Research Article

The Extreme and Support Points of a New Class of Analytic Functions with Positive Real Part

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Suppose that $0 < \alpha < \beta < +\infty$. Let $\mathcal{P}(\alpha, \beta)$ denote the set of functions $p(z)$ that are analytic in $D = \{z : |z| < 1\}$ and satisfy $\operatorname{Re} p(z) > 0 (|z| < 1)$ and $\alpha \leq p(0) \leq \beta$. In this paper, we investigate the extreme points and support points of $\mathcal{P}(\alpha, \beta)$.

1. Introduction

By $\mathcal{A}$, we denote the space of functions analytic in the unit disk $D = \{z : |z| < 1\}$. Endowed with the topology of uniform convergence on compact subsets of the unit disk, $\mathcal{A}$ is a locally convex topological vector space.

Let X be a topological vector space and U a subset of X . If $V \subset U$, then V is called an extremal subset of U provided that whenever $u = tx + (1-t)y$, where $u \in V$, $x, y \in U$ and $0 < t < 1$, then x and y both belong to V . An extremal subset of U consisting of just one point is called an extreme point of U . Thus, an element $x \in U$ is an extreme point of U if and only if x is not a proper convex combination of any two distinct points in U . The set of all extreme points of U is denoted by EU . It is apparent that if V is an extremal subset of U , then $EV \subset EU$. If X is a locally convex topological vector space and U is a nonempty compact subset of X , then EU is nonempty [1, page 44], [2, page 181]. For any subset U of X , we use HU to denote the closed convex hull of U . If U is a compact subset of the locally convex topological vector space X , then, by Krein-Milman theorem [1, page 44], [2, page 182], $HU = EU$.

Let $\mathcal{P}$ be the set of all functions $p(z)$ which are analytic, have positive real part in D , and satisfy $p(0) = 1$. Then $\mathcal{P}$ is a compact subset of $\mathcal{A}$ [1, page 39]. It is well known that $E\mathcal{P} = \{(1+xz)/(1-xz) : |x| = 1\}$ [1, page 48], [3–5]. Bellamy and Tkaczyńska [6] investigated the extreme points of some classes of analytic functions with positive real part and a prescribed set of coefficients. Peng [7] investigated the

extreme points of a class of analytic functions with positive real part and a prescribed set of values.

Suppose that $\mathcal{F}$ is a compact subset of $\mathcal{A}$. A function f is called a support point of $\mathcal{F}$ if $f \in \mathcal{F}$ and there is a continuous linear functional J on $\mathcal{A}$ such that $\operatorname{Re} J$ is nonconstant on $\mathcal{F}$ and

$$\operatorname{Re} J(f) = \max \{ \operatorname{Re} J(g) : g \in \mathcal{F} \}. \quad (1)$$

The set of all support points of $\mathcal{F}$ is denoted by $\operatorname{supp} \mathcal{F}$. Hallenbeck and MacGregor [1, page 94], [8] proved that the set $\operatorname{supp} \mathcal{P}$ consists of all functions which may be written as

$$p(z) = \sum_{k=1}^m \lambda_k \frac{1+x_k z}{1-x_k z}, \quad (2)$$

where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$, and $|x_k| = 1$ ($m = 1, 2, \dots$). The author [9] investigated the support points of a class of analytic functions with positive real part and a prescribed set of coefficients.

Suppose that $0 < \alpha < \beta < +\infty$. Let $\mathcal{P}(\alpha, \beta)$ denote the set of functions $p(z)$ that are analytic in $D = \{z : |z| < 1\}$ and satisfy $\operatorname{Re} p(z) > 0 (|z| < 1)$ and $\alpha \leq p(0) \leq \beta$. It is apparent that $p(z) \in \mathcal{P}(\alpha, \beta)$ if and only if $p(z) = \mu q(z)$ with some $q(z) \in \mathcal{P}$ and $\alpha \leq \mu \leq \beta$. Thus, it is easy to prove that $\mathcal{P}(\alpha, \beta)$ is a compact subset of $\mathcal{A}$. In this paper we investigate the extreme points and support points of $\mathcal{P}(\alpha, \beta)$. In some ways, the results we obtained generalize the results of Holland, Hallenbeck, and MacGregor.

2. Main Results

Theorem 1. $p(z)$ is an extreme point of $\mathcal{P}(\alpha, \beta)$ if and only if

$$p(z) = \alpha \frac{1+xz}{1-xz}, \quad \text{or} \quad \beta \frac{1+xz}{1-xz}, \quad (3)$$

where $|x| = 1$.

