Research Article

The Depth of Generalized Full Terms and Generalized Full Hypersubstitutions

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We generalize the concept of a full term which was introduced by Denecke and Changphas and the concept of the depth of a hypersubstitution which was introduced by Denecke et al. to the depth of a generalized full term and the depth of a generalized full hypersubstitution and then derive the formula for its depth.

1. Introduction

Identities are used to classify algebras into collections called varieties. An important activity is to classify all varieties of algebras of a given type τ . Such as for a type $\tau = (2)$, if the binary operation symbol satisfies the associative law, the algebras are semigroups. Hyperidentities are also used to classify varieties into collections called hypervarieties. Its generalization is called M -hyperidentities and M -solid varieties. The study of hyperidentities is a part of Universal Algebra and also a part of Model Theory of second-order language. The notions of hyperidentities and hypervarieties of a given type τ without nullary operations were originated by Aczél [1], Belousov [2], Neumann [3], and Taylor [4]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution which was introduced by Denecke et al. [5]. In 2000, Leeratanavalee and Denecke generalized the concept of hypersubstitution [6].

Let $X = \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as *letters*, to X as an *alphabet*, and also refer to the set $X_n = \{x_1, x_2, \dots, x_n\}$ as an n -*element alphabet*. Let $(f_i)_{i \in I}$ be an indexed set which is disjoint from X . Each f_i is called n_i -*ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its *arity*. The function τ , on the values of τ written as $(n_i)_{i \in I}$ is called a *type*.

An n -*ary term* of type τ , or simply an n -*ary term* is defined inductively as follows:

- (i) the variables $x_1, \dots, x_n$ are n -*ary terms*.
- (ii) If $t_1, \dots, t_{n_i}$ are n -*ary terms*, then $f_i(t_1, \dots, t_{n_i})$ is an n -*ary term*.

By $W_\tau(X_n)$ we mean the smallest set which contains $x_1, \dots, x_n$ and is closed under finite application of (ii). It is clear that every n -*ary term* is also an m -*ary term* for all $m \geq n$. The set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ over the alphabet X . This set can be used as the universe of an algebra of type τ . For every $i \in I$, an n_i -*ary operation* $\bar{f}_i$ on $W_\tau(X)$ is defined by $\bar{f}_i : W_\tau(X)^{n_i} \rightarrow W_\tau(X)$ with $(t_1, \dots, t_{n_i}) \mapsto f_i(t_1, \dots, t_{n_i})$. The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i \in I})$ is called the *absolutely free algebra of type τ over the set X* . The term algebra $\mathcal{F}_\tau(X)$ is generated by the set X and has the property called *absolute freeness*, meaning that for every algebra $\mathcal{A} \in \text{Alg}(\tau)$ and every mapping $f : X \rightarrow \mathcal{A}$ there exists a unique homomorphism $\hat{f} : \mathcal{F}_\tau(X) \rightarrow \mathcal{A}$ which extends the mapping f and such that $\hat{f} \circ \varphi = f$, where $\varphi : X \rightarrow \mathcal{F}_\tau(X)$ is the embedding of X into $\mathcal{F}_\tau(X)$. This can be shown by Figure 1.

It is clear that the absolutely free algebras $\mathcal{F}_\tau(X)$ and $\mathcal{F}_\tau(Y)$ are isomorphic if X and Y have the same cardinality. One can use trees instead of terms. Trees are connected

FIGURE 1: The embedding of X into $\mathcal{F}_\tau(X)$.

graphs which have no cycles. It can be used to visualize the structure of computer programs. For applications, we can measure the complexity of a tree. Then the depth of a term (or the height of a tree) is defined. Let $t \in W_\tau(X)$, the depth of a term t which is denoted by $\text{depth}(t)$ and is defined as follows:

- (i) if $t = x \in X$, then $\text{depth}(t) := 0$,
- (ii) if $t = f_i(t_1, \dots, t_{n_i})$ where $t_1, \dots, t_{n_i} \in W_\tau(X)$, then $\text{depth}(t) := \max\{\text{depth}(t_j) \mid 1 \leq j \leq n_i\} + 1$.

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ which does not necessarily preserve the arity.

We denoted the set of all generalized hypersubstitutions of type τ by $\text{Hyp}_G(\tau)$. To define a binary operation on $\text{Hyp}_G(\tau)$, we need to define the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ by the following steps.

For any term $t \in W_\tau(X)$,

- (i) if $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$,
- (ii) if $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$,
- (iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)). \quad (1)$$

Then the generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ by the following steps:

- (i) $\hat{\sigma}[x] := x \in X$,
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i -ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

We defined a binary operation $\circ_G$ on $\text{Hyp}_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i -ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. In [6], it is proved that for arbitrary terms $t, t_1, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

- (i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)]$,
- (ii) $(\hat{\sigma}_1 \circ \sigma_2)^\wedge = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

It turns out that $(\text{Hyp}_G(\tau); \circ_G, \sigma_{\text{id}})$ is a monoid where σ_{id} is its identity element. The monoid $(\text{Hyp}(\tau); \circ_h, \sigma_{\text{id}})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(\text{Hyp}_G(\tau); \circ_G, \sigma_{\text{id}})$. In 2001, Denecke et al. introduced the concept of the depth of a hypersubstitution [7]. Also in 2002, Denecke and Changpas defined full terms and studied full hypersubstitutions [8]. In this paper we generalized these concepts to the depth of generalized full terms and generalized full hypersubstitutions.

2. Generalized Full Terms

We use standard terminology established in modern algebra; see [9–11]. In this section, we give the definition of a generalized full term of type τ , or simply a generalized full term, and describe the behavior of the depth of a generalized full term.

Definition 1. Let Ω be the set of operation symbols and $f \in \Omega$ with arity $(f) = n$:

- (i) if $s : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is a permutation, then $f(x_{s(1)}, \dots, x_{s(n)})$ is a generalized full term,
- (ii) if $j_1, j_2, \dots, j_n$ are natural numbers greater than n , then $f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ is a generalized full term where s' is a permutation on $\{j_1, \dots, j_n\}$,
- (iii) if $t_1, \dots, t_n$ are generalized full terms, then $f(t_1, \dots, t_n)$ is a generalized full term.

