Research Article

Isomorphic Digraphs from Affine Maps of Finite Cyclic Groups

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Let n be a positive integer. For any pair of integers k and a , let $G(n, k, a)$ be the digraph whose set of vertices is $\{0, 1, 2, \dots, n-1\}$, and there exists a directed edge from vertex x to vertex y if $kx + a \equiv y \pmod{n}$. In this paper, we obtain a necessary and sufficient condition for which $G(n, k_1, a_1) \cong G(n, k_2, a_2)$.

1. Introduction

Let X be a nonempty set and let f be a mapping from X into itself. Then, we call $G(X, f)$ the iteration digraph from f of X , where the set of vertices of $G(X, f)$ is X and there is a directed edge from a vertex $a \in G(X, f)$ to a vertex $b \in G(X, f)$ if $f(a) = b$. In particular, there are many interesting structures on the digraphs when there is an algebraic structure on X .

Many people study the iteration graphs from endomorphisms of finite abelian groups. The authors of [1] studied the digraph from the endomorphism $f(x) = x^2$ of $(\mathbb{Z}/p\mathbb{Z})^*$, where p is a prime. These results were generalized to the digraph from any endomorphism of $(\mathbb{Z}/p\mathbb{Z})^*$ in [2]. Sha and Hu [3] studied some elementary properties of the digraph from any endomorphism of $\mathbb{F}_q^*$. They also obtained many elementary properties of the digraph from any endomorphism of finite cyclic groups in [3].

Let n be a positive integer and let k, a be two integers. In this paper, we denote by $G(n, k, a)$ the digraph obtained from the affine mapping $x \mapsto kx + a$ of $\mathbb{Z}_n$, the ring of integers modulo n . In particular, we set $G(n, k) = G(n, k, a)$ if $n \mid a$, which is from an endomorphism $x \mapsto kx$.

We are mainly interested in the problem when $G(n, k_1, a_1) \cong G(n, k_2, a_2)$. The authors in [2] gave an example $G(10, 2) \cong G(10, 8)$. Let p be a prime. Sha [4] showed that $G(p, k_1) \cong G(p, k_2)$ if and only if $\text{ord}_p k_1 = \text{ord}_p k_2$. In [5], Deng and Yuan gave a necessary and sufficient condition for $G(p-1, k_1) \cong G(p-1, k_2)$, where p is a prime. In this paper, we obtain a necessary and sufficient condition for when $G(n, k_1, a_1) \cong G(n, k_2, a_2)$.

The paper is organized as follows. In Section 2, we study the $G(n, k, a)$, where n is a power of a prime. The main results are proved in Section 3. We also determined the digraph from projective map on one-dimensional projective space over a finite field in the last section.

2. Digraph Product and the Local Case

Let G be a digraph and let a be a vertex in G . A component of a digraph is a subdigraph which is a maximal connected subgraph. Each component of a finite iteration graph $G(X, f)$ contains a unique cycle. Let O_i denote this i -cycle. Let $A_1(G)$ denote the number of cycle of length 1 in G .

Let $G_1, G_2, \dots, G_r$ be iteration digraphs. The product $G_1 \times G_2 \times \dots \times G_r$ is the digraph whose vertices are the ordered pairs $(a_1, a_2, \dots, a_r)$, where $a_i \in G_i$ and there is a directed edge from $(a_1, a_2, \dots, a_r)$ to $(b_1, b_2, \dots, b_r)$ if and only if there is a directed edge in G_i from a_i to b_i for each $1 \leq i \leq r$. We use the following notation:

$$\prod_{i=1}^r G_i := G_1 \times G_2 \times \dots \times G_r. \quad (1)$$

Lemma 1 (see [6]). $O_s \times O_t$ is isomorphic to $\gcd(s, t) O_{\text{lcm}[s, t]}$, which is a union of $\gcd(s, t)$ cycles, each having length $\text{lcm}[s, t]$.