Proof. Suppose that

$$\alpha \frac{1+xz}{1-xz} = tp_1(z) + (1-t)p_2(z), \quad (4)$$

where $|x| = 1$, $0 < t < 1$, $p_1, p_2 \in \mathcal{P}(\alpha, \beta)$. Then $\alpha = tp_1(0) + (1-t)p_2(0)$. Since $\alpha \leq p_1(0)$, $\alpha \leq p_2(0)$, it follows that $p_1(0) = p_2(0) = \alpha$. Consequently, $p_1(z)/\alpha \in \mathcal{P}$, $p_2(z)/\alpha \in \mathcal{P}$. Notice that (4) is equivalent to

$$\frac{1+xz}{1-xz} = t \frac{1}{\alpha} p_1(z) + (1-t) \frac{1}{\alpha} p_2(z), \quad (5)$$

and that $[1+xz]/[1-xz] \in E\mathcal{P}$, and we have

$$\frac{1}{\alpha} p_1(z) = \frac{1}{\alpha} p_2(z). \quad (6)$$

So, $p_1(z) = p_2(z)$. This proves that $[\alpha(1+xz)]/[1-xz] \in E\mathcal{P}(\alpha, \beta)$. Similarly, we can prove that $[\beta(1+xz)]/[1-xz] \in E\mathcal{P}(\alpha, \beta)$.

Conversely, Suppose $p(z) \in E\mathcal{P}(\alpha, \beta)$. If $\alpha < p(0) < \beta$, then $p(0) = t\alpha + (1-t)\beta$ for some $0 < t < 1$. Since

$$\begin{aligned} p(z) &= t \frac{\alpha}{p(0)} p(z) + (1-t) \frac{\beta}{p(0)} p(z), \\ \frac{[\alpha p(z)]}{p(0)} &\in \mathcal{P}(\alpha, \beta), \quad \frac{[\beta p(z)]}{p(0)} \in \mathcal{P}(\alpha, \beta), \quad (7) \\ \frac{[\alpha p(z)]}{p(0)} &\neq \frac{[\beta p(z)]}{p(0)}, \end{aligned}$$

it follows that $p(z) \notin E\mathcal{P}(\alpha, \beta)$ which contradicts the assumption. Thus $p(0) = \alpha$ or $p(0) = \beta$.

If $p(0) = \alpha$, then $p(z)/\alpha \in \mathcal{P}$. By Herglotz formula [1, page 30], [10, page 22], we have

$$p(z) = \alpha \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x), \quad (8)$$

where μ is a probability measure on the unit circle $|x| = 1$. If μ is not a point mass, then there exist probability measures μ_1 and μ_2 on the unit circle $|x| = 1$ such that $\mu_1 \neq \mu_2$ and $\mu = t\mu_1 + (1-t)\mu_2$ for some t with $0 < t < 1$ [1, page 47]. Let

$$p_1(z) = \alpha \int_{|x|=1} \frac{1+xz}{1-xz} d\mu_1(x), \quad (9)$$

$$p_2(z) = \alpha \int_{|x|=1} \frac{1+xz}{1-xz} d\mu_2(x).$$

Then $p_1, p_2 \in \mathcal{P}(\alpha, \beta)$, $p_1 \neq p_2$, and $p = tp_1 + (1-t)p_2$. This implies that p is not an extreme point of $\mathcal{P}(\alpha, \beta)$. We get a

contradiction. So, μ is a point mass. Without loss of generality, we suppose $\mu(\{x\}) = 1$ for some unit complex number x . Then $p(z) = \alpha(1+xz)/(1-xz)$ with $|x| = 1$. Similarly, if $p(0) = \beta$, then we can prove that $p(z) = \beta(1+xz)/(1-xz)$, where $|x| = 1$. The proof is completed. $\square$

Theorem 2. p is a support point of $\mathcal{P}(\alpha, \beta)$ if and only if p satisfies one of the following conditions:

$$(1) \quad p \in \mathcal{P}(\alpha, \beta), \quad p(0) = \alpha \text{ or } \beta,$$

$$(2)$$

$$p(z) = \alpha \sum_{k=1}^m \lambda_k \frac{1+x_k z}{1-x_k z} + \beta \sum_{k=m+1}^{m+n} \lambda_k \frac{1+x_k z}{1-x_k z}, \quad (10)$$

where $\lambda_k \geq 0$, $\sum_{k=1}^{m+n} \lambda_k = 1$, $|x_k| = 1$, m and n are positive integers.