Let $W_{(n)}^{\text{GF}}(X)$ be the set of all generalized full terms and let P_n be the set of all permutations on $\{1, 2, \dots, n\}$. By Definition 1, the set $W_{(n)}^{\text{GF}}(X)$ is closed under finite application of (iii). Therefore $(W_{(n)}^{\text{GF}}(X); (\bar{f}_i)_{i \in I})$ is a subalgebra of $\mathcal{F}_\tau(X)$.

Proposition 2. Let $p, p_1, p_2, \dots, p_n \in W_{(n)}^{\text{GF}}(X)$. Then $S^n(p, p_1, \dots, p_n)$ is also a generalized full term.

Proof. We give a proof by induction on the complexity of a generalized full term p .

If $p = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$, then:

$$\begin{aligned} S^n(p, p_1, \dots, p_n) &= S^n(f(x_{s(1)}, \dots, x_{s(n)}), p_1, \dots, p_n) \\ &= f(p_{s(1)}, \dots, p_{s(n)}) \end{aligned} \quad (2)$$

which is a generalized full term.

If $p = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then

$$\begin{aligned} S^n(p, p_1, \dots, p_n) &= S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), p_1, \dots, p_n) \\ &= f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \end{aligned} \quad (3)$$

which is a generalized full term.

If $p = f(t_1, \dots, t_n)$ where $t_1, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$ and assume that $S^n(t_j, p_1, \dots, p_n)$ are generalized full terms for all $1 \leq j \leq n$, then

$$\begin{aligned} S^n(p, p_1, \dots, p_n) \\ &= S^n(f(t_1, \dots, t_n), p_1, \dots, p_n) \\ &= f(S^n(t_1, p_1, \dots, p_n), \dots, S^n(t_n, p_1, \dots, p_n)). \end{aligned} \quad (4)$$

By Definition 1(iii), $S^n(p, p_1, \dots, p_n)$ is a generalized full term. $\square$

Now we consider the algebra clone $_{\text{GF}}(n) := (W_{(n)}^{\text{GF}}(X); S^n)$ of type $(n+1)$ with $\mathcal{F}_{\text{GS}(n)} = \{f(x_{s(1)}, \dots, x_{s(n)}) \mid s \in P_n\} \cup \{f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \mid j_1, \dots, j_n > n, s' \text{ is a permutation on } \{j_1, \dots, j_n\}\}$ as a generating system. The algebra clone $_{\text{GF}}(n)$ is called the *clone of generalized full terms of type (n)*. Then we have the following.

Proposition 3. *The algebra clone $_{\text{GF}}(n)$ satisfies the so-called superassociative law (C)*

$$\begin{aligned} \tilde{S}^n(X_0, \tilde{S}^n(Y_1, X_1, \dots, X_n), \dots, \tilde{S}^n(Y_n, X_1, \dots, X_n)) \\ \approx \tilde{S}^n(\tilde{S}^n(X_0, Y_1, \dots, Y_n), X_1, \dots, X_n), \end{aligned} \quad (5)$$

where $\tilde{S}^n$ is an $(n+1)$ -ary operation symbol and X_i, Y_j are variables.

Proof. We give a proof by induction on the complexity of the generalized full term which is substituted for X_0 . Substituting for X_0 a term of the form $f(x_{s(1)}, \dots, x_{s(n)})$ for $s \in P_n$, then

$$\begin{aligned} S^n(f(x_{s(1)}, \dots, x_{s(n)}), S^n(t_1, s_1, \dots, s_n), \dots, \\ S^n(t_n, s_1, \dots, s_n)) \\ &= f(S^n(x_{s(1)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots, \\ &\quad S^n(x_{s(n)}, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))) \\ &= f(S^n(t_{s(1)}, s_1, \dots, s_n), \dots, S^n(t_{s(n)}, s_1, \dots, s_n)) \\ &= S^n(f(t_{s(1)}, \dots, t_{s(n)}), s_1, \dots, s_n) \\ &= S^n(S^n(f(x_{s(1)}, \dots, x_{s(n)}), t_1, \dots, t_n), s_1, \dots, s_n). \end{aligned} \quad (6)$$

If we substitute for X_0 a term $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then

$$\begin{aligned} S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), S^n(t_1, s_1, \dots, s_n), \dots, \\ S^n(t_n, s_1, \dots, s_n)) \\ &= f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \\ &= S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), s_1, \dots, s_n) \\ &= S^n(f(S^n(x_{s'(j_1)}, t_1, \dots, t_n), \dots, \end{aligned}$$

$$\begin{aligned} S^n(x_{s'(j_n)}, t_1, \dots, t_n), s_1, \dots, s_n) \\ &= S^n(S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), t_1, \dots, t_n), s_1, \dots, s_n). \end{aligned} \quad (7)$$

If we substitute for X_0 a term $t = f(r_1, \dots, r_n)$ and assume that (C) is satisfied for $r_1, \dots, r_n \in W_{(n)}^{\text{GF}}(X)$, then

$$\begin{aligned} S^n(f(r_1, \dots, r_n), S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)) \\ &= f(S^n(r_1, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n)), \dots, \\ &\quad S^n(r_n, S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))) \\ &= S^n(f(S^n(r_1, t_1, \dots, t_n), \dots, S^n(r_n, t_1, \dots, t_n)), \\ &\quad s_1, \dots, s_n) \\ &= S^n(S^n(f(r_1, \dots, r_n), t_1, \dots, t_n), s_1, \dots, s_n). \end{aligned} \quad (8)$$

$\square$

An algebra $(M; S^n)$ of type $(n+1)$ which satisfies the condition (C) is called a *Menger algebra of rank n*.