Theorem 2. Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorization of n and let k, a be two integers. Then

$$G(n, k, a) \cong \prod_{i=1}^r G(p_i^{e_i}, k, a). \quad (2)$$

Proof. Let $G = G(n, k, a)$ and $G_i = G(p_i^{e_i}, k, a)$. Assume that $\varphi : \mathbb{Z}_n \rightarrow \mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \times \mathbb{Z}_{p_r^{e_r}}$ is the canonical isomorphism of rings. Then φ induces an isomorphism of digraphs. $\square$

2.1. The Case of Linear Maps. The structure of $G(n, k)$ has been studied in [2, 4, 5]. We present some results here.

Lemma 3 (see [4]). *Let k, n be positive integers. Then one has the following.*

- (i) *If $p \mid k$ for any prime $p \mid n$, then $G(n, k)$ is a tree attached to the fixed point 0.*
- (ii) *If $\gcd(n, k) = 1$, then $G(n, k) \simeq \cup_r a_r O_r$, where $a_r = (1/r) \sum_{i \mid r} \mu(i) \gcd(n, k^{r/i-1})$ and μ is the Möbius function.*

Let $v_p(n)$ be the p -adic valuation of n for any prime p and integer n . It is the largest power of p dividing n for any prime p and integer n . We set $v_p(0) = \infty$ as usual. If X is a set of primes and n is a positive integer, we denote that

$$n_X = \prod_{p \in X} p^{v_p(n)}. \quad (3)$$

Lemma 4. *Let p be a prime. Let k, a be positive integers such that $k, a \leq p^e - 1$. The following statements are equivalent:*

- (i) $G(p^e, k, a) \simeq G(p^e, k)$;
- (ii) $\gcd(p^e, k-1, a) = \gcd(p^e, k-1)$;
- (iii) $v_p(k-1) \leq v_p(a)$;
- (iv) $G(p^e, k, a)$ has a fixed point.

Proof. By hypothesis the statements (ii), (iii), and (iv) are equivalent to that the congruence equation $(k-1)x \equiv a \pmod{p^e}$ that has a solution. Suppose that $G(p^e, k, a) \simeq G(p^e, k)$. Then $G(p^e, k, a)$ has a fixed point z since $G(p^e, k)$ has a fixed point 0. So $kz+a \equiv z \pmod{p^e}$. On the other hand, if $kz+a \equiv z \pmod{p^e}$, then $x \mapsto x+z$ is an isomorphism of $G(p^e, k)$ and $G(p^e, k, a)$. $\square$

Remark 5. Many people showed that the tree attached to any two cycle vertices in $G(n, k)$ is isomorphic. Moreover, Deng and Yuan [5] showed that the tree attached to cycle vertices in $G(p-1, k)$ is totally determined by $\gcd(p-1, k)$, where p is a prime. In fact, if we replace $p-1$ by n in the arguments in [5], then the result is also valid.

2.2. The Case of Affine Maps

Lemma 6. *Let $k = lp^t + 1$ be a positive integer, where p is a prime and $p \nmid l, t \geq 1$.*

- (i) *If $p > 2$ or $t > 1$, then $\text{ord}_p k = \max\{1, p^{r-t}\}$ for any integers $t \geq 1$.*
- (ii) *If $p = 2$ and $t = 1$, assuming that $2^u \parallel k^2 - 1$, then*

$$\text{ord}_2 k = \begin{cases} 1, & r = 1, \\ 2, & 2 \leq r \leq u, \\ 2^{r-u+1}, & r > u. \end{cases} \quad (4)$$

