Research Article

D_a -Homothetic Deformation of K -Contact Manifolds

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We study D_a -homothetic deformations of K -contact manifolds. We prove that D_a -homothetically deformed K -contact manifold is a generalized Sasakian space form if it is conharmonically flat. Further, we find expressions for scalar curvature of D_a -homothetically deformed K -contact manifolds.

1. Introduction

In 1968 Tanno [1] introduced the notion of D_a -homothetic deformations. Carriazo and Martín-Molina [2] studied D_a -homothetic deformation of generalized (k, μ) space forms and gave several examples for manifolds of dimension 3. De and Ghosh [3] studied D_a -homothetic deformation of almost normal contact metric manifolds and prove that $Q\phi\phi Q$ is invariant under such transformation. Bagewadi and Venkatesha [4] studied concircularly semisymmetric trans-Sasakian manifolds and De et al. [5] studied conharmonically semisymmetric, conharmonically flat, ξ -conharmonically flat, and conharmonically recurrent generalized Sasakian space forms. Several authors [6–11] studied K -contact manifolds and proved conditions for these manifolds to be of ξ -conformally flat, ϕ -conformally flat, quasi-conharmonically flat, and ξ -conharmonically flat. Motivated by the above studies, in this paper we study D_a -homothetic deformations of K -contact manifolds by considering conharmonic and projective curvature tensor. The paper is organized as follows. After Preliminaries, we give a brief account of information of D_a -homothetic deformation of K -contact manifolds in Section 3. In Section 4, we study conharmonically flat, semisymmetric, ϕ -conharmonically flat, quasi-conharmonically flat, and ξ -conharmonically flat K -contact manifolds with respect to D_a -homothetic deformation. In the last section, we consider Weyl projective curvature in K -contact manifolds with respect to D_a -homothetic deformation.

2. Preliminaries

Let (M, ϕ, ξ, η, g) be a $(2n + 1)$ -dimensional almost contact metric manifold [12], consisting of a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η , and Riemannian metric g . Then

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (1)$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad (3)$$

$$g(X, \phi X) = 0, \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$. If ξ is a Killing vector field, then M is called a K -contact Riemannian manifold [13]. A K -contact Riemannian manifold is called Sasakian [12], if the relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (4)$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g .

If M^{2n+1} is a K -contact Riemannian manifold, then besides (1), (2), (3), and (4) the following relations hold [14]:

$$\nabla_X \xi = -\phi X, \quad (5)$$

$$(\nabla_X \eta)(Y) = -g(\phi X, Y), \quad (6)$$

$$S(X, \xi) = g(QX, \xi) = 2n\eta(X), \quad (7)$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (8)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (9)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad (10)$$

for any vector fields X and Y , where R and S denote, respectively, the curvature tensor of type (1, 3) and the Ricci tensor of type (0, 2).

Definition 1. A contact metric manifold M is said to be η -Einstein if $S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$, where α and β are smooth functions on M .

3. D_a -Homothetic Deformation of K -Contact Manifolds

Let (M, ϕ, ξ, η, g) be a $(2n + 1)$ -dimensional almost contact metric manifold. A D_a -homothetic deformation is defined by

$$\begin{aligned} \bar{\phi} &= \phi, & \bar{\xi} &= \frac{1}{a}\xi, & \bar{\eta} &= a\eta, \\ \bar{g} &= ag + a(a-1)\eta \otimes \eta, \end{aligned} \quad (11)$$

with a being a positive constant [1].

It is clear that the $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric manifold.

If (M, ϕ, ξ, η, g) is a K -contact manifold with Riemannian connection ∇ , the connection $\bar{\nabla}$ of the D_a -deformed K -contact manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ can be calculated from ∇ and $\bar{g}$. Using Koszul's formula and (5), (6), and (11), $\bar{\nabla}$ of $\bar{g}$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y - a(a-1)[\eta(Y)\phi X + \eta(X)\phi Y]. \quad (12)$$

Using (12), we obtain

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y + (a-1)\eta(Y)\phi^2 X. \quad (13)$$