Lemma 4. *Let $t, t_1, t_2, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$. If $\text{depth}(t) = 1$, then*

$$\begin{aligned} \text{depth}(S^n(t, t_1, \dots, t_n)) = 1 \\ \text{or } \max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + 1. \end{aligned} \quad (9)$$

Proof. Assume that $\text{depth}(t) = 1$. Then $t = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$ or $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$:

if $t = f(x_{s(1)}, \dots, x_{s(n)})$, then

$$\begin{aligned} \text{depth}(S^n(t, t_1, \dots, t_n)) \\ &= \text{depth}(S^n(f(x_{s(1)}, \dots, x_{s(n)}), t_1, \dots, t_n)) \\ &= \text{depth}(f(t_{s(1)}, \dots, t_{s(n)})) \\ &= \max\{\text{depth}(t_{s(1)}), \dots, \text{depth}(t_{s(n)})\} + 1 \\ &= \max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + 1. \end{aligned} \quad (10)$$

If $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$, then

$$\begin{aligned} \text{depth}(S^n(t, t_1, \dots, t_n)) \\ &= \text{depth}(S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), t_1, \dots, t_n)) \\ &= \text{depth}(f(x_{s'(j_1)}, \dots, x_{s'(j_n)})) \\ &= 1. \end{aligned} \quad (11)$$

$\square$

Theorem 5. *Let $t, t_1, t_2, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$. Then $\text{depth}(S^n(t, t_1, \dots, t_n)) = 1$ or $\max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + \text{depth}(t)$.*

Proof. We prove this fact by induction on the complexity of $\text{depth}(t)$.

If $\text{depth}(t) = 1$, then by Lemma 4 we have $\text{depth}(S^n(t, t_1, \dots, t_n)) = 1$ or $\max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + \text{depth}(t)$.

If $\text{depth}(t) > 1$, then we have $t = f(s_1, \dots, s_n)$ and assume that the formula is satisfied for $s_1, \dots, s_n \in W_{(n)}^{\text{GF}}(X)$ that is $\text{depth}(S^n(s_k, t_1, \dots, t_n)) = 1$ or $\max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + \text{depth}(s_k)$ for all $1 \leq k \leq n$. Then we have

$$\begin{aligned}
 & \text{depth}(S^n(t, t_1, \dots, t_n)) \\
 &= \text{depth}(f(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n))) \\
 &= \max\{\text{depth}(S^n(s_1, t_1, \dots, t_n)), \dots, \\
 & \quad \text{depth}(S^n(s_n, t_1, \dots, t_n))\} + 1 \\
 &= \max\{\max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} \\
 & \quad + \text{depth}(s_1), \dots, \\
 & \quad \max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} \\
 & \quad + \text{depth}(s_n)\} + 1 \\
 &= \max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} \\
 & \quad + \max\{\text{depth}(s_1), \dots, \text{depth}(s_n)\} + 1 \\
 &= \max\{\text{depth}(t_1), \dots, \text{depth}(t_n)\} + \text{depth}(t).
 \end{aligned} \tag{12}$$

□

3. The Depth of Generalized Full Terms with respect to k

In this section, we generalize the concept of the depth of a full term t with respect to k which were studied by Denecke et al. [7] to the depth of a generalized full term t with respect to k .

Definition 6. Let $t \in W_{(n)}^{\text{GF}}(X)$ and let k be a fixed element of $\{1, 2, \dots, n\}$ and $\text{var}(t)$ be the set of all variables occurring in t . For each variable x_k , the depth of a generalized full term t with respect to k denoted by $\text{depth}_k(t)$ is defined inductively as follows:

- (i) if $t = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$, then $\text{depth}_k(t) = 1$,
- (ii) if $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, j_2, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then $\text{depth}_k(t) = 0$,
- (iii) if $t = f(t_1, \dots, t_n)$ and assume that $\text{depth}_k(t_j)$, $1 \leq j \leq n$ are already defined, then

$$\begin{aligned}
 & \text{depth}_k(t) \\
 &= \begin{cases} 0, & \text{if } x_k \notin \text{var}(t); \\ \max\{\text{depth}_k(t_j) \mid 1 \leq j \leq n; \\ \quad x_k \in \text{var}(t_j)\} + 1. & \text{otherwise} \end{cases}
 \end{aligned} \tag{13}$$

Theorem 7. Let $t, t_1, t_2, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$. Then

$$\begin{aligned}
 & \text{depth}(S^n(t, t_1, \dots, t_n)) \\
 & \geq \max\{\text{depth}_j(t) + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(t)\}.
 \end{aligned} \tag{14}$$

Proof. We prove this fact by induction on the complexity of $\text{depth}(t)$. Suppose that $\text{depth}(t) = 1$. Then $t = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$ or $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, j_2, \dots, j_n$ are natural numbers greater than n , and s' is a permutation on $\{j_1, \dots, j_n\}$.

If $t = f(x_{s(1)}, \dots, x_{s(n)})$, then $S^n(t, t_1, \dots, t_n) = S^n(f(x_{s(1)}, \dots, x_{s(n)}), t_1, \dots, t_n) = f(t_{s(1)}, \dots, t_{s(n)})$. Therefore,

$$\begin{aligned}
 & \text{depth}(S^n(t, t_1, \dots, t_n)) \\
 &= \text{depth}(f(t_{s(1)}, \dots, t_{s(n)})) \\
 &= \max\{\text{depth}(t_{s(1)}), \dots, \text{depth}(t_{s(n)})\} + 1, \\
 & \max\{\text{depth}_j(t) + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(t)\} \\
 &= \max\{1 + \text{depth}(t_1), \dots, 1 + \text{depth}(t_n)\} \\
 &= \max\{\text{depth}(t_{s(1)}), \dots, \text{depth}(t_{s(n)})\} + 1.
 \end{aligned} \tag{15}$$

If $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$, then $S^n(t, t_1, \dots, t_n) = S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), t_1, \dots, t_n) = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$. Therefore,

$$\begin{aligned}
 & \text{depth}(S^n(t, t_1, \dots, t_n)) = \text{depth}(f(x_{s'(j_1)}, \dots, x_{s'(j_n)})) \\
 &= 1, \\
 & \max\{\text{depth}_j(t) + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(t)\} = 0.
 \end{aligned} \tag{16}$$