Proof. If $r \leq t$, then it is clear that $\text{ord}_p k = 1 = \max\{1, p^{r-t}\}$. Firstly, we assume that $r > t$. By computation, $p^{m-s_i} \parallel \binom{p^m}{i}$ for any $1 \leq i \leq p^m - 1$ and $p^{s_i} \parallel i$. But

$$k^{p^{r-t}} - 1 = (lp^t + 1)^{p^{r-t}} - 1 = \sum_{i=1}^{p^{r-t}} \binom{p^{r-t}}{i} (lp^t)^i; \quad (5)$$

thus $p^{r-t-s_i+ti} \parallel \binom{p^{r-t}}{i} (lp^t)^i$. It is clear that $r-t-s_i+ti \geq r$. So $p^r \mid k^{p^{r-t}} - 1$; considering that $\text{ord}_p k \mid p^{r-t}$. So $\text{ord}_p k = p^r$ if $r-t=1$. Suppose that $r-t > 1$. If $p > 2$ or $t > 1$, then

$$\begin{aligned} k^{p^{r-t-1}} - 1 &= (lp^t + 1)^{p^{r-t-1}} - 1 \\ &= \sum_{i=1}^{p^{r-t-1}} \binom{p^{r-t-1}}{i} (lp^t)^i \\ &= lp^{r-1} + \sum_{i=2}^{p^{r-t-1}} \binom{p^{r-t-1}}{i} (lp^t)^i. \end{aligned} \quad (6)$$

Therefore, $r-t-1-s_i+ti \geq r$ for any $i \geq 2$. Hence, $p^r \mid \binom{p^{r-t-1}}{i} (lp^t)^i$ for $i \geq 2$ and $p^r \nmid k^{p^{r-t-1}} - 1$. Thus, $\text{ord}_p k = p^{r-t}$ in this case.

Suppose that $2 \parallel k-1$ and $2^u \parallel k^2 - 1, u \geq 3$. It remains to show that $\text{ord}_2 k = 2^{r-u+1}$ for $r > u$. Let $k^2 = q2^u + 1$, where $2 \nmid q$. By the discussion above we get $2^r \parallel k^{2^{r-u+1}} - 1$ and $2^{r-1} \parallel k^{2^{r-u}} - 1$. Therefore, $\text{ord}_2 k = 2^{r-u+1}$ for $r > u$. This finishes the proof. $\square$

Lemma 7. *Let p be a prime. Let $k, a \leq p^e - 1$ be two positive integers such that $v_p(k-1) > v_p(a)$. Then $G(p^e, k, a) \simeq p^{e-s} O_{p^s}$, where*

$$p^s = \text{ord}_{p^{e+v_p(k-1)-v_p(a)}} k = \begin{cases} p^{e-v_p(a)}, & 2 \nmid p \text{ or } v_p(k-1) > 1, \\ 2, & p = 2, v_2(k-1) = 1, \text{ and } e+1 \leq v_2(k-1), \\ 2^{e+2-v_2(k^2-1)}, & p = 2, v_2(k-1) = 1, \text{ and } e+1 > v_2(k-1). \end{cases} \quad (7)$$

Proof. Let $f(x) = kx + a$. Then $f^t(x) = k^t x + (\sum_{i=0}^{t-1} k^i) a$. We consider the following congruence equation

$$k^t x + \left(\sum_{i=0}^{t-1} k^i \right) a \equiv x \pmod{n}. \quad (8)$$

Equation (8) has a solution if and only if $\gcd(p^e, k^t - 1) \mid \gcd(p^e, (k^t - 1)/(k - 1) \cdot a)$. In this case, $p^e \mid k^t - 1$ and $p^e \mid (k^t - 1)/(k - 1) \cdot a$, since $v_p(k-1) > v_p(a)$. Since t is the minimal positive integer such that $p^e \mid (k^t - 1)/(k - 1) \cdot a$, we get $t = \text{ord}_{p^{e+v_p(k-1)-v_p(a)}} k = p^s$. Moreover, each integer is a solution of

(8). Thus, $G(p^e, k, a)$ is a union of cycles, each having length p^s . The rest of the proof follows from Lemma 6. $\square$

The structure of digraph $G(p^e, k, a)$ is clear now. We classify it as follows.

Type (1): $G(p^e, k, a) \simeq G(p^e, k)$ is a tree attached to a fixed point, which is determined by $\gcd(p^e, k)$.