The curvature tensor $\bar{R}$ of $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - (a-1) \\ &\times (g(\phi Y, Z)\phi X + g(\phi Z, X)\phi Y + 2g(\phi Y, X)\phi Z \\ &+ [g(X, Z)\xi - \eta(Z)X]\eta(Y) \\ &- [g(Y, Z)\xi - \eta(Z)Y]\eta(X) \\ &+ a[\eta(Y)X - \eta(X)Y]\eta(Z)). \end{aligned} \quad (14)$$

Using (9), (10), and (14), we have

$$\bar{R}(X, Y)\bar{\xi} = (2-a)[\eta(Y)X - \eta(X)Y],$$

$$\begin{aligned} \bar{R}(\bar{\xi}, Y)Z &= [g(Y, Z)\xi - \eta(Z)Y] \\ &- (a-1)[\eta(Y)\xi - Y]\eta(Z), \end{aligned}$$

$$\bar{R}(\bar{\xi}, Y)\bar{\xi} = \frac{(2-a)}{a}[\eta(Y)\xi - Y],$$

$$\bar{\eta}(\bar{R}(X, Y)Z) = a^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \quad (15)$$

From (14), we get

$$\begin{aligned} \bar{S}(Y, Z) &= aS(Y, Z) - a(a-1) \\ &\times ((2-a)g(Y, Z) + [2n(a-1) + a-2]\eta(Y)\eta(Z)), \end{aligned} \quad (16)$$

where $\bar{S}$ and S are the Ricci tensors of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ and (M, ϕ, ξ, η, g) , respectively.

It follows from (16) that

$$\bar{S}(Y, \bar{\xi}) = 2na(2-a)\eta(Y), \quad (17)$$

$$\bar{S}(\phi Y, \phi Z) = \bar{S}(Y, Z) - 2na^2(2-a)\eta(Y)\eta(Z). \quad (18)$$

Again contracting (16) over Y, Z , we get

$$\bar{r} = ar - 2na(a-1), \quad (19)$$

where $\bar{r}$ and r are the scalar curvatures of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ and (M, ϕ, ξ, η, g) , respectively.

4. Conharmonic Curvature Tensor in D_a -Homothetically Deformed K -Contact Manifolds

The conharmonic tensor of a D_a -homothetically deformed K -contact manifold is defined by [15]

$$\begin{aligned} \bar{K}(X, Y)Z &= \bar{R}(X, Y)Z - \frac{1}{2n-1} \\ &\times [\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + \bar{g}(Y, Z)\bar{Q}X \\ &- \bar{g}(X, Z)\bar{Q}Y], \end{aligned} \quad (20)$$

for $X, Y, Z \in TM$, where $\bar{R}$, $\bar{S}$, and $\bar{Q}$ are the Riemannian curvature tensor, Ricci tensor, and Ricci operator of $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$.

Definition 2. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be

(1) conharmonically flat if

$$K(X, Y)Z = 0, \quad (21)$$

(2) conharmonically semisymmetric if

$$R \cdot K = 0, \quad (22)$$

(3) ϕ -conharmonically flat if

$$g(K(\phi X, \phi Y)\phi Z, \phi W) = 0, \quad (23)$$

(4) quasi-conharmonically flat if

$$g(K(X, Y)Z, \phi W) = 0, \quad (24)$$

(5) ξ -conharmonically flat if

$$K(X, Y)\xi = 0, \quad (25)$$

for all vector fields X, Y , and Z .

Assume that $\bar{M}$ is conharmonically flat K -contact manifold with respect to D_a -homothetic deformation. So, we have $\bar{K}(X, Y)Z = 0$.

Then from (20), we have

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{1}{2n-1} \\ &\times [\bar{S}(Y, Z)X - \bar{S}(X, Z)Y + \bar{g}(Y, Z)\bar{Q}X \\ &\quad - \bar{g}(X, Z)\bar{Q}Y]. \end{aligned} \quad (26)$$

Setting $Z = \bar{\xi}$, contracting (26) with W , and using (7), (9), (14), and (16), we obtain

$$\begin{aligned} (2-a)(2n-1-2na)[\eta(Y)\bar{g}(X, W) - \eta(X)\bar{g}(Y, W)] \\ = \bar{\eta}(Y)\bar{S}(X, W) - \bar{\eta}(X)\bar{S}(Y, W). \end{aligned} \quad (27)$$

Taking $Y = \bar{\xi}$ in (27) and using (1), (7), and (16), it follows that

$$\begin{aligned} \bar{S}(X, W) &= \frac{(2-a)(2n-1-2na)}{a}\bar{g}(X, W) \\ &+ \frac{(2-a)(4na-2n+1)}{a}\bar{\eta}(X)\bar{\eta}(W). \end{aligned} \quad (28)$$

Thus, $\bar{M}$ is η -Einstein.