Proof. Let p_0 be a support point of $\mathcal{P}(\alpha, \beta)$. Then there is a continuous linear functional J defined on $\mathcal{A}$ such that $\operatorname{Re} J$ is not constant on $\mathcal{P}(\alpha, \beta)$ and

$$\operatorname{Re} J(p_0) = \max \{ \operatorname{Re} J(p) : p \in \mathcal{P}(\alpha, \beta) \}. \quad (11)$$

Suppose that $\operatorname{Re} J(p_0) = M$. Let $\mathcal{Q} = \{q : q \in \mathcal{P}(\alpha, \beta), \operatorname{Re} J(q) = M\}$. Since $p_0 \in \mathcal{Q}$ and J is continuous, $\mathcal{Q}$ is a nonvacuous closed convex subset of $\mathcal{P}(\alpha, \beta)$. As $\mathcal{P}(\alpha, \beta)$ is a compact subset of $\mathcal{A}$, so is $\mathcal{Q}$. By Krein-Milman theorem [1, page 44], [2, page 182], $E\mathcal{Q}$ is nonempty, and $\mathcal{Q} = HE\mathcal{Q}$. Suppose that $q \in \mathcal{Q}$ and $q = tq_1 + (1-t)q_2$, where $q_1, q_2 \in \mathcal{P}(\alpha, \beta)$ and $0 < t < 1$. Then

$$\begin{aligned} M = \operatorname{Re} J(q) &= t \operatorname{Re} J(q_1) + (1-t) \operatorname{Re} J(q_2) \\ &\leq tM + (1-t)M = M. \end{aligned} \quad (12)$$

Since $\operatorname{Re} J(q_1) \leq M$, $\operatorname{Re} J(q_2) \leq M$, it follows from (12) that

$$\operatorname{Re} J(q_1) = \operatorname{Re} J(q_2) = M. \quad (13)$$

This implies that $q_1, q_2 \in \mathcal{Q}$. Thus $\mathcal{Q}$ is an extremal subset of $\mathcal{P}(\alpha, \beta)$ and so $E\mathcal{Q} \subset E\mathcal{P}(\alpha, \beta)$.

Since J is a continuous linear functional on $\mathcal{A}$, there is a sequence $\{b_n\}$ of complex numbers satisfying

$$\overline{\lim}_{n \rightarrow \infty} (|b_n|)^{1/n} < 1, \quad (14)$$

such that

$$J(f) = \sum_{n=0}^{\infty} b_n a_n, \quad (15)$$

where $f \in \mathcal{A}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ [1, page 42]. Let

$$G(x) = J\left(\frac{1+xz}{1-xz}\right) = b_0 + \sum_{n=1}^{\infty} 2b_n x^n. \quad (16)$$

Then $G(x)$ is analytic in $\overline{D} = \{x : |x| \leq 1\}$.

If $E\mathcal{Q}$ has infinitely many elements, then there must be infinitely many $[\alpha(1+xz)]/(1-xz)$ or infinitely many $[\beta(1+xz)]/(1-xz)$ in $E\mathcal{Q}$, where $|x| = 1$. Without loss of generality, we assume that there are infinitely many $[\alpha(1+xz)]/(1-xz)$ in $E\mathcal{Q}$ with $|x| = 1$. Then $\operatorname{Re} G(x) = M/\alpha$ has infinitely many solutions which implies that $G(x)$ is constant in $\overline{D} = \{z : |z| \leq 1\}$. So, $E\mathcal{Q}$ contains all $[\alpha(1+xz)]/(1-xz)$ with $|x| = 1$ but contains no elements such as $[\beta(1+xz)]/(1-xz)$ with $|x| = 1$. This prove that

$$E\mathcal{Q} = \left\{ \frac{[\alpha(1+xz)]}{1-xz} : |x| = 1 \right\}, \quad (17)$$

and thus

$$\mathcal{Q} = HE\mathcal{Q} = \{p : p \in \mathcal{P}(\alpha, \beta), p(0) = \alpha\}. \quad (18)$$

In particular, $p_0 \in \mathcal{P}(\alpha, \beta)$ and $p_0(0) = \alpha$. Similarly, if we assume that there are infinitely many $[\beta(1+xz)]/(1-xz)$ in $E\mathcal{Q}$ with $|x| = 1$, then we can prove that $p_0 \in \mathcal{P}(\alpha, \beta)$ and $p_0(0) = \beta$.