Type (2): $G(p^e, k, a) \simeq G(p^e, k)$ is a union of cycles, and the minimal length of these cycles is 1.

Type (3): $G(p^e, k, a) \simeq p^{e-s} O_{p^s}$, where $1 \leq s \leq e$.

Let $X_i = \{p \mid G(p^{v_p(n)}, k, a) \text{ is type (i)}\}$. By the properties of digraph product we see that

$$G(n, k, a) \simeq G(n_{X_1}, k, a) \times G(n_{X_2}, k, a) \times G(n_{X_3}, k, a), \quad (9)$$

where $G(n_{X_1}, k, a)$ is a tree attached to a fixed point, $G(n_{X_2}, k, a)$ is a union of cycles, and the minimal length of these cycles is 1, and $G_3 \simeq (n_{X_3}/l) O_l$.

3. The Main Results

We begin with some lemmas before we prove our main results.

Lemma 8. Suppose that f is a permutation of a finite set X and $G(X, f) \simeq \cup_{i=1}^n a_i O_i$. Then $G(X, f^r) \simeq \cup_{i=1}^n \gcd(r, i) a_i O_{i/\gcd(r, i)}$ for each $r \geq 1$.

Proof. Assume that $E = \{x_1, x_2, \dots, x_l\}$ is a cycle of length l in $G(X, f)$ such that $f(x_i) = x_{i+1}$, for $i = 1, 2, \dots, l-1$ and $f(x_l) = x_1$. Suppose that x_i is contained in a cycle of length m in $G(X, f^r)$. Then $m = \min\{j : f^{rj}(x_i) = x_i\} = \min\{j : l \mid rj\} = l/\gcd(r, l)$. Hence, E splits into $\gcd(r, l)$ cycles with the same length $l/\gcd(r, l)$. This completes the proof. $\square$

Lemma 9. Let f_i be a permutation of a finite set X_i , $i = 1, 2$. Then

- (i) $G(X_1, f_1) \simeq G(X_2, f_2)$ if and only if the numbers of fixed point of $G(X_1, f_1^r)$ and $G(X_2, f_2^r)$ are equal for each $r \geq 1$;
- (ii) $G(X_1, f_1) \times O_l \simeq G(X_2, f_2) \times O_l$ if and only if $G(X_1, f_1^l) \simeq G(X_2, f_2^l)$ for each $l \geq 1$.

Proof. Suppose that $G(X_1, f_1) \simeq \cup a_i O_i$ and $G(X_2, f_2) \simeq \cup b_j O_j$.

Assume that $\varphi: G(X_1, f_1) \rightarrow G(X_2, f_2)$ is an isomorphism of digraphs. Then $f_2(\varphi(x)) = \varphi(f_1(x))$ for any $x \in X_1$. By induction, $f_2^r(\varphi(x)) = \varphi(f_1^r(x))$. Hence, φ is also an isomorphism of $G(X_1, f_1^r)$ and $G(X_2, f_2^r)$. Thus, $A_1(G(X, f_1^r)) = A_1(G(X, f_2^r))$. Conversely, if $G(X_1, f_1^r) \simeq G(X_2, f_2^r)$, then $a_1 = b_1$. Assume that $a_i = b_i$ for any $i < r$. By Lemma 8, $A_1(G(X_1, f_1^r)) = \sum_{i|r} i a_i$ and $A_1(G(X_2, f_2^r)) = \sum_{j|r} j b_j$. Hence, $\sum_{i|r} i a_i = \sum_{j|r} j b_j$ and $a_r = b_r$. The proof of statement (i) is complete by induction.