Using (28) in (26), we obtain

$$\begin{aligned} \bar{R}(X, Y, Z, W) &= \frac{2(2-a)(2n-1-2na)}{a(2n-1)} \\ &\times [\bar{g}(Y, Z)\bar{g}(X, W) - \bar{g}(X, Z)\bar{g}(Y, W)] \\ &+ \frac{(2-a)(4na-2n+1)}{a(2n-1)} \\ &\times ([\bar{g}(X, W)\bar{\eta}(Y) - \bar{g}(Y, W)\bar{\eta}(X)]\bar{\eta}(Z) \\ &\quad + [\bar{g}(Y, Z)\bar{\eta}(X) - \bar{g}(X, Z)\bar{\eta}(Y)]\bar{\eta}(W)). \end{aligned} \quad (29)$$

From (29), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{2(2-a)(2n-1-2na)}{a(2n-1)} \\ &\times [\bar{g}(Y, Z)X - \bar{g}(X, Z)Y] \\ &\quad - \frac{(2-a)(4na-2n+1)}{a(2n-1)} \\ &\times [\bar{\eta}(X)\bar{\eta}(Z)Y - \bar{\eta}(Y)\bar{\eta}(Z)X \\ &\quad + \bar{g}(X, Z)\bar{\eta}(Y)\xi - \bar{g}(Y, Z)\bar{\eta}(X)\xi]. \end{aligned} \quad (30)$$

Hence, it reduces to a generalized Sasakian space form with $f_1 = 2(2-a)(2n-1-2na)/a(2n-1)$, $f_2 = 0$, and $f_3 = -(2-a)(4na-2n+1)/a(2n-1)$. Thus, (30) leads to the following.

Theorem 3. A conharmonically flat K -contact manifold admitting D_a -homothetic deformation reduces to a generalized Sasakian space form with associated functions $f_1 = 2(2-a)(2n-1-2na)/a(2n-1)$, $f_2 = 0$, and $f_3 = -(2-a)(4na-2n+1)/a(2n-1)$.

Let us now consider a conharmonically semisymmetric K -contact manifold admitting D_a -homothetic deformation. Then the condition

$$\bar{R}(X, Y) \cdot \bar{K} = 0 \quad (31)$$

holds on $\bar{M}$ for all vector fields X, Y .

From (8), (14), (17), and (20), we obtain

$$\begin{aligned} \bar{\eta}(\bar{K}(X, Y)Z) &= a^2 [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - \frac{1}{(2n-1)} \\ &\times [(\bar{S}(Y, Z)\bar{\eta}(X) - \bar{S}(X, Z)\bar{\eta}(Y)) \\ &\quad + 2na(2-a)(\bar{g}(Y, Z)\eta(X) - \bar{g}(X, Z)\eta(Y))]. \end{aligned} \quad (32)$$

Setting $Z = \bar{\xi}$, in (32), we get

$$\bar{\eta}(\bar{K}(X, Y)\bar{\xi}) = 0. \quad (33)$$

Again taking $X = \bar{\xi}$ in (32) and using (17), we obtain

$$\begin{aligned} \bar{\eta}(\bar{K}(\bar{\xi}, Y)Z) &= \frac{-1}{2n-1}\bar{S}(Y, Z) + \left[1 - \frac{2n(2-a)}{2n-1}\right]\bar{g}(Y, Z) \\ &\quad + \left[-1 + \frac{4n(2-a)}{2n-1}\right]\bar{\eta}(Z)\bar{\eta}(Y). \end{aligned} \quad (34)$$