In the case that $E\mathcal{Q}$ has only a finite number of elements, say, $E\mathcal{Q} = \{[\alpha(1+x_kz)]/(1-x_kz), [\beta(1+x_jz)]/(1-x_jz) : |x_k| = |x_j| = 1, k = 1, \dots, m; j = 1, \dots, n\}$. Then $\mathcal{Q} = HE\mathcal{Q}$ consists of functions given by (10). p_0 especially must have the form given by (10).

Conversely, Suppose that $p_0 \in \mathcal{P}(\alpha, \beta)$ and that $p_0(0) = \alpha$. Define a continuous linear functional J on $\mathcal{A}$ by $J(f) = -a_0$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $\alpha < \beta$, it is clear that $\operatorname{Re} J$ is not constant on $\mathcal{P}(\alpha, \beta)$ and $\operatorname{Re} J(p_0) = \max\{\operatorname{Re} J(p) : p \in \mathcal{P}(\alpha, \beta)\}$. So, $p_0 \in \operatorname{supp} \mathcal{P}(\alpha, \beta)$. Similarly, we can prove that $p_0 \in \operatorname{supp} \mathcal{P}(\alpha, \beta)$ if $p \in \mathcal{P}(\alpha, \beta)$ and $p(0) = \beta$.

Now suppose that p_0 has the form (10). Then, by Lemma 7.2 in [1], there is a function F analytic on $\overline{D} = \{z : |z| \leq 1\}$ such that $\operatorname{Re} F(z) \leq 0$ when $|z| \leq 1$ and $\operatorname{Re} F(z) = 0$ if and only if $z = x_k$ ($k = 1, 2, \dots, m+n$). Suppose that $F(z) = \sum_{n=0}^{\infty} d_n z^n$. Let

$$b_0 = d_0, \quad b_n = \frac{d_n}{2} \quad (n = 1, 2, \dots). \quad (19)$$

Then $\lim_{n \rightarrow \infty} |b_n|^{1/n} < 1$. Define a linear functional J on $\mathcal{A}$ by

$$J(f) = \sum_{n=0}^{\infty} b_n a_n, \quad (20)$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}. \quad (21)$$

Then J is continuous [1, page 42]. Since

$$\begin{aligned} J\left(\frac{\alpha(1+xz)}{(1-xz)}\right) &= \alpha F(x), \\ J\left(\frac{\beta(1+xz)}{(1-xz)}\right) &= \beta F(x), \end{aligned} \quad (22)$$

it follows that [1, page 44]

$$\begin{aligned} \max\{\operatorname{Re} J(p) : p \in \mathcal{P}(\alpha, \beta)\} \\ = \max\{\operatorname{Re} J(p) : p \in E\mathcal{P}(\alpha, \beta)\} = 0. \end{aligned} \quad (23)$$

Note that

$$\begin{aligned} \operatorname{Re} J\left(\frac{(1+x_kz)}{(1-x_kz)}\right) &= \operatorname{Re} F(x_k) = 0 \\ (k = 1, 2, \dots, m+n), \end{aligned} \quad (24)$$

we have $\operatorname{Re} J(p_0) = 0$. If $\operatorname{Re} J$ is constant on $\mathcal{P}(\alpha, \beta)$, then $\operatorname{Re} F(x) = 0$ when $|x| = 1$. But it is not the case. Therefore $p_0 \in \operatorname{supp} \mathcal{P}(\alpha, \beta)$. $\square$

Remark 3. Though we assume that $\alpha < \beta$ in Theorem 1, it is easy to see that Theorem 1 is valid for $\alpha = \beta = 1$, which is just the result of Holland. For $\alpha = \beta = 1$, Theorem 2 is invalid, since $p \in \mathcal{P}(1, 1) = \mathcal{P}$ and $p(0) = 1$ does not imply that p is a support point of $\mathcal{P}$. In the case where $\alpha = \beta = 1$, Theorem 2 should be stated as follows.

The set $\operatorname{supp} \mathcal{P}$ consists of all functions which may be written as

$$p(z) = \sum_{k=1}^m \lambda_k \frac{1+x_kz}{1-x_kz}, \quad (25)$$

where $\lambda_k \geq 0$, $\sum_{k=1}^m \lambda_k = 1$, and $|x_k| = 1$ ($m = 1, 2, \dots$).

This is the result of Hallenbeck and MacGregor [1, page 94], [8]. It is easy to see that Theorem 2 generalizes the result of Hallenbeck and MacGregor in some sense.

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