It remains to prove (ii). Assume that $G(Z, g) \simeq O_l$. Then $G(X_i, f_i) \times O_l \simeq G(X_i \times Z, f_i \times g)$, where $f_i \times g$ maps (x, z) to $(f_i(x), g(z))$ for any $(x, z) \in X_i \times Z$. Let $G_i^r = G(X_i \times Z, (f_i \times g)^r)$. By Lemma 8,

$$\begin{aligned} G_i^r &\simeq G(X_i, f_i^r) \times G(Z, g^r) \\ &\simeq G(X_i, f_i^r) \times \gcd(r, l) O_{l/\gcd(r, l)}. \end{aligned} \quad (10)$$

If $G(X_1, f_1) \times O_l \simeq G(X_2, f_2) \times O_l$, then $G_1^r \simeq G_2^r$ for each $r \geq 1$. Let $r = l$ in (10). We get $G(X_1, f_1^l) \simeq G(X_2, f_2^l)$. Conversely, if $G(X_1, f_1^l) \simeq G(X_2, f_2^l)$, then by (10) $A_1(G_i^r) = 0$ if $l \nmid r$. If $l \mid r$, then

$$G_i^r \simeq G(X_i, f_i^r) \times \gcd(r, l) O_{l/\gcd(r, l)} \simeq l G\left(X_i, (f_i^l)^{r/l}\right). \quad (11)$$

Hence, $G_1^r \simeq G_2^r$ if $l \mid r$. Therefore, $A_1(G_1^r) = A_1(G_2^r)$ for any $r \geq 1$. By statement (i), $G(X_1, f_1) \times O_l \simeq G(X_2, f_2) \times O_l$. $\square$

In [3], Sha and Hu showed that $G(p, k_1) \simeq G(p, k_2)$ if and only if $\text{ord}_p k_1 = \text{ord}_p k_2$, where p is a prime and $p \nmid k_i$. We generalized this result to the following.

Lemma 10. Let n, k_1 and k_2 be positive integers such that $\gcd(n, k_i) = 1$, $i = 1, 2$. Then $G(n, k_1) \simeq G(n, k_2)$ if and only if $\text{ord}_d k_1 = \text{ord}_d k_2$ for any divisor d of n .

Proof. Let $G_i^r = G(n, k_i^r)$. By Lemma 3 and Möbius inverse transform $A_1(G_i^r) = \gcd(k_i^r - 1, n)$ for any $r \geq 1$. Suppose that $\text{ord}_t k_1 = \text{ord}_t k_2$ for any divisor t of n . If $t_1 \mid \gcd(k_1^r - 1, n)$, let $s = \text{ord}_{t_1} k_1 = \text{ord}_{t_1} k_2$; then $s \mid r$. Thus $t_1 \mid \gcd(k_2^r - 1, n)$. By symmetry, we have $t_2 \mid \gcd(k_1^r - 1, n)$ if $t_2 \mid \gcd(k_2^r - 1, n)$. Thus, $\gcd(k_1^r - 1, n) = \gcd(k_2^r - 1, n)$. By Lemma 9, $G(n, k_1) \simeq G(n, k_2)$.

Conversely, if $G(n, k_1) \simeq G(n, k_2)$, then $G_1^r \simeq G_2^r$ and $\gcd(k_1^r - 1, n) = \gcd(k_2^r - 1, n)$ for any $r \geq 1$. If $t \mid n$, let $s_i = \text{ord}_t k_i$; then $t \mid \gcd(k_1^{s_1} - 1, n) = \gcd(k_2^{s_1} - 1, n)$. Hence, $s_2 \mid s_1$. By symmetry, $s_1 \mid s_2$. Thus, $s_1 = s_2$. The proof is complete. $\square$

Theorem 11. Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorization of n and let k_i, a_i be integers, $i = 1, 2$. Suppose that $Y_j = \{p_i \mid G(p_i^{e_i}, k_j, a_j) \text{ is Type (2)}\}$ and that the minimal length of cycles in $G(n, k_i, a_i)$ is l_i . Then $G(n, k_1, a_1) \simeq G(n, k_2, a_2)$ if and only if the following two conditions are satisfied.