Now, (21) yields

$$\begin{aligned} \bar{R}(X, Y)\bar{K}(U, V)Z - \bar{K}(\bar{R}(X, Y)U, V)Z \\ - \bar{K}(U, \bar{R}(X, Y)V)Z - \bar{K}(U, V)\bar{R}(X, Y)Z = 0. \end{aligned} \quad (35)$$

Therefore,

$$\begin{aligned} & \bar{g}(\bar{R}(\bar{\xi}, Y) \bar{K}(U, V) Z, \bar{\xi}) - \bar{g}(\bar{K}(\bar{R}(\bar{\xi}, Y) U, V) Z, \bar{\xi}) \\ & - \bar{g}(\bar{K}(U, \bar{R}(\bar{\xi}, Y) V) Z, \bar{\xi}) \\ & - \bar{g}(\bar{K}(U, V) \bar{R}(\bar{\xi}, Y) Z, \bar{\xi}) = 0. \end{aligned} \quad (36)$$

From this it follows that

$$\begin{aligned} & -\bar{K}(U, V, Z, Y) + \eta(\bar{K}(U, V) Z) \eta(Y) \\ & + [g(Y, U) - (a-1) \eta(U) \eta(Y)] \eta(\bar{K}(\xi, V) Z) \\ & + (a-2) \eta(U) \eta(\bar{K}(Y, V) Z) \\ & + [g(Y, V) - (a-1) \eta(V) \eta(Y)] \eta(\bar{K}(U, \xi) Z) \\ & + (a-2) \eta(V) \eta(\bar{K}(U, Y) Z) \\ & + [g(Y, Z) - (a-1) \eta(Z) \eta(Y)] \eta(\bar{K}(U, V) \xi) \\ & + (a-2) \eta(Z) \eta(\bar{K}(U, V) Y) = 0, \end{aligned} \quad (37)$$

where

$$\bar{K}(U, V, Z, Y) = \bar{g}(\bar{K}(U, V) Z, Y). \quad (38)$$

Taking $Y = U$ in (37) and making use of (32) and (33), we obtain

$$\begin{aligned} & -\bar{K}(U, V, Z, U) + (a-1) \eta(U) \eta(\bar{K}(U, V) Z) \\ & + [g(U, U) - (a-1) \eta(U) \eta(U)] \eta(\bar{K}(\xi, V) Z) \\ & + [g(U, V) - (a-1) \eta(U) \eta(V)] \eta(\bar{K}(U, \xi) Z) \\ & + (a-2) \eta(Z) \eta(\bar{K}(U, V) U) = 0. \end{aligned} \quad (39)$$

If $\{e_1, e_2, \dots, e_{2n}, \bar{\xi}\}$ is a local orthonormal basis of vector fields in $\bar{M}$, then, from (39), we get

$$\begin{aligned} & \sum_{i=1}^{2n} \bar{K}(e_i, V, Z, e_i) \\ & = (a-1) \sum_{i=1}^{2n} \eta(e_i) \eta(\bar{K}(e_i, V) Z) \\ & + \sum_{i=1}^{2n} [g(e_i, e_i) - (a-1) \eta(e_i) \eta(e_i)] \eta(\bar{K}(\xi, V) Z) \\ & + \sum_{i=1}^{2n} [g(e_i, V) - (a-1) \eta(e_i) \eta(V)] \eta(\bar{K}(e_i, \xi) Z) \\ & + (a-2) \sum_{i=1}^{2n} \eta(\bar{K}(e_i, V) e_i) \eta(Z). \end{aligned} \quad (40)$$

