Research Article

Arc Length Inequality for a Certain Class of Analytic Functions Related to Conic Regions

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In our present investigation, we introduce a subclass of analytic function associated with conic regions which is a form of generalized close-to-convexity. The arc-length inequality for a class of analytic function is well known. We derive this inequality for the newly defined class and also study some of its interesting consequences.

1. Introduction

Let \( \mathcal{A} \) denote the class of functions \( f : \mathcal{A} \)

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]  

(1)

which are analytic in the unit disc \( \mathcal{U} = \{ z : |z| < 1 \} \). Let \( \mathcal{S} \) denote the class of all functions in \( \mathcal{A} \) which are univalent. Also let \( \mathcal{S}^*, \mathcal{E} \), and \( \mathcal{K} \) be the well-known subclasses of \( \mathcal{S} \) consisting of all functions which are, respectively, of starlike, convex, and close-to-convex.

Kanas and Wiśniowska [1, 2] studied the classes of \( k \)-uniformly convex denoted by \( k-\mathcal{UCV} \) and the corresponding class \( k-\mathcal{ST} \) related by the Alexander type relation. Later Acu [3] considered the class \( k \)-uniformly close-to-convex denoted by \( k-\mathcal{UK} \) to be defined as

\[ k-\mathcal{UK} = \left\{ f(z) \in \mathcal{A} : \Re \left( \frac{zf'(z)}{g(z)} \right) > k \left( \frac{zf'(z)}{g(z)} - 1 \right), g(z) \in k-\mathcal{ST}, z \in E \right\}; \]  

(2)

for more detail see [4–6].

In [7], the conic domain \( \Omega_{k, \gamma} \) with complex order is defined as

\[ \Omega_{k, \gamma} = \gamma \Omega_k + (1 - \gamma), \quad 0 < \Re \gamma \leq k + 1, \]  

(3)

where

\[ \Omega_k = \left\{ u + iv : u > k \sqrt{(u - 1)^2 + v^2} \right\}. \]  

(4)

The domain \( \Omega_{k, \gamma} \) is elliptic for \( k > 1 \), hyperbolic when \( 0 < k < 1 \), parabolic for \( k = 1 \), and right half plane when \( k = 0 \). The functions which play the role of extremal functions for the conic regions of complex order are given as

\[ p_{k, \gamma}(z) = \begin{cases} 
 1 + \frac{(2\gamma - 1)z}{1 - z}, & k = 0, \\
 1 + \frac{2\gamma}{\pi^2} \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, & k = 1, \\
 1 + \frac{2\gamma}{k^2 - 1} \sinh^2 \left( \frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z}, & 0 < k < 1, \\
 1 + \frac{\gamma}{k^2 - 1} \sin \left( \frac{\pi}{2R(t)} \int_0^{\sqrt{u(z)/\sqrt{t}}} \frac{1}{\sqrt{1-x^2}} \sqrt{1-(tx)^2} \right) dx, & k > 1,
\end{cases} \]  

(5)

where \( u(z) = (z - \sqrt{t})/(1 - \sqrt{t}z) \), \( t \in (0, 1) \), \( z \in E \), and \( z \) is chosen such that \( k = \cosh(\pi R(t)/4R(t)) \), where \( R(t) \) is the
Legendre’s complete elliptic integral of the first kind and $R(\tau)$ is complementary integral of $R(t)$, see [1, 2].

Let $P = \{ p(z) : p(0) = 1 \text{ and } \text{Re} \ p(z) > 0, \ z \in E \}$ be the class of functions with positive real part, and let $k - P(\gamma)$

$$k - P_m(\gamma) = \left\{ p(z) : p(0) = 1 \text{ and } \exists p_1(z), p_2(z) \text{ in } k - P(\gamma) \text{ such that } p_1(z) - \left( \frac{m}{4} - \frac{1}{2} \right) p_2(z), z \in E \right\}.$$  \hfill (6)

Note that $k - P_2(\gamma) = k - P(\gamma)$ and $0 - P_m(0) = P_m$, the class introduced and studied by Pinchuk [9].

