Research Article

A Common Fixed Point Theorem for Two Hybrid Pairs of Mappings in b -Metric Spaces

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We introduce a condition (W.C.C) and prove a common fixed point theorem for two hybrid pairs of mappings in b -metric spaces.

1. Introduction and Preliminaries

The study of fixed points for set-valued contraction maps using the Hausdorff metric was initiated by Nadler [1] in 1969, who extended the Banach contraction principle to set-valued mappings. The theory of set-valued maps has many applications in control theory, convex optimization, differential equations, and economics.

Recently Amini-Harandi [2] proved a set-valued version of Ciric's [3] theorem as follows.

Theorem 1. Let (X, d) be a complete metric space. Suppose $T : X \rightarrow CB(X)$ satisfy

$$H(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (1)$$

for all $x, y \in X$, where $0 \leq q < 1/2$. Then T has a fixed point in X .

Aydi et al. [4] proved a slight version of the above theorem in b -metric spaces as follows.

Theorem 2. Let (X, d) be a complete b -metric space. Suppose $T : X \rightarrow CB(X)$ satisfy

$$H(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (2)$$

for all $x, y \in X$, where $0 \leq q < 1/(s^2 + s)$. Then T has a fixed point in X .

In this paper, we introduce a new condition, namely, "condition (W.C.C)" of mappings which are not continuous and non commuting pairwise. Using this new condition we prove a common fixed point theorem for two pairs of hybrid mappings in b -metric spaces.

For the sequel, we need the following concepts.

Czerwik [5] presented a generalization of the well-known Banach fixed point theorem in b -metric spaces by defining the following.

Definition 3. Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is called a b -metric if the following axioms are satisfied for all $x, y, z \in X$:

$$(b_1) \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$(b_2) \quad d(x, y) = d(y, x),$$

$$(b_3) \quad d(x, y) \leq s[d(x, z) + d(z, y)].$$

Let (X, d) be a b -metric space. Let $CB(X)$ be the collection of all non-empty closed and bounded subsets of X . For $A, B \in CB(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \quad (3)$$

where

$$\begin{aligned}\delta(A, B) &= \sup \{d(a, B) : a \in A\}, \\ \delta(B, A) &= \sup \{d(b, A) : b \in B\},\end{aligned}\quad (4)$$

with

$$d(a, B) = \inf \{d(a, b) : b \in B\}. \quad (5)$$

Then H is called the Hausdorff metric induced by the b -metric.

We cite the following lemmas from Czerwik [6]. For more details we refer to [7, 8].

Lemma 4. Let (X, d) be a b -metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following:

- (i) $d(x, B) \leq d(x, b)$ for any $b \in B$,
- (ii) $\delta(A, B) \leq H(A, B)$,
- (iii) $d(x, B) \leq H(A, B)$ for any $x \in A$,
- (iv) $H(A, A) = 0$,
- (v) $H(A, B) = H(B, A)$,
- (vi) $H(A, C) \leq s[H(A, B) + H(B, C)]$,
- (vii) $d(x, A) \leq s[d(x, y) + d(y, A)]$.

Lemma 5. Let (X, d) be a b -metric space. Let $A, B \in CB(X)$. Then for each $\epsilon > 0$ and for all $b \in B$ there exists $a \in A$ such that $d(a, b) \leq H(A, B) + \epsilon$.

Lemma 6. Let (X, d) be a b -metric space. For $A \in CB(X)$ and $x \in X$, we have $d(x, A) = 0 \Leftrightarrow x \in \bar{A} = A$.

In this paper, we introduce the following condition on three mappings without continuity and commutativity (W.C.C).

Definition 7. Let (X, d) be a b -metric space. The mappings $(f, g; S)$ where $f, g : X \rightarrow X$ and $S : X \rightarrow CB(X)$ are said to satisfy the condition (W.C.C) if $d(fx, gy) \leq d(y, Sx)$ for all $x, y \in X$.

Remark 8. The mappings $(f, g; S)$ satisfying the condition (W.C.C) need not be continuous and commutative in view of the following examples.

Example 9. Let (X, d) be a b -metric space, where $X = [0, 1]$ and $d(x, y) = |x - y|^2$.

Define $f, g, S : X \rightarrow X$ by

$$fx = \begin{cases} \frac{x}{12} & \text{if } x \in [0, 1), \\ \frac{1}{24} & \text{if } x = 1, \end{cases} \quad (6)$$

$$gx = \begin{cases} \frac{x}{6} & \text{if } x \in [0, 1), \\ \frac{1}{12} & \text{if } x = 1, \end{cases} \quad (7)$$

$$Sx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1), \\ \frac{1}{4} & \text{if } x = 1. \end{cases} \quad (8)$$

We distinguish the following cases.

