 1) - (k - 1) f_{z,r,k} \geq (r - 1) \left(\frac{r + k - 1}{r - 1}\right)^2 + z.$$ (10)

Set $\mu(r, k) := (r - 1)(r + k - 1) - k^2$ and $z > 0$, to get a contradiction, it is sufficient to prove that $\mu(r, k) \geq 0$. First assume $r = 4$. We have $8 \mu(4, k) = (k + 3)(k + 1)(k - 1) - 8(k^2 + k - 1)$. Obviously $8 \mu(4, k) \geq 0$ if $k \geq 6$. We have $\mu(4, 5) = 210 - 5\cdot 29, \mu(4, 4) = 105 - 76 > 0$, and $\mu(4, 3) = 45 - 33 > 0$.

We have $\mu(r, 1) - \mu(r, k) = k(k + 1) - (r + 1)^2 = (r + 1)(r + 1)[(k(r + 1)/(r + k)) - 1] > 0$. By induction on $r$, we get the lemma for all $k \geq 5$.

Lemma 11. Fix integers $r \geq 4, c > 0, z \geq 0$, and $x \leq 0$. Then $e_{z,c,x} \geq e_{z,c-1,x} + k$ for all $k \geq 2$.

Proof. Assume $e_{z,c,x} \leq e_{z,c-1,x} + k - 1$. From (3), we get

$$e_{z,c-1,x} + (k + 1)(k + 1) - f_{z,c,x} \geq f_{z,c-1,x} \geq \left(\frac{r + k - 1}{r - 1}\right)^2.$$ (11)

From (2) for the integer $k' := k - 1$, we get

$$e_{z,c-1,x} = \left(\frac{(r+k-1)}{k} + x - z - f_{z,c-1,x}\right).$$ (12)
Since \( f_{z, r, k} \leq k, \ (c+1)t \leq (\frac{r_c}{r^2}), \ k \ (\frac{rk-1}{r-1}) = r \ (\frac{rk-1}{r}), \ z \geq 0, \) and \( x \leq 0, \) from (11) and (12), we get
\[
k(k+1)(k-1)-(k-1) f_{z, r, k-1} \geq (r-1) \left( \frac{r+k-1}{r} \right).
\]
\[
(13)
\]
Set \( \phi(r, k) := (r-1) \ (\frac{rk-1}{r}) - k(k+1)(k-1). \) Since \( f_{z, r, k-1} \geq 0, \) to get a contradiction it is sufficient to prove that \( \phi(r, k) > 0. \) First assume \( r = 4. \) We have \( 8\phi(4, k) = (k+3)(k-1)k - 8(k+1)(k-1) = (k+1)(k^2 - 3k + 8). \) Hence \( \phi(4, k) > 0 \) for all \( k \geq 2. \) Now assume \( r > 4. \) We have \( \phi(r+1, k) - \phi(r, k) = k \ (\frac{rk-1}{r}) - (\frac{rk}{r+1}) \ (k(r+1)/(r+k)] - 1) > 0. \) By induction on \( r, \) we get the lemma.

Lemma 12. Assume \( r \geq 4. \) Fix integers \( c, z \) such that \( c > 0 \) and \( 0 \leq z \leq (\frac{c}{c+1}). \) One has \( e_{z, r, k} \geq e_{z, r, k-1} \) for all \( k > c. \)

Proof. By Lemma 10, it would be sufficient to do the case \( k = c + 1. \) Assume \( e_{z, r, k} \leq e_{z, r, k-1} - 1. \) From (3) and (12), we get
\[
\left( \frac{r+k-1}{r} \right) - k(k+1) - (k-1) f_{z, r, k-1} \geq k \left( \frac{r+k-1}{r} \right)
\]
which is obviously false.

Lemma 13. Assume \( r \geq 4, \ k > c > 0, \) and \( f_{z, r, k} < f_{z, r, k-1}. \) If \( r = 4, \) assume \( k \geq 9. \) Then \( e_{z, r, k}^e - e_{z, r, k-1}^e \geq 2(f_{z, r, k-1} - f_{z, r, k}). \)

Proof. Assume \( e_{z, r, k}^e - e_{z, r, k-1}^e \leq 2(f_{z, r, k-1} - f_{z, r, k}) - 1. \) From (3), we get
\[
e_{z, r, k-1} + (2k+1)(f_{z, r, k-1} - f_{z, r, k}) - 1 - k \geq \left( \frac{r+k-1}{r-1} \right).
\]
\[
(15)
\]
Since \( e_{z, r, k-1} \leq \left[ (\frac{k+1}{r-1}) - z \right] - f_{z, r, k-1}/k, k \ (\frac{rk-1}{r}) = r \ (\frac{rk-1}{r}), \) and \( f_{z, r, k} \geq 0, \) from (15), we get
\[
(2k^2 + k - 1) f_{z, r, k-1} - k(k+1) \geq (r-1) \left( \frac{r+k-1}{r} \right) + z.
\]
\[
(16)
\]
Since \( f_{z, r, k-1} \leq k, \) we get
\[
2k(k+1)(k-1) \geq (r-1) \left( \frac{r+k-1}{r} \right) + z
\]
\[
(17)
\]
Set \( \alpha(r, k) := (r-1) \ (\frac{rk-1}{r}) - k(2k^2 - 2). \) Since \( z > 0, \) to get a contradiction and hence to prove the lemma, it is sufficient to prove that \( \alpha(r, k) > 0. \) First assume \( r = 4. \) We have \( 8\alpha(4, k) = (k+3)(k+1)k - 8(2k^2 - 2) - k(k+1) \ (k+3)(k+2) - 16(k+1) \geq 16. \) Hence \( \alpha(4, k) \geq 0 \) if and only if \( k \geq 9. \)

