Research Article

Multidecompositions of the Balanced Complete Bipartite Graph into Paths and Stars

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Let P_k and S_k denote a path and a star with k edges, respectively. For graphs F , G , and H , a (G, H) -multidecomposition of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H . In this paper, necessary and sufficient conditions for the existence of the (P_k, S_k) -multidecomposition of the balanced complete bipartite graph are given.

1. Introduction

Let F , G , and H be graphs. A G -decomposition of F is a partition of the edge set of F into copies of G . If F has a G -decomposition, we say that F is G -decomposable and write $G \mid F$. A (G, H) -multidecomposition of F is a partition of the edge set of F into copies of G and copies of H with at least one copy of G and at least one copy of H . If F has a (G, H) -multidecomposition, we say that F is (G, H) -multidecomposable and write $(G, H) \mid F$.

For positive integers m and n , $K_{m,n}$ denotes the complete bipartite graph with parts of sizes m and n . A complete bipartite graph is balanced if $m = n$. A k -path, denoted by P_k , is a path with k edges. A k -star, denoted by S_k , is the complete bipartite graph $K_{1,k}$. A k -cycle, denoted by C_k , is a cycle of length k .

P_k -decompositions of graphs have been a popular topic of research in graph theory. Articles of interest include [1–11]. The reader can refer to [12] for an excellent survey of this topic. Decompositions of graphs into k -stars have also attracted a fair share of interest. Articles of interest include [13–18]. The study of the (G, H) -multidecomposition was introduced by Abueida and Daven in [19]. Abueida and Daven [20] investigated the problem of the (K_k, S_k) -multidecomposition of the complete graph K_n . Abueida and Daven [21] investigated the problem of the (C_4, E_2) -multidecomposition of several graph products

where E_2 denotes two vertex disjoint edges. Abueida and O'Neil [22] settled the existence problem of the (C_k, S_{k-1}) -multidecomposition of the complete multigraph λK_n for $k = 3, 4$, and 5 . In [23], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of the (G_n, H_n) -multidecomposition of λK_n where $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$. Furthermore, Shyu [24] investigated the problem of decomposing K_n into k -paths and k -stars, and gave a necessary and sufficient condition for $k = 3$. In [25], Shyu considered the existence of a decomposition of K_n into k -paths and k -cycles and established a necessary and sufficient condition for $k = 4$. He also gave criteria for the existence of a decomposition of K_n into 3-paths and 3 cycles in [26]. Shyu [27] investigated the problem of decomposing K_n into k -cycles and k -stars and settled the case $k = 4$. Recently, Lee [28] established necessary and sufficient conditions for the existence of the (C_k, S_k) -multidecomposition of a complete bipartite graph.

In this paper, we investigate the problem of the (P_k, S_k) -multidecomposition of the balanced complete bipartite graph and give necessary and sufficient conditions for such a multidecomposition to exist.

2. Preliminaries

For our discussions, some terminologies and notations are needed. Let G be a graph. The degree of a vertex x of G ,

denoted by $\deg_G x$, is the number of edges incident with x . A graph is r -regular if each vertex is of degree r . The vertex of degree k in S_k is called the center of S_k . Let V and E be subsets of the vertex set and the edge set of G , respectively. We use $G[V]$ to denote the subgraph of G induced by V and $G - E$ to denote the subgraph obtained from G by deleting E . Suppose that $G_1, G_2, \dots, G_t$ are edge-disjoint-graphs. Then, $G_1 + G_2 + \dots + G_t$, or $\sum_{i=1}^t G_i$, denotes the graph H with vertex set $V(H) = \bigcup_{i=1}^t V(G_i)$, and edge set $E(H) = \bigcup_{i=1}^t E(G_i)$. Thus, if a graph H can be decomposed into subgraphs $G_1, G_2, \dots, G_t$, we write $H = G_1 + G_2 + \dots + G_t$, or $H = \sum_{i=1}^t G_i$. Moreover, $\lceil x \rceil$ denotes the smallest integer not less than x and $\lfloor x \rfloor$ denotes the largest integer not greater than x . Let $v_0 v_1 v_2, \dots, v_{k-1} v_k$ denote the k -path with edges $v_0 v_1, v_1 v_2, \dots, v_{k-1} v_k$, and $(v_1, v_2, \dots, v_k)$ denote the k -cycle with edges $v_1 v_2, v_2 v_3, \dots, v_{k-1} v_k, v_k v_1$. Throughout the paper, (A, B) denotes the bipartition of $K_{n,n}$, where $A = \{a_0, a_1, \dots, a_{n-1}\}$ and $B = \{b_0, b_1, \dots, b_{n-1}\}$.