If $\text{depth}(t) > 1$, then $t = f(p_1, \dots, p_n)$ where $p_1, \dots, p_n \in W_{(n)}^{\text{GF}}(X)$ and assume that the formula is satisfied for $p_1, \dots, p_n$. Thus

$$\begin{aligned}
 & S^n(t, t_1, \dots, t_n) \\
 &= S^n(f(p_1, \dots, p_n), t_1, \dots, t_n) \\
 &= f(S^n(p_1, t_1, \dots, t_n), \dots, S^n(p_n, t_1, \dots, t_n)), \\
 & \text{depth}(S^n(t, t_1, \dots, t_n)) \\
 &= \max\{\text{depth}(S^n(p_1, t_1, \dots, t_n)), \dots, \\
 & \quad \text{depth}(S^n(p_n, t_1, \dots, t_n))\} + 1
 \end{aligned}$$

$$\begin{aligned}
&\geq \max \left\{ \max \left\{ \text{depth}_j(p_1) \right. \right. \\
&\quad \left. \left. + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(p_1) \right\}, \dots, \right. \\
&\quad \left. \max \left\{ \text{depth}_j(p_n) \right. \right. \\
&\quad \left. \left. + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(p_n) \right\} \right\} + 1 \\
&= \max \left\{ \max \left\{ \text{depth}_j(p_k) \mid 1 \leq j \leq n, \right. \right. \\
&\quad \left. \left. 1 \leq k \leq n; x_j \in \text{var}(p_k) \right\} + 1 \right. \\
&\quad \left. + \text{depth}(t_j) \mid 1 \leq j \leq n, \right. \\
&\quad \left. x_j \in \cup \left\{ \text{var}(p_k) \mid 1 \leq k \leq n \right\} \right\} \\
&= \max \left\{ \max \left\{ \text{depth}_j(p_k) \mid 1 \leq j \leq n, \right. \right. \\
&\quad \left. \left. 1 \leq k \leq n; x_j \in \text{var}(p_k) \right\} + 1 \right. \\
&\quad \left. + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(t) \right\} \\
&= \max \left\{ \text{depth}_j(t) + \text{depth}(t_j) \mid 1 \leq j \leq n; x_j \in \text{var}(t) \right\}.
\end{aligned} \tag{17}$$

□

Example 8. Let $\tau = (3)$ be a type; that is, we have only one ternary operation symbol, say f . Let $t, t_1, t_2 \in W_{(3)}^{\text{GF}}(X)$ where $t = f(f(x_2, x_1, x_3), f(x_7, x_4, x_5), f(x_3, x_2, x_1))$, $t_1 = f(x_9, x_5, x_4)$, $t_2 = f(f(x_2, x_3, x_1), f(x_2, x_3, x_1), f(x_2, x_3, x_1))$ and $t_3 = f(x_6, x_4, x_5)$. Then we have $\text{depth}(t) = 2$, $\text{depth}(t_1) = 1$, $\text{depth}(t_2) = 2$, and $\text{depth}(t_3) = 1$. Thus $\text{depth}(S^3(t, t_1, t_2, t_3)) = 4$. Consider $\max\{\text{depth}_1(t) + \text{depth}(t_1), \text{depth}_2(t) + \text{depth}(t_2), \text{depth}_3(t) + \text{depth}(t_3)\} = \max\{2 + 1, 2 + 2, 2 + 1\} = 4$.

4. Generalized Full Hypersubstitutions and Substitutions of $\text{clone}_{\text{GF}}(n)$

In this section, we will generalize the concept of full hypersubstitution. For any generalized full term t , we define the generalized full term t_s arising from t by mapping all variables corresponding to a permutation $s \in P_n$ inductively by the following steps:

- (i) if $t = f(x_{r(1)}, \dots, x_{r(n)})$ where $r \in P_n$, then $t_s = f(x_{s(r(1))}, \dots, x_{s(r(n))})$,
- (ii) if $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then $t_s = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$,
- (iii) if $t = f(t_1, \dots, t_n)$ where $t_1, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$, then $t_s = f((t_1)_s, \dots, (t_n)_s)$.

It is clear that t_s is a generalized full term for any generalized full term t and for any $s \in P_n$.

Definition 9. A mapping $\sigma : \Omega \rightarrow W_{(n)}^{\text{GF}}(X)$ is called a *generalized full hypersubstitution*.

By $\text{Hyp}_{\text{GF}}(n)$ we denote the set of all generalized full hypersubstitutions of type (n) . Every generalized full hypersubstitution σ can be extended to a map $\widehat{\sigma}$ defined on $W_{(n)}^{\text{GF}}(X)$ as follows:

- (i) $\widehat{\sigma}[f(x_{s(1)}, \dots, x_{s(n)})] := \sigma(f)_s$,
- (ii) $\widehat{\sigma}[f(x_{s'(j_1)}, \dots, x_{s'(j_n)})] := S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)})$,
- (iii) $\widehat{\sigma}[f_i(t_1, \dots, t_n)] := S^n(\sigma(f), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n])$.

We define a binary operation $\circ_{\text{GF}}$ on $\text{Hyp}_{\text{GF}}(n)$ by $\sigma_1 \circ_{\text{GF}} \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ denotes the usual composition of functions. Together with the hypersubstitution σ_{id} defined by $\sigma_{\text{id}}(f) := f(x_1, \dots, x_n)$, one has a monoid $(\text{Hyp}_{\text{GF}}(n); \circ_{\text{GF}}, \sigma_{\text{id}})$. Then we have the following.

Lemma 10. Let $t \in W_{(n)}^{\text{GF}}(X)$, $s \in P_n$ and $\sigma \in \text{Hyp}_{\text{GF}}(n)$. Then

$$S^n(t, \widehat{\sigma}[t_{s(1)}], \dots, \widehat{\sigma}[t_{s(n)}]) = S^n(t_s, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]). \tag{18}$$

Proof. We give a proof by induction on the complexity of a generalized full term t .