From (20), it follows that

$$\begin{aligned} \sum_{i=1}^{2n} \bar{K}(e_i, V, Z, e_i) & = \frac{1}{2n-1} \bar{S}(V, Z) \\ & - \left[1 - \frac{\bar{r} + 2n(2-a)}{2n-1} \right] \bar{g}(V, Z) \\ & + \left[1 - \frac{4n(2-a)}{2n-1} \right] \bar{\eta}(V) \bar{\eta}(Z), \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{2n} \eta(e_i) \eta(\bar{K}(e_i, V) Z) \\ & = \frac{1-a^2}{a^2(2n-1)} \bar{S}(V, Z) \\ & + \left[\frac{a^2-1}{a^2} + \frac{2n(2-a)(1-a^2)}{a^2(2n-1)} \right] \bar{g}(V, Z) \\ & + \frac{1-a^2}{a^2} \bar{\eta}(V) \bar{\eta}(Z), \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^{2n} [g(e_i, V) - (a-1) \eta(e_i) \eta(V)] \eta(\bar{K}(e_i, \xi) Z) \\ & = \frac{1}{2n-1} \bar{S}(V, Z) + \left[-a^2 + \frac{2na^2(2-a)}{2n-1} \right] g(V, Z) \\ & + \left[a^2 - \frac{2na(2-a)(a+1)(a^2-a+1)}{2n-1} \right] \eta(V) \eta(Z), \\ & \sum_{i=1}^{2n} \eta(\bar{K}(e_i, V) Z) \eta(Z) = \left[\frac{\bar{r}}{2n-1} + 4n(1-a) \right] \eta(V) \eta(Z). \end{aligned} \quad (41)$$

Using (41) in (40), we obtain

$$\begin{aligned} & (2n+2-a) \eta(\bar{K}(\bar{\xi}, V) Z) \\ & = \frac{(a-1)(a^2-1)}{a^2(2n-1)} \bar{S}(V, Z) \\ & + \left[\frac{\bar{r}}{2n-1} + \frac{2na^2-a(1+4n)+1+2n}{a^2(2n-1)} \right] \bar{g}(V, Z) \\ & + \left[\frac{(a-2)\bar{r}}{a^2(2n-1)} + \frac{p}{a^2(2n-1)} \right] \bar{\eta}(V) \bar{\eta}(Z), \end{aligned} \quad (42)$$

where

$$\begin{aligned} p = & -2na^5 + 2na^4 + 10na^3 + a^2(8n^2 - 18n) \\ & + a(14n - 24n^2 + 1) + 16n^2 - 6n - 1. \end{aligned} \quad (43)$$

In view of (34), (42) yields

$$\bar{S}(V, Z) = -\frac{1}{a^2(2n+1) + 1 - a} [\alpha \bar{g}(V, Z) + \beta \bar{\eta}(V) \bar{\eta}(Z)], \quad (44)$$

where

$$\begin{aligned} \alpha &= \bar{r}a^2 + 2na^4 + a^3(-4n^2 - 6n - 1) \\ &\quad + a^2(4n^2 + 8n + 2) + a(-1 - 4n) + 1 + 2n, \\ \beta &= (a - 2)\bar{r} - 2na^5 - 2na^4 + a^3(8n^2 + 24n + 1) \\ &\quad + a^2(-4n^2 - 32n - 2) + a(14n - 24n^2 + 1) \\ &\quad + 16n^2 - 6n - 1. \end{aligned} \quad (45)$$

Thus, $\bar{M}$ is η -Einstein.

If $\{e_1, e_2, \dots, e_{2n}, \bar{\xi}\}$ is a local orthonormal basis of vector fields in $\bar{M}$, then, from (44), we get

$$\begin{aligned} \bar{r} &= (-2na^5 + 4n^2a^4 + a^3l + a^2m \\ &\quad + a(-32n^2 + 8n) + 20n^2 + 2n) \\ &\quad \times (1 - 2a^2(2n + 1))^{-1}, \end{aligned} \quad (46)$$

where

$$\begin{aligned} l &= -8n^3 - 8n^2 + 16n, \\ m &= 8n^3 + 16n^2 - 20n. \end{aligned} \quad (47)$$

So, we can state the following.

Theorem 4. In a $(2n + 1)$ -dimensional conharmonically semisymmetric K -contact manifold admitting D_a -homothetic deformation, scalar curvature $\bar{r}$ is given by (46).

Analogous to the definition of ϕ -conharmonically flat K -contact manifolds [8], we define ϕ -conharmonically flat K -contact manifolds with respect to D_a -homothetic deformation. Let us assume that M is a ϕ -conharmonically flat K -contact manifold with respect to D_a -homothetic deformation. It can be easily seen that

$$\bar{g}(\bar{K}(\phi X, \phi Y)\phi Z, \phi W) = 0, \quad (48)$$

where $X, Y, Z, W \in T\bar{M}$.