- (i) $\gcd(n, k_1) = \gcd(n, k_2)$.
- (ii) $Y_1 = Y_2 = Y$, $l_1 = l_2 = l$, and $\text{ord}_d k_1^l = \text{ord}_d k_2^l$ for any divisor d of n_Y .

Proof. Suppose that $X_j = \{p_i \mid G(p_i^{e_i}, k_j, a_j) \text{ is Type (1)}\}$ and that $Z_j = \{p_i \mid G(p_i^{e_i}, k_j, a_j) \text{ is Type (3)}\}$. We have the following decomposition:

$$\begin{aligned} G(n, k_i, a_i) &\simeq G(n_{X_i}, k_i, a_i) \times G(n_{Y_i}, k_i, a_i) \times G(n_{Z_i}, k_i, a_i) \\ &\simeq G(n_{X_i}, k_i) \times G(n_{Y_i}, k_i) \times \frac{n_{Z_i}}{l_i} O_{l_i}, \end{aligned} \quad (12)$$

where $G(n_{X_i}, k_i, a_i)$ is a tree attached to a fixed point, $G(n_{Y_i}, k_i, a_i)$ is a union of cycles, and the minimal length of these cycles is 1.

Therefore, $G(n, k_1, a_1) \simeq G(n, k_2, a_2)$ if and only if the following two conditions hold.

- (iii) $G(n_{X_1}, k_1) \simeq G(n_{X_2}, k_2)$.
- (iv) $n_{Z_1}/l_1 = n_{Z_2}/l_2$ and $G(n_{Y_1}, k_1) \times O_{l_1} \simeq G(n_{Y_2}, k_2) \times O_{l_2}$.

However, by Remark 5, (i) is equivalent to (iii). It remains to show that (ii) is equivalent to (iv). If (ii) is satisfied, then $Z_1 = Z_2$ and by Lemma 10 $G(n_Y, k_1^l) \simeq G(n_Y, k_2^l)$. By Lemma 9, $G(n_Y, k_1) \times O_l \simeq G(n_Y, k_2) \times O_l$. Conversely, if (iv) holds, we have $l_1 = l_2 = l$ since the minimal length of cycles in $G(n_{Y_i}, k_i) \times O_{l_i}$ is l_i . So $n_{Z_1} = n_{Z_2}$, $Z_1 = Z_2$, $Y_1 = Y_2 = Y$, and $G(n_Y, k_1) \times O_l \simeq G(n_Y, k_2) \times O_l$. The rest of proof follows from Lemmas 9 and 10. $\square$

Remark 12. Suppose that $G(n, k_1, a_1) \simeq G(n, k_2, a_2)$. By Theorem 11, we see that both $G(p^{v_p(n)}, k_1, a_1)$ and $G(p^{v_p(n)}, k_2, a_2)$ are the same types for any prime $p \mid n$. Moreover, they are isomorphic if they are Type (1) or (3).

Example 13. Consider $G(40, 37, 1)$, $G(40, 29, 1)$, and $G(40, 29, 3)$. One has

$$\begin{aligned}
 G(40, 37, 1) &\simeq G(5, 37, 1) \times G(8, 37, 1) \\
 &\simeq G(5, 2, 1) \times G(8, 5, 1) \\
 &\simeq (O_1 \cup O_4) \times O_8 \\
 &\simeq 5O_8, \\
 G(40, 29, i) &\simeq G(5, 29, i) \times G(8, 29, i) \\
 &\simeq G(5, 4, i) \times G(8, 5, i) \\
 &\simeq (O_1 \cup 2O_2) \times O_8 \\
 &\simeq 5O_8,
 \end{aligned} \tag{13}$$

where $i = 1, 3$. Hence, $G(40, 37, 1) \simeq G(40, 29, 1)$, but $G(5, 37, 1) \neq G(5, 29, 1)$. This shows that even if $G(n, k_1, a_1) \simeq G(n, k_2, a_2)$, it may happen that $G(p^{v_p(n)}, k_1, a_1) \neq G(p^{v_p(n)}, k_2, a_2)$. But we also see that $G(40, 29, 1) \simeq G(40, 29, 3)$, $G(5, 29, 1) \simeq G(5, 29, 3)$, and $G(8, 29, 1) \simeq G(8, 29, 3)$. In fact, we have the following result if k is fixed.