If $t = f(x_{r(1)}, \dots, x_{r(n)})$ where $r \in P_n$, then

$$\begin{aligned}
&S^n(t, \widehat{\sigma}[t_{s(1)}], \dots, \widehat{\sigma}[t_{s(n)}]) \\
&= S^n(f(x_{r(1)}, \dots, x_{r(n)}), \widehat{\sigma}[t_{s(1)}], \dots, \widehat{\sigma}[t_{s(n)}]) \\
&= f(\widehat{\sigma}[t_{s(r(1))}], \dots, \widehat{\sigma}[t_{s(r(n))}]), \\
&S^n(t_s, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\
&= S^n(f(x_{r(1)}, \dots, x_{r(n)}), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\
&= S^n(f(x_{s(r(1))}, \dots, x_{s(r(n))}), \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\
&= f(\widehat{\sigma}[t_{s(r(1))}], \dots, \widehat{\sigma}[t_{s(r(n))}]).
\end{aligned} \tag{19}$$

If $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then

$$\begin{aligned}
&S^n(t, \widehat{\sigma}[t_{s(1)}], \dots, \widehat{\sigma}[t_{s(n)}]) \\
&= S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), \widehat{\sigma}[t_{s(1)}], \dots, \widehat{\sigma}[t_{s(n)}]) \\
&= f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), \\
&S^n(t_s, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\
&= S^n((f(x_{s'(j_1)}, \dots, x_{s'(j_n)}))_s, \widehat{\sigma}[t_1], \dots, \widehat{\sigma}[t_n]) \\
&= f(x_{s'(j_1)}, \dots, x_{s'(j_n)}).
\end{aligned} \tag{20}$$

If $t = f(t'_1, \dots, t'_n)$ where $t'_1, \dots, t'_n \in W_{(n)}^{\text{GF}}(X)$ and assume that the formula is satisfied for $t'_1, \dots, t'_n$, then

$$\begin{aligned}
& S^n(t, \hat{\sigma}[t_{s(1)}], \dots, \hat{\sigma}[t_{s(n)}]) \\
&= S^n(f(t'_1, \dots, t'_n), \hat{\sigma}[t_{s(1)}], \dots, \hat{\sigma}[t_{s(n)}]) \\
&= f(S^n(t'_1, \hat{\sigma}[t_{s(1)}], \dots, \hat{\sigma}[t_{s(n)}]), \dots, \\
&\quad S^n(t'_n, \hat{\sigma}[t_{s(1)}], \dots, \hat{\sigma}[t_{s(n)}])) \\
&= f(S^n((t'_1)_s, \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]), \dots, \\
&\quad S^n((t'_n)_s, \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])) \\
&= S^n(f((t'_1)_s, \dots, (t'_n)_s), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\
&= S^n((f(t'_1, \dots, t'_n))_s, \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\
&= S^n(t_s, \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]).
\end{aligned} \tag{21}$$

□

Lemma 11. Let $\sigma \in \text{Hyp}_{\text{GF}}(n)$ and p be a generalized full term. Then $\hat{\sigma}[p]$ is also a generalized full term.

Proof. We give a proof by induction on the complexity of a generalized full term p .

If $p = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$, then $\hat{\sigma}[p] = \sigma(f)_s$. Since $\sigma(f)_s$ is a generalized full term, $\hat{\sigma}[p]$ is also a generalized full term.

If $p = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then $\hat{\sigma}[p] = \hat{\sigma}[f(x_{s'(j_1)}, \dots, x_{s'(j_n)})] = S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)})$. Since $S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)})$ is a generalized full term, $\hat{\sigma}[p]$ is also a generalized full term.

If $p = f(p_1, \dots, p_n)$ where p_j are generalized full terms for all $1 \leq j \leq n$ and assume that $\hat{\sigma}[p_j]$ are generalized full terms for all $1 \leq j \leq n$, then $\hat{\sigma}[p] = S^n(\sigma(f), \hat{\sigma}[p_1], \dots, \hat{\sigma}[p_n])$ is also a generalized full term. □

Lemma 12. $\text{Hyp}_{\text{GF}}(n)$ forms a submonoid of the monoid $\text{Hyp}_{\text{G}}(n)$ of all generalized hypersubstitutions.

Proof. Since $\sigma_{\text{id}}(f) = f(x_1, \dots, x_n)$ is a generalized full term, the identity-generalized hypersubstitution is a generalized full hypersubstitution.

Assume that $\sigma_1, \sigma_2 \in \text{Hyp}_{\text{GF}}(n)$. Since $(\sigma_1 \circ_{\text{GF}} \sigma_2)(f) = \hat{\sigma}_1[\sigma_2(f)]$ and $\sigma_1(f), \sigma_2(f)$ are generalized full terms, then by Lemma 11 we have that $(\sigma_1 \circ_{\text{GF}} \sigma_2)(f)$ is a generalized full term. Hence $\sigma_1 \circ_{\text{GF}} \sigma_2 \in \text{Hyp}_{\text{GF}}(n)$ and therefore $\text{Hyp}_{\text{GF}}(n)$ forms a submonoid of $\text{Hyp}_{\text{G}}(n)$. □

Definition 13. Let σ be a generalized full hypersubstitution of type (n) , and then we define

$$\text{depth}(\sigma) = \max \{ \text{depth}(\sigma(f)) \mid f \in \Omega \}. \tag{22}$$

Corollary 14. Let $p = f(p_1, \dots, p_n) \in W_{(n)}^{\text{GF}}(X)$, $\sigma \in \text{Hyp}_{\text{GF}}(n)$. Then

$$\text{depth}(\hat{\sigma}[p]) = \text{depth}(\sigma(f)) \text{ depth}(p). \tag{23}$$

Proof. We give a proof by induction on $\text{depth}(p)$.

If $\text{depth}(p) = 1$, then $p = f(x_{s(1)}, \dots, x_{s(n)})$ where $s \in P_n$ or $p = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$.