Using (20), (48) yields

$$\begin{aligned} &\bar{g}(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) \\ &= \frac{1}{2n-1} (\bar{S}(\phi Y, \phi Z)\bar{g}(\phi X, \phi W) \\ &\quad - \bar{S}(\phi X, \phi Z)\bar{g}(\phi Y, \phi W) \\ &\quad + \bar{S}(\phi X, \phi W)\bar{g}(\phi Y, \phi Z) \\ &\quad - \bar{S}(\phi Y, \phi W)\bar{g}(\phi X, \phi Z)), \end{aligned} \quad (49)$$

for all $X, Y, Z, W \in T\bar{M}$.

If $\{e_1, e_2, \dots, e_{2n}, \bar{\xi}\}$ is a local orthonormal basis of vector fields in $\bar{M}$, then $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \bar{\xi}\}$ is also a local orthonormal basis. So, using (1), (6), (14), (16), and (18), it can be easily verified that

$$\begin{aligned} \sum_{i=1}^{2n} \bar{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) &= \bar{S}(Y, Z) - ag(Y, Z) \\ &\quad + [a - 2na(2 - a)]\eta(Y)\eta(Z), \\ \sum_{i=1}^{2n} \bar{g}(\phi e_i, \phi e_i) &= 2n, \\ \sum_{i=1}^{2n} \bar{S}(\phi e_i, \phi e_i) &= \bar{r} - 2n(2 - a), \\ \sum_{i=1}^{2n} \bar{S}(\phi Y, \phi e_i)\bar{g}(\phi e_i, \phi Z) &= \bar{S}(\phi Y, \phi Z). \end{aligned} \quad (50)$$

For a local orthonormal basis $\{\phi e_1, \phi e_2, \dots, \phi e_{2n}, \bar{\xi}\}$ of vector fields in $\bar{M}$, putting $X = W = e_i$ in (49) and summing up with respect to $i = 1, 2, \dots, 2n + 1$, we have

$$\begin{aligned} &\sum_{i=1}^{2n} \bar{g}(\bar{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) \\ &= \frac{1}{2n-1} \sum_{i=1}^{2n} (\bar{S}(\phi Y, \phi Z)\bar{g}(\phi e_i, \phi e_i) \\ &\quad - \bar{S}(\phi e_i, \phi Z)\bar{g}(\phi Y, \phi e_i) \\ &\quad + \bar{S}(\phi e_i, \phi e_i)\bar{g}(\phi Y, \phi Z) \\ &\quad - \bar{S}(\phi Y, \phi e_i)\bar{g}(\phi e_i, \phi Z)), \end{aligned} \quad (51)$$

for all $Y, Z \in T\bar{M}$. The previous equation, in view of (50), becomes

$$\begin{aligned} \bar{S}(\phi Y, \phi Z) &= [a(2n-1) + a(\bar{r} - 2n(2-a))] \\ &\quad \times [g(Y, Z) - \eta(Y)\eta(Z)], \end{aligned} \quad (52)$$

for all $Y, Z \in T\bar{M}$.

Using (2) and (18), (52) reduces to

$$\begin{aligned} \bar{S}(Y, Z) &= [\bar{r} + 2n - 1 - 2n(2-a)]\bar{g}(Y, Z) \\ &\quad + [2n(2-a)(a^2 + 2a - 1) \\ &\quad - (2a - 1)(\bar{r} + 2n - 1)]\bar{\eta}(Y)\bar{\eta}(Z). \end{aligned} \quad (53)$$

Setting $Y = Z = e_i$ in (53), summing up with respect to $i = 1, 2, \dots, 2n + 1$, and using (19), we obtain

$$\bar{r} = \frac{-2na^3 + a(4n^2 + 8n + 2) - 4n^2 - 6n - 2}{2a - 2n - 1}. \quad (54)$$

Replacing X by ϕX and Y by ϕY in (53) and using (54), we obtain $\bar{S}(\phi X, \phi Y) = (\bar{r} + 2n - 1 - 2n(2 - a))\bar{g}(\phi X, \phi Y)$ for all $X, Y \in T\bar{M}$.