For the edge $a_i b_j$ in $K_{n,n}$, the label of $a_i b_j$ is $j - i \pmod n$. For example, in $K_{9,9}$, the labels of $a_3 b_7$ and $a_8 b_2$ are 4 and 3, respectively. Note that each vertex of $K_{n,n}$ is incident with exactly one edge with label i for $i = 0, 1, 2, \dots, n-1$. Let G be a subgraph of $K_{n,n}$ and t a nonnegative integer. We use G_{+t} to denote the graph with vertex set $\{a_i : a_i \in V(G)\} \cup \{b_{j+t} : b_j \in V(G)\}$ and edge set $\{a_i b_{j+t} : a_i b_j \in E(G)\}$, where the subscripts of b are taken modulo n . In particular, $G_{+0} = G$.

The following results due to Yamamoto et al. and Parker, respectively, are essential for our discussions.

Proposition 1 (see [18]). *Let $m \geq n \geq 1$ be integers. Then, $K_{m,n}$ is S_k -decomposable if and only if $m \geq k$ and*

$$\begin{aligned} m &\equiv 0 \pmod k & \text{if } n < k, \\ mn &\equiv 0 \pmod k & \text{if } n \geq k. \end{aligned} \quad (1)$$

Proposition 2 (see [7]). *There exists a P_k -decomposition of $K_{m,n}$ if and only if $mn \equiv 0 \pmod k$, and one of aforementioned (see Table 1) cases occurs.*

3. Main Results

We first give necessary conditions of the (P_k, S_k) -multidecomposition of $K_{n,n}$.

Lemma 3. *Let k and n be positive integers. If there exists a (P_k, S_k) -multidecomposition of $K_{n,n}$, then $k \leq n$ and $n^2 \equiv 0 \pmod k$.*

Proof. The result follows from the fact that the maximum size of a star in $K_{n,n}$ is n , the size of each member in the multidecomposition is k , and $|E(K_{n,n})| = n^2$. $\square$

We now show that the necessary conditions are also sufficient. Since $P_k = S_k$ for $k = 1, 2$, the result holds for $k = 1, 2$ by Proposition 1. So it remains to consider the case $k \geq 3$. The proof is divided into cases $n \geq 2k$, $n = k$ and $k < n < 2k$, which are treated in Lemmas 4, 5, and 6, respectively.

TABLE 1

Case	k	m	n	Conditions
1	Even	Even	Even	$k \leq 2m, k \leq 2n$, both are not equalities
2	Even	Even	Odd	$k \leq 2m - 2, k \leq 2n$
3	Even	Odd	Even	$k \leq 2m, k \leq 2n - 2$
4	Odd	Even	Even	$k \leq 2m - 1, k \leq 2n - 1$
5	Odd	Even	Odd	$k \leq 2m - 1, k \leq n$
6	Odd	Odd	Even	$k \leq m, k \leq 2n - 1$
7	Odd	Odd	Odd	$k \leq m, k \leq n$

Lemma 4. *Let k and n be positive integers with $n \geq 2k \geq 6$. If $n^2 \equiv 0 \pmod k$, then $K_{n,n}$ is (P_k, S_k) -multidecomposable.*

Proof. Let $n = qk + r$ where q and r are integers with $0 \leq r < k$. Then, $q \geq 2$ from the assumption $n \geq 2k$. Note that

$$\begin{aligned} K_{n,n} &= K_{qk+r,n} \\ &= K_{(q-1)k,n} + K_{k+r,n}. \end{aligned} \quad (2)$$

By Proposition 1, $K_{(q-1)k,n}$ is S_k -decomposable. On the other hand, trivially, $k \leq \min\{k+r, n\}$, and $|E(K_{k+r,n})| = (k+r)n \equiv 0 \pmod k$ from the assumption $n^2 \equiv 0 \pmod k$. This implies that $K_{k+r,n}$ is P_k -decomposable by Proposition 2. Hence, $K_{n,n}$ is (P_k, S_k) -multidecomposable. $\square$

Lemma 5. *Let k be a positive integer with $k \geq 3$. Then, $K_{k,k}$ is (P_k, S_k) -multidecomposable.*

Proof. Note that $K_{k,k} = K_{1,k} + K_{k-1,k}$. Trivially, $K_{1,k} = S_k$. On the other hand, $|E(K_{k-1,k})| = (k-1)k \equiv 0 \pmod k$. Furthermore, $k \leq 2(k-1) - 1$ for odd k , and $k \leq 2(k-1) = 2k-2$ for even k . Hence, $K_{k-1,k}$ is P_k -decomposable by Proposition 2, and $K_{k,k}$ is (P_k, S_k) -multidecomposable. $\square$

Lemma 6. *Let k and n be integers with $3 \leq k < n < 2k$. If $n^2 \equiv 0 \pmod k$, then $K_{n,n}$ is (P_k, S_k) -multidecomposable.*