If $p = f(x_{s(1)}, \dots, x_{s(n)})$, then

$$\begin{aligned}
\text{depth}(\hat{\sigma}[p]) &= \text{depth}(\hat{\sigma}[f(x_{s(1)}, \dots, x_{s(n)})]) \\
&= \text{depth}(\sigma(f)_s) \\
&= \text{depth}(\sigma(f)) \\
&= \text{depth}(\sigma(f)) \text{ depth}(p).
\end{aligned} \tag{24}$$

If $p = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$, then

$$\begin{aligned}
\text{depth}(\hat{\sigma}[p]) &= \text{depth}(\hat{\sigma}[f(x_{s'(j_1)}, \dots, x_{s'(j_n)})]) \\
&= \text{depth}(S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)})) \\
&= \max \{ \text{depth}(x_{s'(j_1)}), \dots, \text{depth}(x_{s'(j_n)}) \} \\
&\quad + \text{depth}(\sigma(f)) \\
&= 0 + \text{depth}(\sigma(f)) \\
&= \text{depth}(\sigma(f)) \\
&= \text{depth}(\sigma(f)) \text{ depth}(p).
\end{aligned} \tag{25}$$

Assume that the $\text{depth}(\hat{\sigma}[p_k]) = \text{depth}(\sigma(f)) \text{ depth}(p_k)$ for all $1 \leq k \leq n$ and $p = f(p_1, \dots, p_n)$. Then

$$\begin{aligned}
\text{depth}(\hat{\sigma}[p]) &= \text{depth}(S^n(\sigma(f), \hat{\sigma}[p_1], \dots, \hat{\sigma}[p_n])) \\
&= \text{depth}(\sigma(f)) + \max \{ \text{depth}(\hat{\sigma}[p_k]) \mid 1 \leq k \leq n \} \\
&= \text{depth}(\sigma(f)) \\
&\quad + \max \{ \text{depth}(\sigma(f)) \text{ depth}(p_k) \mid 1 \leq k \leq n \} \\
&= \text{depth}(\sigma(f)) \\
&\quad + \text{depth}(\sigma(f)) \max \{ \text{depth}(p_k) \mid 1 \leq k \leq n \} \\
&= \text{depth}(\sigma(f)) (1 + \max \{ \text{depth}(p_k) \mid 1 \leq k \leq n \}) \\
&= \text{depth}(\sigma(f)) \text{ depth}(p).
\end{aligned} \tag{26}$$

□

Corollary 15. The function $\text{depth}^{\text{GF}} : \text{Hyp}_{\text{GF}}(n) \rightarrow \mathbb{N}^+$ with $\sigma \mapsto \text{depth}(\sigma)$ is a homomorphism from the monoid $(\text{Hyp}_{\text{GF}}(n); \circ_{\text{GF}}, \sigma_{\text{id}})$ onto the monoid $(\mathbb{N}^+; \cdot, 1)$.

Proof. The mapping depth^{GF} is well defined and surjective since for every $n \in \mathbb{N}^+$ there exists a generalized full term

p with $\text{depth}(p) = n$. Furthermore, we have $\text{depth}^{\text{GF}}(\sigma_{\text{id}}) = \text{depth}(\sigma_{\text{id}}(f)) = \text{depth}(f(x_1, \dots, x_n)) = 1$ and

$$\begin{aligned} \text{depth}((\sigma_1 \circ_{\text{GF}} \sigma_2)(f)) &= \text{depth}(\hat{\sigma}_1[\sigma_2(f)]) \\ &= \text{depth}(\sigma_1(f)) \text{depth}(\sigma_2(f)) \\ &= \text{depth}^{\text{GF}}(\sigma_1) \text{depth}^{\text{GF}}(\sigma_2). \end{aligned} \quad (27)$$

Proposition 16. *Let $\sigma \in \text{Hyp}_{\text{GF}}(n)$. Then $\hat{\sigma}$ is an endomorphism on the algebra $(W_{(n)}^{\text{GF}}(X); S^n)$.*

Proof. Indeed $\hat{\sigma} : (W_{(n)}^{\text{GF}}(X); S^n) \rightarrow (W_{(n)}^{\text{GF}}(X); S^n)$ is a function. Now we give a proof by induction on the complexity of a term t_0 that, for any $t_1, \dots, t_n \in W_{(n)}^{\text{GF}}(X)$,

$$\hat{\sigma}[S^n(t_0, t_1, \dots, t_n)] = S^n(\hat{\sigma}[t_0], \dots, \hat{\sigma}[t_n]). \quad (28)$$

First consider the case that $t_0 = f(x_{s(1)}, \dots, x_{s(n)})$ where $i \in I$ and $s \in P_n$. Then

$$\begin{aligned} \hat{\sigma}[S^n(t_0, t_1, \dots, t_n)] &= \hat{\sigma}[S^n(f(x_{s(1)}, \dots, x_{s(n)}), t_1, \dots, t_n)] \\ &= \hat{\sigma}[f(t_{s(1)}, \dots, t_{s(n)})] \\ &= S^n(\sigma(f), \hat{\sigma}[t_{s(1)}], \dots, \hat{\sigma}[t_{s(n)}]) \\ &= S^n(\sigma(f)_s, \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\ &= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]). \end{aligned} \quad (29)$$

If $t_0 = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then

$$\begin{aligned} \hat{\sigma}[S^n(t_0, t_1, \dots, t_n)] &= \hat{\sigma}[S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), t_1, \dots, t_n)] \\ &= \hat{\sigma}[f(x_{s'(j_1)}, \dots, x_{s'(j_n)})] \\ &= S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)}) \\ &= S^n(\sigma(f), \hat{\sigma}[S^n(x_{s'(j_1)}, t_1, \dots, t_n)], \dots, \\ &\quad \hat{\sigma}[S^n(x_{s'(j_n)}, t_1, \dots, t_n)]) \\ &= S^n(S^n(\sigma(f), x_{s'(j_1)}, \dots, x_{s'(j_n)}), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\ &= S^n(\hat{\sigma}[f(x_{s'(j_1)}, \dots, x_{s'(j_n)})], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\ &= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]). \end{aligned} \quad (30)$$