Now using the previous expression in (49), we obtain

$$\begin{aligned} \bar{R}(\phi X, \phi Y, \phi Z, \phi W) \\ = \frac{2(\bar{r} + 2n - 1 - 2n(2 - a))}{2n - 1} \\ \times (\bar{g}(\phi Y, \phi Z)\bar{g}(\phi X, \phi W) - \bar{g}(\phi X, \phi Z)\bar{g}(\phi Y, \phi W)), \end{aligned} \quad (55)$$

for all $X, Y, Z, W \in T\bar{M}$.

The converse is obvious. Thus we have the following.

Theorem 5. A $(2n + 1)$ -dimensional K -contact manifold is ϕ -conharmonically flat with respect to D_a -homothetic deformation if and only if $\bar{M}$ satisfies (55).

From (20), we obtain

$$\begin{aligned} \bar{g}(\bar{K}(X, Y)Z, \phi W) \\ = \bar{R}(X, Y, Z, \phi W) - \frac{1}{2n - 1} \\ \times (\bar{S}(Y, Z)\bar{g}(X, \phi W) - \bar{S}(X, Z)\bar{g}(Y, \phi W) \\ + \bar{S}(X, \phi W)\bar{g}(Y, Z) - \bar{S}(Y, \phi W)\bar{g}(X, \phi Z)), \end{aligned} \quad (56)$$

for all $X, Y, Z, W \in T\bar{M}$.

Suppose that M is quasi-conharmonically flat K -contact manifold with respect to D_a -homothetic deformation; that is,

$$\bar{g}(\bar{K}(X, Y)Z, \phi W) = 0. \quad (57)$$

Then (56) reduces to

$$\begin{aligned} \bar{R}(X, Y, Z, \phi W) \\ = \frac{1}{2n - 1} (\bar{S}(Y, Z)\bar{g}(X, \phi W) - \bar{S}(X, Z)\bar{g}(Y, \phi W) \\ + \bar{S}(X, \phi W)\bar{g}(Y, Z) - \bar{S}(Y, \phi W)\bar{g}(X, \phi Z)). \end{aligned} \quad (58)$$

For a local orthonormal basis $\{e_1, e_2, \dots, e_{2n}, \bar{\xi}\}$ of vector fields in $\bar{M}$, putting $X = \phi e_i$ and $W = e_i$ in (58) and summing up with respect to $i = 1, 2, \dots, 2n + 1$, we obtain

$$\begin{aligned} \sum_{i=1}^{2n} \bar{R}(\phi e_i, Y, Z, \phi e_i) \\ = \frac{1}{2n - 1} \sum_{i=1}^{2n} (\bar{S}(Y, Z)\bar{g}(\phi e_i, \phi e_i) \\ - \bar{S}(\phi e_i, Z)\bar{g}(Y, \phi e_i) \\ + \bar{S}(\phi e_i, \phi e_i)\bar{g}(Y, Z) \\ - \bar{S}(Y, \phi e_i)\bar{g}(\phi e_i, Z)). \end{aligned} \quad (59)$$

Using (2), (10), (14), and (17) in (59), we obtain

$$\begin{aligned} \bar{S}(Y, Z) &= [\bar{r} + 2n - 1 - 2n(2 - a)]\bar{g}(Y, Z) \\ &\quad - (2n - 1)\bar{\eta}(Y)\bar{\eta}(Z). \end{aligned} \quad (60)$$

Taking $Z = \bar{\xi}$ and using (1) and (17), we obtain

$$\bar{r} = 2n(2 - a)(a + 1), \quad (61)$$

and using (61) in (19), we obtain

$$r = \frac{4n}{a}. \quad (62)$$

Hence, we can state the following.

Theorem 6. Let M be a K -contact manifold. Suppose that $\bar{M}$ is obtained from M by D_a -homothetic deformation. If $\bar{M}$ is quasi-conharmonically flat, then the scalar curvatures $\bar{r}$ and r of $\bar{M}$ and M are, respectively, given by (61) and (62).