Proof. Suppose that $n = k + r$. Then, $0 < r < k$ from the assumption $k < n < 2k$. Let $A_1 = \{a_0, a_1, \dots, a_{k-1}\}$, $B_1 = \{b_0, b_1, \dots, b_{k-1}\}$, $A_2 = \{a_k, a_{k+1}, \dots, a_{k+r-1}\}$, and $B_2 = \{b_k, b_{k+1}, \dots, b_{k+r-1}\}$. Let $G_i = K_{n,n}[A_i \cup B_1]$ for $i = 1, 2$ and $H = K_{n,n}[A \cup B_2]$. Then, $K_{n,n} = G_1 + G_2 + H$. Note that G_1 is isomorphic to $K_{k,k}$, H is isomorphic to $K_{k+r,r}$, and G_2 is isomorphic to $K_{r,k}$, which is S_k -decomposable by Proposition 1. Hence, it is sufficient to show that $G_1 + H$ is (P_k, S_k) -multidecomposable.

Let $t = r^2/k$. Since $k \mid n^2$, we have $k \mid r^2$, which implies that t is a positive integer. Let $C = (b_0, a_0, b_1, a_1, \dots, b_{k-2}, a_{k-2}, b_{k-1}, a_{k-1})$. Then, C is a $2k$ -cycle in G_1 . Let $p = \lfloor t/2 \rfloor$. For odd t , define a k -path P in G_1 as follows:

$$P = \begin{cases} b_{2p} a_0 b_{2p+1} a_1 \cdots b_{2p-1+k/2} a_{k/2-1} b_{2p+k/2} & \text{if } k \text{ is even,} \\ b_{2p} a_0 b_{2p+1} a_1 \cdots b_{2p+(k-1)/2} a_{(k-1)/2} & \text{if } k \text{ is odd,} \end{cases} \quad (3)$$

where the subscripts of b are taken modulo k . Since $t = r^2/k$ and $r < k$, we have

$$t \leq r - 1 \leq k - 2. \quad (4)$$

Thus, $2p + 1 = t \leq k - 2$, which implies the labels of the edges in P are $2p$ and $2p + 1$. Note that for $i = 0, 1, \dots, p - 1$, C_{+2i} is a $2k$ -cycle which consists of all of the edges with labels $2i$ and $2i + 1$ in G_1 . Thus, $C, C_{+2}, \dots, C_{+2(p-1)}$ and P are edge-disjoint in G_1 .

Define a subgraph W of G_1 as follows:

$$W = \begin{cases} \sum_{i=0}^{p-1} C_{+2i} & \text{for even } t, \\ \sum_{i=0}^{p-1} C_{+2i} + P & \text{for odd } t \geq 3, \\ P & \text{for } t = 1. \end{cases} \quad (5)$$

Since C_{2k} can be decomposed into 2 copies of P_k and $2p = t$ for even t as well as $2p + 1 = t$ for odd t , W can be decomposed into t copies of P_k . Let $\delta = 1$ for even k and $\delta = 0$ for odd k . Note that for even t , $\deg_{G_1-E(W)} a_i = k - 2p = k - t$, and for odd t ,

$$\begin{aligned} & \deg_{G_1-E(W)} a_i \\ &= \begin{cases} k - t - 1 & \text{if } i = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor - 2, \\ k - t - \delta & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor - 1, \\ k - t + 1 & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, k - 1. \end{cases} \end{aligned} \quad (6)$$

Let $X_i = (G_1 - E(W))[\{a_i\} \cup B_1]$ for $i = 0, 1, \dots, k - 1$. Then for even t , $X_i = S_{k-t}$, and for odd t ,

$$X_i = \begin{cases} S_{k-t-1} & \text{if } i = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor - 2, \\ S_{k-t-\delta} & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor - 1, \\ S_{k-t+1} & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, k - 1 \end{cases} \quad (7)$$

with the center at a_i . In the following, we will show that H can be decomposed into r copies of S_k with centers in B_2 , and into k copies of S_t with centers in A_1 for even t , and into $\lfloor k/2 \rfloor - 1$ copies of S_{t+1} with centers in $\{a_0, a_1, \dots, a_{\lfloor k/2 \rfloor - 2}\}$, an $S_{t+\delta}$ with the center at $a_{\lfloor k/2 \rfloor - 1}$, and $k - \lfloor k/2 \rfloor$ copies of S_{t-1} with centers in $\{a_{\lfloor k/2 \rfloor}, a_{\lfloor k/2 \rfloor + 1}, \dots, a_{k-1}\}$ for odd t .