Case 1. Consider $0 \leq x < 1$ and $0 \leq y < 1$

$$\text{Then } d(fx, gy) = \left| \frac{x}{12} - \frac{y}{6} \right|^2 = \frac{1}{36} \left| \frac{x}{2} - y \right|^2 = \frac{1}{36} d(y, Sx). \quad (9)$$

Case 2. Consider $0 \leq x < 1$ and $y = 1$

$$\text{Then } d(fx, gy) = \left| \frac{x}{12} - \frac{1}{12} \right|^2 = \frac{1}{36} \left| \frac{1}{2} - \frac{x}{2} \right|^2 < \frac{1}{36} d(y, Sx). \quad (10)$$

Case 3. Consider $x = 1$ and $0 \leq y < 1$

$$\text{Then } d(fx, gy) = \left| \frac{1}{24} - \frac{y}{6} \right|^2 = \frac{1}{36} \left| y - \frac{1}{4} \right|^2 = \frac{1}{36} d(y, Sx). \quad (11)$$

Case 4. Consider $x = 1$ and $y = 1$

$$\text{Then } d(fx, gy) = \left| \frac{1}{24} - \frac{1}{12} \right|^2 = \frac{1}{576} < \frac{9}{16} = d(y, Sx). \quad (12)$$

Thus $d(fx, gy) \leq d(y, Sx)$ for all $x, y \in X$ is satisfied.

In this example all the mappings are discontinuous.

Example 10. Let $d(x, y) = |x - y|^2$ be a b -metric on a set $A = \{x, y\}$. Define $fx = fy = y$, $gx = y$, $gy = x$, and $Sx = Sy = x$. Clearly the mappings $(f, g; S)$ satisfy the condition (W.C.C) and the pairs (f, g) , (f, S) , and (g, S) are not commuting.

Now, we prove our main result with the condition (W.C.C) for two hybrid pairs of mappings in b -metric spaces.

2. Main Result

Theorem 11. Let (X, d) be a complete b -metric space. Let $f, g : X \rightarrow X$ and $S, T : X \rightarrow CB(X)$ be mappings satisfying the following:

(2.1.1) $H(Sx, Ty) \leq qM(x, y)$, where $M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\}$ for all $x, y \in X$, where $0 \leq q < 1/(s^2 + s)$,

(2.1.2) $\bigcup_{x \in X} Sx \subseteq g(X)$ and $\bigcup_{x \in X} Tx \subseteq f(X)$,

(2.1.3) the mappings $(f, g; S)$ or the mappings $(f, g; T)$ satisfy the condition (W.C.C).

Then either the pairs (f, S) , and (g, T) have coincidence points or the mappings S, T, f and g have a common fixed point in X .

Proof. Suppose $M(x, y) = 0$ for some $x, y \in X$.

Then $fx \in Sx$ and $gy \in Ty$. Thus x is a coincidence point of f and S , and y is a coincidence point of g and T .

Assume that $M(x, y) > 0$ for all $x, y \in X$.

Let $\epsilon = (1/2)(1/(s^2 + s) - q)$ and $\beta = q + \epsilon = (1/2)(1/(s^2 + s) + q)$.

Then $\epsilon > 0$ and $0 < \beta < 1$.

Let $x_0 \in X$. From (2.1.2), there exist $x_1, y_1 \in X$ such that $y_1 = gx_1 \in Sx_0$.

From (2.1.2), there exist $x_2, y_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and from Lemma 5, we have

$$\begin{aligned} d(y_1, y_2) &\leq H(Sx_0, Tx_1) + \epsilon M(x_0, x_1) \\ &\leq qM(x_0, x_1) + \epsilon M(x_0, x_1), \quad \text{from (2.1.1)} \\ &= \beta M(x_0, x_1). \end{aligned} \quad (13)$$

Again from (2.1.2), there exist $x_3, y_3 \in X$ such that $y_3 = gx_3 \in Sx_2$ and from Lemma 5, we have

$$\begin{aligned} d(y_2, y_3) &\leq H(Sx_2, Tx_1) + \epsilon M(x_2, x_1) \\ &\leq qM(x_2, x_1) + \epsilon M(x_2, x_1), \quad \text{from (2.1.1)} \\ &= \beta M(x_2, x_1). \end{aligned} \quad (14)$$

Continuing in this way, we get the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n+1} = gx_{2n+1} \in Sx_{2n}$, $y_{2n+2} = fx_{2n+2} \in Tx_{2n+1}$ for all $n = 0, 1, 2, 3, \dots$ and

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &\leq \beta M(x_{2n}, x_{2n-1}), \\ d(y_{2n+1}, y_{2n+2}) &\leq \beta M(x_{2n}, x_{2n+1}) \quad \text{for } n = 1, 2, 3, \dots \end{aligned} \quad (15)$$

For simplification, write $d_n = d(y_n, y_{n+1})$.

Now

$$\begin{aligned} d_{2n+1} &= d(y_{2n+1}, y_{2n+2}) \\ &\leq \beta M(x_{2n}, x_{2n+1}) \\ &= \beta \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+2}), \\ &\quad d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), 0\} \\ &\leq \beta \max\{d_{2n}, d_{2n+1}, s[d_{2n} + d_{2n+1}]\} \end{aligned} \quad (16)$$

which in turn yields

$$d_{2n+1} \leq \gamma d_{2n}, \quad (17)$$

where $\gamma = \max\{\beta, s\beta/(1 - s\beta)\}$.