Suppose that $\bar{M}$ is ξ -conharmonically flat. Then from (20), we have

$$\begin{aligned} \bar{R}(X, Y)\bar{\xi} &= \frac{1}{2n - 1} \\ &\times (2na(2 - a)[\eta(Y)X - \eta(X)Y] \\ &\quad + [\bar{Q}X\bar{\eta}(Y) - \bar{Q}Y\bar{\eta}(X)]). \end{aligned} \quad (63)$$

Contracting the above equation with respect to W , we obtain

$$\begin{aligned} \bar{R}(X, Y, \bar{\xi}, W) \\ = \frac{1}{2n - 1} (2na(2 - a)[\eta(Y)\bar{g}(X, W) - \eta(X)\bar{g}(Y, W)] \\ + [\bar{S}(X, W)\bar{\eta}(Y) - \bar{S}(Y, W)\bar{\eta}(X)]), \end{aligned} \quad (64)$$

for all $X, Y, Z, W \in T\bar{M}$.

For a local orthonormal basis $\{e_1, e_2, \dots, e_{2n}, \bar{\xi}\}$ of vector fields in $\bar{M}$, using (64), we obtain

$$\begin{aligned} \sum_{i=1}^{2n} \bar{R}(e_i, Y, \bar{\xi}, e_i) \\ = \frac{1}{2n - 1} \sum_{i=1}^{2n} (2na(2 - a)[\eta(Y)\bar{g}(e_i, e_i) - \eta(e_i)\bar{g}(Y, e_i)] \\ + [\bar{S}(e_i, e_i)\bar{\eta}(Y) - \bar{S}(Y, e_i)\bar{\eta}(e_i)]). \end{aligned} \quad (65)$$

Therefore,

$$2n\bar{S}(Y, \bar{\xi}) = [4n^2(2 - a) + \bar{r}]\bar{\eta}(Y). \quad (66)$$

Using (17) in (66), we obtain

$$\bar{r} = 0. \quad (67)$$

Taking $Y = \bar{\xi}$ in (64) and using (10), (14), and (17), we obtain

$$\begin{aligned} \bar{S}(X, W) = & [2na - 4n - 1] \bar{g}(X, W) \\ & + [6n - 4na + 1] \bar{\eta}(X) \bar{\eta}(W). \end{aligned} \quad (68)$$

From this we can conclude that M is η -Einstein. Thus, we have the following.

Theorem 7. *A D_a -Homothetically deformed ξ -conharmonically flat K -contact manifold is η -Einstein and its scalar curvature vanishes.*

5. D_a -Homothetic Deformation of ξ -Weyl Projectively Flat K -Contact Manifolds

Suppose that, in a $(2n + 1)$ -dimensional K -contact manifold M with D_a -homothetic deformation, the Ricci tensor vanishes; that is,

$$\bar{S}(X, Y) = 0. \quad (69)$$

Then from (16), we have

$$\begin{aligned} S(Y, Z) = & (a - 1) (3g(\phi Y, \phi Z) - (1 + a) g(Y, Z) \\ & + [2n(a - 1) + 1 + a] \eta(Y) \eta(Z)). \end{aligned} \quad (70)$$

The Weyl projective curvature tensor of M is given by [16]

$$W(X, Y)Z = R(X, Y)Z - \frac{1}{2n} [S(Y, Z)X - S(X, Z)Y]. \quad (71)$$

If $W(X, Y)\xi = 0$, then (71) reduces to

$$R(X, Y)\xi = \frac{1}{2n} [S(Y, \xi)X - S(X, \xi)Y]. \quad (72)$$

Using (70) in (72), we obtain

$$R(X, Y)\xi = (a - 1)^2 [\eta(Y)X - \eta(X)Y]. \quad (73)$$

The Weyl projective curvature tensor of $\bar{M}$ with respect to D_a -homothetic deformation is given by

$$\bar{W}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{2n} [\bar{S}(Y, Z)X - \bar{S}(X, Z)Y]. \quad (74)$$

Now using (14) and (69) in (74), we get

$$\bar{W}(X, Y)\bar{\xi} = R(X, Y)\bar{\xi} - \frac{(a - 1)^2}{a} [\eta(Y)X - \eta(X)Y]. \quad (75)$$

From (73), the (75) reduces to

$$\bar{W}(X, Y)\bar{\xi} = 0. \quad (76)$$

Thus, we can state the following.

Theorem 8. *Let $\bar{M}$ be obtained from a K -contact manifold M by D_a -homothetic deformation. If the Ricci tensor of $\bar{M}$ vanishes, then it is ξ -Weyl projectively flat.*

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