Now assume \( r > 4. \) We have \( \alpha(r, k) - \alpha(r-1, k) = k \ (\frac{rk-1}{r}) - (\frac{r^2k}{r-1}) = \ (\frac{r^2k}{r-1}) [(rk/(k-1)-1] > 0 \) for all \( k \geq 2. \) We have \( \alpha(5, k) = 3 - 21 - 48 > 0, \) and \( \alpha(5, k) \) is an increasing function of \( k. \) Hence \( \alpha(r, k) \geq 0 \) if either \( r \geq 5 \) or \( k \geq 9. \)

Lemma 14. Assume \( r \geq 4. \) Fix integers \( k, \ c, \ z, \ x, \) and \( x' \) such that \( k > c > 0, \ 0 \leq z \leq (\frac{c}{c+1}), \ x' < 0, \) and \( x = x' + 1. \) One has \( e_{z, r, k}^e \geq e_{z, r, k-1}^e + x' + 2. \)

Proof. Assume \( e_{z, r, k}^e \leq e_{z, r, k-1}^e + x' + 1. \) By (7), we get
\[
e_{z, r, k}^e + f_{z, r, k} - f_{z, r, k-1} \geq \left( \frac{r+k-1}{r} \right).
\]
\[
(18)
\]
We have \( f_{z, r, k} - f_{z, r, k-1} \leq k. \) Since \( x' \leq 0, \) we have \( e_{z, r, k-1}^e + f_{z, r, k} - f_{z, r, k-1} \geq (\frac{r+k-1}{r}) \) / \( k. \) Hence
\[
\left( \frac{r+k-1}{r} \right) + k(2k+2) \geq k \left( \frac{r+k-1}{r-1} \right).
\]
\[
(19)
\]
Since \( k \ (\frac{rk-1}{r-1}) = r \ (\frac{rk-1}{r-1}), \) \( r \geq 4, \) and \( k \geq 2, \) we get a contradiction.

Lemma 15. Take the setup of Theorem 2. For each integer \( y > c \) with \( y \geq 0, \) then \( e_{z, r, y} \geq 2y. \)

Proof. By Lemma 14, it is sufficient to do the case \( y = c + 1. \) In this case, we have \( y = (\frac{z}{z+1}) - z. \) Assume \( e_{z, r, y} \geq 2c + 1. \) Since \( f_{z, r, x-1, y} \leq c + 1, \) we get \( (c+2)(2c+1) + c+1 \geq (\frac{c+1}{c+2}) \) / \( r. \) The right-hand side of this inequality is an increasing function of \( r. \) For \( r = 4, \) this inequality fails for all \( c \geq 2, \) because \( 24(2c^2 + 6c + 3) < (c+5)(c+4)(c+3)(c+2) \) for all \( c \geq 2. \)

Lemma 16. Take the setup of Theorem 2. Assume \( r \geq 4, \ k > c > 0, \) \( x_{k-1} < 0, \) and \( f_{z, r, k} < f_{z, r, k-1, x_{k-1}}. \) Then \( e_{z, r, k} - e_{z, r, k-1, x_{k-1}} \geq 2( f_{z, r, k} - f_{z, r, k-1} ) + 2. \)

Proof. Assume \( e_{z, r, k} - e_{z, r, k-1, x_{k-1}} \leq 2( f_{z, r, k} - f_{z, r, k-1} ) + 2. \) First with \( x_{k-1} = 1, \) we get
\[
\left( \frac{r+k-1}{r} \right) + (2k+1)(f_{z, r, k-1, x_{k-1}} - f_{z, r, k,x_{k-1}}) + (k+1) - 1 \geq \left( \frac{r+k-1}{r} \right).
\]
\[
(20)
\]
Since \( k \ (\frac{rk-1}{r-1}) = r \ (\frac{rk-1}{r-1}), \) we get
\[
2k^2(k+1) > (r-1) \left( \frac{r+k-1}{r} \right).
\]
\[
(21)
\]
This inequality is false if either \( r = 4 \) and \( k \geq 11 \) or \( r \geq 5 \) and \( k \geq 2. \)

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References