Theorem 14. Let $n = \prod_{i=1}^r p_i^{e_i}$ be the prime factorization of n . The following three statements are equivalent for any integers k, a_1 , and a_2 :

- (i) $G(n, k, a_1) \simeq G(n, k, a_2)$;
- (ii) $\gcd(n, k-1, a_1) = \gcd(n, k-1, a_2)$;
- (iii) $G(p_i^{e_i}, k, a_1) \simeq G(p_i^{e_i}, k, a_2)$ for each $i \in \{1, 2, \dots, r\}$.

Proof. Let $X_j = \{p_i \mid G(p_i^{e_i}, k, a_j) \text{ is Type (1)}\}$, $Y_j = \{p_i \mid G(p_i^{e_i}, k, a_j) \text{ is Type (2)}\}$, and $Z_j = \{p_i \mid G(p_i^{e_i}, k, a_j) \text{ is Type (3)}\}$.

(i) $\Rightarrow$ (ii). By Theorem 11 $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$, and $Z_1 = Z_2 = Z$. If $p_i \in X \cup Y$, then $\gcd(p_i^{e_i}, k-1, a_j) = \gcd(p_i^{e_i}, k-1)$, $j = 1, 2$. If $p_i \in Z$, Suppose $G(p_i^{e_i}, k, a_j) \simeq p_i^{e_i-s_i} O_{p_i^{s_i}}$ for any $p_i \in Z$, where $p_i^{s_i} = \text{ord}_{p_i}^{e_i+v_{p_i}(k-1)-v_{p_i}(a_j)} k$. By Lemma 7, $v_{p_i}(a_1) = v_{p_i}(a_2)$. Thus, $\gcd(p_i^{e_i}, k-1, a_1) = \gcd(p_i^{e_i}, k-1, a_2)$ for any i . The result follows from that $\gcd(n, k-1, a_j) = \prod_{i=1}^r \gcd(p_i^{e_i}, k, a_j)$, $j = 1, 2$.

(ii) $\Rightarrow$ (iii). Assume that $\gcd(p_i^{e_i}, k-1, a_1) = \gcd(p_i^{e_i}, k-1, a_2) = p_i^{u_i}$ for each i with $u_i \leq e_i$. If $v_{p_i}(k-1) = u_i$, then by Lemma 4 $G(p_i^{e_i}, k, a_1) \simeq G(p_i^{e_i}, k, a_2) \simeq G(p_i^{e_i}, k)$. If $v_{p_i}(k-1) > u_i$, then $v_{p_i}(a_1) = v_{p_i}(a_2) = u_i$. By Lemma 7 again $G(p_i^{e_i}, k, a_j) \simeq p_i^{e_i-s_i} O_{p_i^{s_i}}$, where $p_i^{s_i} = \text{ord}_{p_i}^{e_i+v_{p_i}(k-1)-u_i} k$.

(iii) $\Rightarrow$ (i) follows immediately from (2). $\square$

4. The Digraph from Projective Map

Let $\mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q^2 \setminus \{(0, 0)\} / \sim$ be the one-dimensional projective space over $\mathbb{F}_q$, where $(a_1, a_2) \sim (b_1, b_2)$ if and only if there exists $\lambda \in \mathbb{F}_q^*$ such that $\lambda a_i = b_i$. With any invertible 2×2 matrix A over $\mathbb{F}_q$, we associate the Möbius transformation of $\mathbb{P}^1(\mathbb{F}_q)$ as $z \mapsto zA$. Let $f(x)$ be a polynomial over $\mathbb{F}_q$ with $f(0) \neq 0$. Recall that $o(f)$, the order of $f(x)$, is the minimal positive integer d such that $f(x) \mid x^d - 1$. We classify the digraph from Möbius transformation of $\mathbb{P}^1(\mathbb{F}_q)$ as follows.