We define the following class:

$$k - \mathcal{U}_m(\gamma) = \left\{ f(z) \in \mathcal{A} : f'(z) \in P, g(z) \in k - \mathcal{U}_m(\gamma), z \in E \right\},$$  \hfill (7)

where

$$k - \mathcal{U}_m(\gamma) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in k - P_m(\gamma), z \in E \right\}.$$  \hfill (8)

Geometrically, a function $f(z) \in \mathcal{U}_m(\gamma)$ means that the functional $1 + \left( \frac{zf''(z)}{f'(z)} \right)$ takes all the values in conic domain $\Omega_{k,\gamma}$ and its boundary rotation is at most $\pi m \gamma$. We note that class $k - \mathcal{U}_m(\gamma)$ coincides with already known classes of analytic functions by choosing special values for the involved parameters. For example, for $k = 0, \gamma = 1$, we have the class $\mathcal{F}_m$ introduced and studied by Noor [10], and further along with this by taking $m = 2$, we obtain the well-known class $\mathcal{K}$ of close-to-convex functions. The purpose of this paper is to investigate some interesting properties of class $k - \mathcal{U}_m(\gamma)$. For this, we require the following results.

**Lemma 1.** A function $f \in k - \mathcal{U}_m(\gamma)$ if and only if

(i) $f'(z) = \left[ f_1(z) \right]^{(1+k)}$, $f_1(z) \in \mathcal{V}'_m$

(ii) there exist two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that

$$f'(z) = \left[ \frac{(s_1(z)/z)^{(k/k)+(1/2)}}{(s_2(z)/z)^{(k/k)-(1/2)}} \right]^{(1+k)}.$$  \hfill (9)

The above lemma can be proved by using the similar procedure as in [11]; also see [8].

**Lemma 2** (see [12]). Let $h \in P$ with $z = re^{\theta}$. Then,

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 \, d\theta \leq \frac{1 + 3r^2}{1 - r^2}.$$  \hfill (10)

2. Some Properties of the Class $k - \mathcal{U}_m(\gamma)$

In this section, we provide some of the interesting properties of class $k - \mathcal{U}_m(\gamma)$ such as radius of convexity problem, arc length, and growth rate of its coefficients. The following theorem is readily seen when we proceed on similar lines as in [13].

**Theorem 3.** The function $f(z) \in k - \mathcal{U}_m(\gamma)$ if and only if

$$f'(z) = \frac{f_1(z)^{(m/4)+(1/2)} \gamma}{f_2(z)^{(m/4)-(1/2)} \gamma},$$  \hfill (11)

where $f_1(z)$ and $f_2(z)$ are close-to-convex functions.

**Theorem 4.** Let $f \in k - \mathcal{U}_m(\rho, \gamma)$ in $E$. Then, $f \in C$ for $|z| < r_0$, where

$$r_0 = \frac{2(1+k)}{m \gamma + 2k + \sqrt{(m \gamma + 2k + 2)^2 - 4(1+k)(2\gamma - k - 1)}}.$$  \hfill (12)

This result is sharp.

**Proof.** We can write

$$f'(z) = g'(z) h(z), \quad g(z) \in k - \mathcal{U}_m(\gamma), \quad h(z) \in P.$$  \hfill (13)

Using Lemma 1(ii), we get

$$f'(z) = \left[ \frac{(s_1(z)/z)^{(m/4)+(1/2)}}{(s_2(z)/z)^{(m/4)-(1/2)}} \right]^{(1+k)} h(z),$$  \hfill (14)

where $s_1$ and $s_2$ are starlike functions. Logarithmic differentiation of (14) gives us

$$zf''(z)$$  \hfill (15)

$$= \frac{1}{1+k} \left[ -1 + \left( \frac{m}{4} + \frac{1}{2} \right) \frac{zs_1(z)}{s_1(z)} - \left( \frac{m}{4} - \frac{1}{2} \right) \frac{zs_2(z)}{s_2(z)} \right] + \frac{zh'(z)}{h(z)}.$$
This implies that
\[
1 + \frac{zf''(z)}{f'(z)} = \frac{1+k-\gamma}{1+k} + \frac{\gamma}{1+k}\left[\left(\frac{m}{4} + \frac{1}{2}\right) \frac{z s'_1(z)}{s_1(z)} - \left(\frac{m}{4} - \frac{1}{2}\right) \frac{z s'_2(z)}{s_2(z)}\right] + \frac{zh'(z)}{h(z)}.
\]