Similarly, we can show that

$$d_{2n} \leq \gamma d_{2n-1}. \quad (18)$$

Thus, from (17) and (18), we have $d_n \leq \gamma d_{n-1}$ for all $n = 1, 2, 3, \dots$, so that

$$d_n \leq \gamma^n d_0 \quad \forall n = 1, 2, 3, \dots \quad (19)$$

If $\gamma = \beta$, then $\gamma s = \beta s = (s/2)(1/(s^2 + s) + q) < s/(s^2 + s) = 1/(s + 1) \leq 1/2 < 1$.

If $\gamma = s\beta/(1 - s\beta)$, then $\gamma s = s(s\beta/(1 - s\beta)) < s[1/(s + 1)]/1 - 1/(s + 1) = 1$.

Thus

$$\gamma s < 1. \quad (20)$$

Since $\gamma < 1$, from (19), we have

$$d_n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (21)$$

Now for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(y_n, y_m) &\leq sd(y_n, y_{n+1}) \\ &\quad + s^2 d(y_{n+1}, y_{n+2}) + \dots + s^{m-n} d(y_{m-1}, y_m) \\ &\leq (s\gamma^n + s^2\gamma^{n+1} + \dots + s^{m-n}\gamma^{m-1}) d_0, \quad \text{from (19)} \\ &\leq s\gamma^n (1 + s\gamma + \dots + s^{m-n-1}\gamma^{m-n-1} + \dots) d_0 \\ &= \frac{\gamma^n s}{1 - \gamma s} d_0 \rightarrow 0 \quad \text{as } n, m \rightarrow \infty. \end{aligned} \quad (22)$$

Hence $\{y_n\}$ is a Cauchy sequence in X . Since X is complete, there exists $z \in X$ such that $y_n \rightarrow z$ which in turn yields $fx_{2n} \rightarrow z$ and $gx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$.

Suppose the mappings $(f, g; S)$ satisfy the condition (W.C.C). Then

$$d(fx, gy) \leq d(y, Sx) \quad \forall x, y \in X. \quad (23)$$

From (23), we have

$$d(fx_{2n}, gz) \leq d(z, Sx_{2n}) \leq d(z, gx_{2n+1}). \quad (24)$$

Letting $n \rightarrow \infty$, we get

$$d(z, gz) \leq 0 \quad \text{so that } gz = z. \quad (25)$$

Now

$$\begin{aligned}
 & \frac{1}{s}d(z, Tz) \\
 & \leq d(z, gx_{2n+1}) + d(gx_{2n+1}, Tz) \\
 & \leq d(z, gx_{2n+1}) + H(Sx_{2n}, Tz) \\
 & \leq d(z, y_{2n+1}) \\
 & \quad + q \max \left\{ \begin{array}{l} d(fx_{2n}, gz), d(fx_{2n}, Sx_{2n}), d(gz, Tz), \\ d(fx_{2n}, Tz), d(gz, Sx_{2n}) \end{array} \right\} \\
 & \leq d(z, y_{2n+1}) \\
 & \quad + q \max \left\{ \begin{array}{l} d(y_{2n}, z), d(y_{2n}, y_{2n+1}), d(z, Tz), \\ d(y_{2n}, Tz), d(z, y_{2n+1}) \end{array} \right\}. \quad (26)
 \end{aligned}$$

Using (vii) of Lemma 4 and letting $n \rightarrow \infty$ and using (21), we get

$$\frac{1}{s}d(z, Tz) \leq 0 + qsd(z, Tz). \quad (27)$$

Thus we have

$$d(z, Tz) \leq qs^2 d(z, Tz), \quad (28)$$

so that

$$d(z, Tz) = 0 \quad \text{since } qs^2 < 1. \quad (29)$$

From Lemma 6, we have $z \in Tz$. Thus

$$gz = z \in Tz. \quad (30)$$

From (23), we get

$$d(fz, z) = d(fz, gz) \leq d(z, Sz). \quad (31)$$

Now from (2.1.1), we have

$$\begin{aligned}
 d(z, Sz) & \leq H(Sz, Tz) \\
 & \leq q \max \{d(fz, gz), d(fz, Sz), d(gz, Tz), \\
 & \quad d(fz, Tz), d(gz, Sz)\} \\
 & \leq q \max \{d(z, Sz), s[d(z, Sz) + d(z, Sz)], \\
 & \quad 0, sd(z, Sz), d(z, Sz)\} \\
 & \quad \text{from (30) and (31)} \\
 & = 2qsd(z, Sz).
 \end{aligned} \quad (32)$$

Hence $z \in Sz$, since $2qs < 1$.

From (31), we have $fz = z$. Thus

$$fz = z \in Sz. \quad (33)$$

From (30) and (33), it follows that z is a common fixed point of f , g , S , and T . $\square$

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