Theorem 15. Let $G = G(\mathbb{P}^1(\mathbb{F}_q), A)$. Suppose that $f(x)$ and $m(x)$ are the characteristic polynomial and minimal polynomial of A , respectively. Then

- (i) if $f(x) = (x-\lambda)^2$ and $m(x) = x-\lambda$, then $G \simeq (q+1)O_1$;
- (ii) if $f(x) = m(x) = (x-\lambda)^2$, then $G \simeq O_1 \cup (q/p)O_p$;
- (iii) if $f(x) = (x-\lambda_1)(x-\lambda_2)$ and $\lambda_1 \neq \lambda_2$, then $G \simeq 2O_1 \cup ((q-1)/s)O_s$, where s is the multiplicative order of $\lambda_1 \lambda_2^{-1}$;
- (iv) if $f(x)$ is irreducible over $\mathbb{F}_q[x]$, then $G \simeq ((q+1)/s)O_s$, where $s = o(f)/\gcd(o(f), q-1)$.

Proof. (i)–(iii). We see that A is similar to λI_2 , $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ in cases (i), (ii), and (iii), respectively. But the digraphs are isomorphic from similar matrices. The results follows from a simple calculation.

It remains to prove (iv). Let $\mathbb{K} = \mathbb{F}_q[x]/(f(x))$. We can consider G as a subdigraph of $G_1 = G(\mathbb{K}, A)$ under the natural inclusion $\mathbb{F}_q \subseteq \mathbb{K}$. Then $f(x) = (x-\lambda)(x-\lambda^q)$ with some λ in $\mathbb{K}$. By (iii), $G_1 \simeq 2O_1 \cup ((q^2-1)/t)O_t$, where t is the multiplicative order of λ^{q-1} in $\mathbb{K}$. By a direct computation, we see that $zA \in \mathbb{F}_q$ for $z \in \mathbb{K}$ if and only if $z \in \mathbb{F}_q$. Hence, G is a union of cycles of G_1 . However, G has no fixed point since $f(x)$ is irreducible. So the length of each cycle in G is t , and $G \simeq ((q+1)/t)O_t$. Note that $t = \min\{n > 0 \mid \lambda^{n(q-1)} = 1\}$, and that $o(f) = \min\{n > 0 \mid \lambda^n = 1\}$. Therefore, $t = o(f)/\gcd(o(f), q-1)$. This finishes the proof. $\square$

Corollary 16. $G(\mathbb{P}^1(\mathbb{F}_q), A_1) \simeq G(\mathbb{P}^1(\mathbb{F}_q), A_2)$ if and only if they have the same number of fixed points.

References

- [1] T. D. Rogers, “The graph of the square mapping on the prime fields,” *Discrete Mathematics*, vol. 148, no. 1–3, pp. 317–324, 1996.
- [2] C. Lucheta, E. Miller, and C. Reiter, “Digraphs from powers modulo p ,” *The Fibonacci Quarterly*, vol. 34, no. 3, pp. 226–239, 1996.
- [3] M. Sha and S. Hu, “Monomial dynamical systems of dimension one over finite fields,” *Acta Arithmetica*, vol. 148, no. 4, pp. 309–331, 2011.
- [4] M. Sha, “Digraphs from endomorphisms of finite cyclic groups,” *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 83, pp. 105–120, 2012.
- [5] G. Deng and P. Yuan, “Isomorphic digraphs from powers modulo p ,” *Czechoslovak Mathematical Journal*, vol. 61, no. 3, pp. 771–779, 2011.
- [6] R. A. Hernández Toledo, “Linear finite dynamical systems,” *Communications in Algebra*, vol. 33, no. 9, pp. 2977–2989, 2005.