If $t_0 = f(r_1, \dots, r_n)$ where $r_1, \dots, r_n \in W_{(n)}^{\text{GF}}(X)$ and assume that the formula is satisfied for $r_1, \dots, r_n$, then

$$\begin{aligned} \hat{\sigma}[S^n(t_0, t_1, \dots, t_n)] &= \hat{\sigma}[S^n(f(r_1, \dots, r_n), t_1, \dots, t_n)] \\ &= \hat{\sigma}[f(S^n(r_1, t_1, \dots, t_n), \dots, S^n(r_n, t_1, \dots, t_n))] \\ &= S^n(\sigma(f), \hat{\sigma}[S^n(r_1, t_1, \dots, t_n)], \dots, \\ &\quad \hat{\sigma}[S^n(r_n, t_1, \dots, t_n)]) \\ &= S^n(\sigma(f), S^n(\hat{\sigma}[r_1], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]), \dots, \\ &\quad S^n(\hat{\sigma}[r_n], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])) \\ &= S^n(S^n(\sigma(f), \hat{\sigma}[r_1], \dots, \hat{\sigma}[r_n]), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) \\ &= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]). \end{aligned} \quad (31)$$

Since the free algebra $\text{clone}_{\text{GF}}(n)$ is generated by the set

$$\begin{aligned} \{f(x_{s(1)}, \dots, x_{s(n)}) \mid s \in P_n\} \\ \cup \{f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \mid j_1, \dots, j_n > n, s' \\ \text{is a permutation on } \{j_1, \dots, j_n\}\}, \end{aligned} \quad (32)$$

therefore any mapping η from $\mathcal{F}_{\text{GS}(n)}$ into $W_{(n)}^{\text{GF}}(X)$ can be uniquely extended to an endomorphism $\bar{\eta}$ of $\text{clone}_{\text{GF}}(n)$. These mappings are called *generalized full clone substitutions*. We denote the set of all generalized full clone substitutions by $\text{Subst}_{\text{GFC}}$. The set $\text{Subst}_{\text{GFC}}$ with a binary operation $\odot$ is defined by $\eta_1 \odot \eta_2 := \bar{\eta}_1 \circ \eta_2$, where $\circ$ is the usual composition of functions, and, together with $\text{id}_{\mathcal{F}_{\text{GS}(n)}}$, the identity mapping on $\mathcal{F}_{\text{GS}(n)}$ is a monoid. Then we have the following proposition.

Proposition 17. *The monoid $(\text{Hyp}_{\text{GF}}(n); \circ_{\text{GF}}, \sigma_{\text{id}})$ can be embedded into the monoid $(\text{Subst}_{\text{GFC}}; \odot, \text{id}_{\mathcal{F}_{\text{GS}(n)}})$.*

Proof. Let $\sigma \in \text{Hyp}_{\text{GF}}(n)$. Then by Proposition 16, $\hat{\sigma}$ is an endomorphism on the algebra $\text{clone}_{\text{GF}}(n)$. Since $\mathcal{F}_{\text{GS}(n)} = \{f(x_{s(1)}, \dots, x_{s(n)}) \mid s \in P_n\} \cup \{f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \mid j_1, \dots, j_n > n, s' \text{ is a permutation on } \{j_1, \dots, j_n\}\}$ is a generating set of $\text{clone}_{\text{GF}}(n)$, the mapping $\hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}$ is a substitution with $\overline{\hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}} = \hat{\sigma}$. We define the mapping $\psi : \text{Hyp}_{\text{GF}}(n) \rightarrow \text{Subst}_{\text{GFC}}$ by $\psi(\sigma) = \hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}$. Injectivity of ψ is clear. We show that ψ is a homomorphism. Let $\sigma_1, \sigma_2 \in \text{Hyp}_{\text{GF}}(n)$. Then $\psi(\sigma_1 \circ_{\text{GF}} \sigma_2) = (\hat{\sigma}_1 \circ \hat{\sigma}_2)|_{\mathcal{F}_{\text{GS}(n)}} = (\hat{\sigma}_1 \circ \hat{\sigma}_2)|_{\mathcal{F}_{\text{GS}(n)}} = \hat{\sigma}_1 \circ \hat{\sigma}_2|_{\mathcal{F}_{\text{GS}(n)}} = \overline{\hat{\sigma}_1|_{\mathcal{F}_{\text{GS}(n)}}} \circ \overline{\hat{\sigma}_2|_{\mathcal{F}_{\text{GS}(n)}}} = \psi(\sigma_1) \circ \psi(\sigma_2) = \psi(\sigma_1) \odot \psi(\sigma_2)$. Clearly, the mapping ψ preserves the identity element. $\square$

5. Full Strong Hyperidentities and Identities in $\text{clone}_{\text{GF}}(n)$

Let V be the variety of type (n) and let $\text{Id}^{\text{GF}}V := W_{(n)}^{\text{GF}}(X)^2 \cap \text{Id}V$ be the set of all identities of V consisting of generalized full terms. Then we have the following.

Proposition 18. $\text{Id}^{\text{GF}}V$ is a congruence on $\text{clone}_{\text{GF}}(n)$.

Proof. We will prove that if $r \approx t$, $r_k \approx t_k \in \text{Id}^{\text{GF}}V$, $k = 1, 2, \dots, n$, then $S^n(r, r_1, \dots, r_n) \approx S^n(t, t_1, \dots, t_n) \in \text{Id}^{\text{GF}}V$. Firstly, we give a proof by induction on the complexity of a term $t \in W_{(n)}^{\text{GF}}(X)$ that, for every $n \in \mathbb{N}^+$ from $t_k \approx r_k \in \text{Id}^{\text{GF}}V$, $k = 1, 2, \dots, n$, there follows $S^n(t, t_1, \dots, t_n) \approx S^n(r, r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V$.

If $t = f(x_{s(1)}, \dots, x_{s(n)})$ and $s \in P_n$, then

$$\begin{aligned} S^n(f(x_{s(1)}, \dots, x_{s(n)}), t_1, \dots, t_n) \\ &= f(t_{s(1)}, \dots, t_{s(n)}) \\ &\approx f(r_{s(1)}, \dots, r_{s(n)}) \\ &= S^n(f(x_{s(1)}, \dots, x_{s(n)}), r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V, \end{aligned} \quad (33)$$

where $\text{Id}V$ is compatible with the operation $\bar{f}$ of the absolutely free algebra $\mathcal{F}_{(n)}(X)$ and by the definition of generalized full terms.