We show the required star decomposition of H by orienting the edges of H . For any vertex x of H , the outdegree $\deg_H^+ x$ (indegree $\deg_H^- x$, resp.) of x in an orientation of H is the number of arcs incident from (to, resp.) x . It is sufficient to show that there exists an orientation of H such that

$$\deg_H^+ b_j = k, \quad (8)$$

where $j = k, k + 1, \dots, k + r - 1$, and for even t

$$\deg_H^+ a_i = t, \quad (9)$$

where $i = 0, 1, \dots, k - 1$, and for odd t

$$\deg_H^+ a_i = \begin{cases} t + 1 & \text{if } i = 0, 1, \dots, \left\lfloor \frac{k}{2} \right\rfloor - 2, \\ t + \delta & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor - 1, \\ t - 1 & \text{if } i = \left\lfloor \frac{k}{2} \right\rfloor, \left\lfloor \frac{k}{2} \right\rfloor + 1, \dots, k - 1. \end{cases} \quad (10)$$

We first consider the edges oriented outward from A_1 according to the parity of t . Let $\beta = k + (t + 1)(\lfloor k/2 \rfloor - 1)$ and $\gamma = \beta + t + \delta$. If t is even, then the edges $a_i b_{k+ti}, a_i b_{k+ti+1}, \dots, a_i b_{k+ti+t-1}$ are all oriented outward from a_i , where $i = 0, 1, \dots, k - 1$. If t is odd, then the edges $a_i b_{k+(t+1)i}, a_i b_{k+(t+1)i+1}, \dots, a_i b_{k+(t+1)i+t}$ for $i = 0, 1, \dots, \lfloor k/2 \rfloor - 2$, and $a_{\lfloor k/2 \rfloor - 1} b_\beta, a_{\lfloor k/2 \rfloor - 1} b_{\beta+1}, \dots, a_{\lfloor k/2 \rfloor - 1} b_{\beta+t+\delta-1}$, as well as $a_i b_{(t-1)(i-\lfloor k/2 \rfloor)+\gamma}, a_i b_{(t-1)(i-\lfloor k/2 \rfloor)+\gamma+1}, \dots, a_i b_{(t-1)(i-\lfloor k/2 \rfloor)+\gamma+t-2}$ for $i = \lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 1, \dots, k - 1$ are all oriented outward from a_i . In both cases, the subscripts of b are taken modulo r in the set of numbers $\{k, k + 1, \dots, k + r - 1\}$. Note that for even t we orient t edges from each a_i and for odd t we orient at most $t + 1$ edges from a_i . By inequality (4), we have $t + 1 \leq r$, which assures us that there are enough edges for the above orientation.

Finally, the edges which are not oriented yet are all oriented from B_2 to A . From the construction of the orientation, it is easy to see that (9) and (10) are satisfied, and for all $b_w, b_{w'} \in B_2$, we have

$$|\deg_H^- b_w - \deg_H^- b_{w'}| \leq 1. \quad (11)$$

So, we only need to check (8).

Since $\deg_H^+ b_w + \deg_H^- b_w = k + r$ for $b_w \in B_2$, it follows from (11) that

$$|\deg_H^+ b_w - \deg_H^+ b_{w'}| \leq 1 \quad (12)$$

for $b_w, b_{w'} \in B_2$. Note that for even t , $\sum_{i=0}^{k-1} \deg_H^+ a_i = tk$, and for odd t ,

$$\begin{aligned} \sum_{i=0}^{k-1} \deg_H^+ a_i &= (t + 1) \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) + t + \delta + (t - 1) \left(k - \left\lfloor \frac{k}{2} \right\rfloor \right) \\ &= (t - 1)k + 2 \left(\left\lfloor \frac{k}{2} \right\rfloor \right) - 1 + \delta \\ &= \begin{cases} (t - 1)k + 2 \left(\frac{k}{2} \right) - 1 + 1 & \text{if } k \text{ is even,} \\ (t - 1)k + \frac{2(k+1)}{2} - 1 & \text{if } k \text{ is odd.} \end{cases} \\ &= tk. \end{aligned} \quad (13)$$

Thus,

$$\begin{aligned}
 \sum_{w=k}^{k+r-1} \deg_H^+ b_w &= |E(K_{k+r,r})| - \sum_{i=0}^{k-1} \deg_H^+ a_i \\
 &= (k+r)r - tk \\
 &= kr + r^2 - r^2 \\
 &= kr.
 \end{aligned} \tag{14}$$

Therefore from (12), we have $\deg_H^+ b_w = k$ for $b_w \in B_2$. This establishes (8). Hence, there exists the required decomposition $\mathcal{D}$ of H . Let X'_i be the star with center at a_i in $\mathcal{D}$ for $i = 0, 1, \dots, k-1$. Then, $X_i + X'_i$ is a k -star. This completes the proof. $\square$

Now, we are ready for the main result. It is obtained from the arguments above, Lemma 4 and Lemmas 3, 4, 5, and 6.

Theorem 7. *Let k and n be positive integers. Then, $K_{n,n}$ has a (P_k, S_k) -multidecomposition if and only if $k \leq n$ and $n^2 \equiv 0 \pmod{k}$.*

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