If $t = f(x_{s'(j_1)}, \dots, x_{s'(j_n)})$ where $j_1, \dots, j_n$ are natural numbers greater than n and s' is a permutation on $\{j_1, \dots, j_n\}$, then

$$\begin{aligned} S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), t_1, \dots, t_n) \\ &= f(x_{s'(j_1)}, \dots, x_{s'(j_n)}) \\ &= S^n(f(x_{s'(j_1)}, \dots, x_{s'(j_n)}), r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V. \end{aligned} \quad (34)$$

Assume now that $t = f(l_1, \dots, l_n) \in W_{(n)}^{\text{GF}}(X)$ and, for l_j , $1 \leq j \leq n$, we have already

$$S^n(l_j, t_1, \dots, t_n) \approx S^n(r_j, r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V. \quad (35)$$

Then

$$\begin{aligned} S^n(f(l_1, \dots, l_n), t_1, \dots, t_n) \\ &= f(S^n(l_1, t_1, \dots, t_n), \dots, S^n(l_n, t_1, \dots, t_n)) \\ &\approx f(S^n(l_1, r_1, \dots, r_n), \dots, S^n(l_n, r_1, \dots, r_n)) \\ &= S^n(f(l_1, \dots, l_n), r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V. \end{aligned} \quad (36)$$

Now we prove the implication

$$\begin{aligned} t \approx r \in \text{Id}^{\text{GF}}V &\implies S^n(t, r_1, \dots, r_n) \\ &\approx S^n(r, r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V. \end{aligned} \quad (37)$$

This is a consequence of the fully invariance of $\text{Id}V$ as a congruence on the absolutely free algebra $\mathcal{F}_{(n)}(X)$ and the

definition of a generalized full term. Assume now that $t \approx r$, $t_k \approx r_k \in \text{Id}^{\text{GF}}V$. Then

$$\begin{aligned} S^n(t, t_1, \dots, t_n) &\approx S^n(r, t_1, \dots, t_n) \\ &\approx S^n(r, r_1, \dots, r_n) \in \text{Id}^{\text{GF}}V. \end{aligned} \quad (38)$$

□

Definition 19. Let V be a variety of type (n) and let $\text{Id}^{\text{GF}}V$ be the set of all identities of V consisting of generalized full terms. Then $s \approx t \in \text{Id}^{\text{GF}}V$ is called an Ω -GF-hyperidentity in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id}^{\text{GF}}V$ for every $\sigma \in \text{Hyp}_{\text{GF}}(n)$. If every identity in $\text{Id}^{\text{GF}}V$ is satisfied as an Ω -GF-hyperidentity, the variety V is called Ω -GF-solid.

Proposition 20. If $\text{Id}^{\text{GF}}V$ is a fully invariant congruence relation on $\text{clone}_{\text{GF}}(n)$, then the variety V is Ω -GF-solid.

Proof. Let $s \approx t \in \text{Id}^{\text{GF}}V$ and let $\sigma \in \text{Hyp}_{\text{GF}}(n)$ be a generalized full hypersubstitution. Since by Proposition 16 the extension $\hat{\sigma}$ of σ is an endomorphism of $\text{clone}_{\text{GF}}(n)$, we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \text{Id}^{\text{GF}}V$. □

By Proposition 18 we can form a quotient algebra:

$$\text{clone}_{\text{GF}}V := \text{clone}_{\text{GF}}(n) / \text{Id}^{\text{GF}}V \quad (39)$$

which belongs to a variety of a Menger algebra of rank n . There is the following connection between clone identities and Ω -GF-hyperidentities in V .

Proposition 21. Let V be the variety of type (n) and let $s \approx t \in \text{Id}^{\text{GF}}V$. If $s \approx t$ is an identity in $\text{clone}_{\text{GF}}(n)$, then it is an Ω -GF-hyperidentity in V .

Proof. Let $s \approx t \in \text{Id}^{\text{GF}}V$ be an identity in $\text{clone}_{\text{GF}}(n)$ and let $\sigma \in \text{Hyp}_{\text{GF}}(n)$. Then $\hat{\sigma} \in \text{End}(\text{clone}_{\text{GF}}\tau)$ the set of all endomorphisms of $\text{clone}_{\text{GF}}(n)$ and $\hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}} \in \text{Subst}_{\text{GFC}}$ with $\overline{\hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}} = \hat{\sigma}$. By the natural mapping $\text{natId}^{\text{GF}}V$ we have

$$\text{natId}^{\text{GF}}V \circ \hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}} : \mathcal{F}_{\text{GS}(n)} \longrightarrow \text{clone}_{\text{GF}}(n), \quad (40)$$

and this is a valuation mapping with

$$\overline{\text{natId}^{\text{GF}}V \circ \hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}} = \text{natId}^{\text{GF}}V \circ \hat{\sigma}. \quad (41)$$

Then

$$\begin{aligned} s \approx t \in \text{Id}(\text{clone}_{\text{GF}}V) \\ \implies \overline{\text{natId}^{\text{GF}}V \circ \hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}}(s) &= \overline{\text{natId}^{\text{GF}}V \circ \hat{\sigma}|_{\mathcal{F}_{\text{GS}(n)}}}(t) \\ \implies \text{natId}^{\text{GF}}V \circ \hat{\sigma}(s) &= \text{natId}^{\text{GF}}V \circ \hat{\sigma}(t) \\ \implies [\hat{\sigma}[s]]_{\text{Id}^{\text{GF}}V} &= [\hat{\sigma}[t]]_{\text{Id}^{\text{GF}}V} \\ \implies \hat{\sigma}[s] \approx \hat{\sigma}[t] &\in \text{Id}^{\text{GF}}V. \end{aligned} \quad (42)$$

This means that $s \approx t$ is satisfied as an Ω -GF-hyperidentity in V . □

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