Now using distortion results for the class \(P\), we have
\[
\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \frac{1+k-\gamma}{1+k + \gamma}\left[\left(m/4 + \frac{1}{2}\right) \frac{1-r}{1+r} - \left(m/4 - \frac{1}{2}\right) \frac{1+r}{1-r}\right] - 2r \frac{1-r}{1-r^2}
= \frac{(1+k-\gamma)(1-r^2) + y\left[1 + r^2 - mr\right] - 2r(1+k)}{(1+k)(1-r^2)}.
\]

The right hand side of (17) is positive for \(|z| < r_0\), where \(r_0\) is given by (12). The sharpness can be viewed from the function \(f_0 \in k - \mathcal{UT}_m(\gamma)\), given by
\[
f'_0(z) = \frac{(1+z)^{((m+2)/(1+1))(1/(1+k))}}{(1-z)^{((m+2)/(1+1))(1/(1+k))}}, \quad z \in E.
\]

We note the following interesting special cases:
(i) For \(\gamma = 1\), we have the radius of convexity for class \(k - \mathcal{UT}_m\).
(ii) For \(\gamma = 1\) and \(k = 0\), we have the radius of convexity for class \(\mathcal{T}_m\), proved by Noor [10].
(iii) For \(\gamma = 1\), \(k = 0\) and \(m = 2\), we have radius of convexity for close-to-convex functions which is well known.

\[\square\]

**Theorem 5.** Let \(f \in k - \mathcal{UT}_m(\gamma)\) with \(k \geq 0\), \(m \geq 2\), and \((m+2)/(1+k)) \text{ Re } \gamma > 1\). Then,
\[
L_r(f) \leq \mathcal{A}(k, \gamma, m)\left(\frac{1}{1-r}\right)^{(1/2)((m+2)/(1+k)) \text{ Re } \gamma}.
\]

The exponent \((1/2)((m+2)/(1+k)) \text{ Re } \gamma\) is sharp.

**Proof.** Let \(f \in k - \mathcal{UT}_m(\rho, \gamma)\). Then, there exists \(g(z) \in k - \mathcal{UT}_m(\gamma)\) such that
\[
f'(z) = g'(z)h(z), \quad h \in P.
\]

From the definition of \(k - \mathcal{UT}_m(\gamma)\), one can deduce that \(g(z) \in k - \mathcal{UT}_m(1)\) implies that \(g(z) \in \mathcal{V}_m(k/(k+1))\).

Now using (20), Lemma 1(ii), and distortion theorems for starlike functions, we have
\[
L_r(f) \leq \int_0^{2\pi} |zg'(z)|h(z)|\,d\theta,
\]
\[
g(z) \in k - \mathcal{UT}_m(\gamma), \quad h(z) \in P
\]
\[
= \int_0^{2\pi} \left| \frac{z(L_1(z)/z_2(\gamma))^{((m+2)/(1+1))(1/(1+k))\gamma}}{(L_2(z)/(1+1)(1+k))\gamma}\right| \,d\theta,
\]
\[
\times |h(z)| \,d\theta.
\]

By using Hölder’s inequality, this gives
\[
L_r(f) \leq 2^{((m+2)/(1+1))(1/(1+k)) \text{ Re } \gamma - 1} \times \left(\frac{1}{2\pi} \int_0^{2\pi} |z|^{((m+2)/(1+1))(1/(1+k))\gamma} \,d\theta \right)^{1/2}
\]
\[
\times \left(\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 \,d\theta \right)^{1/2}.
\]

Since \((m+2)/(1+k)) \text{ Re } \gamma > 1\), therefore subordination for starlike functions and Lemma 2 give us
\[
L_r(f) \leq \mathcal{A}(k, \gamma, m)\left(\frac{1}{1-r}\right)^{(1/2)((m+2)/(1+k)) \text{ Re } \gamma}.
\]

The function \(F_0(z) \in k - \mathcal{UT}_m(\gamma)\) is defined by
\[
F_0'(z) = G_0'(z)h_0(z),
\]
where
\[
G_0'(z) = \frac{(1+z)^{(m+2)/(1+1)(1+k))\gamma}}{(1-z)^{(m+2)/(1+1)(1+k))\gamma}}, \quad h_0(z) = \frac{1+z}{1-z}.
\]

Some special choices in the above theorem give us the following interesting results.

**Corollary 6.** Let \(f \in k - \mathcal{UT}_2(1)\). Then
\[
L_r(f) \leq \mathcal{A}(k)\left(\frac{1}{1-r}\right)^{(m+2)/(1+k)}.
\]

**Corollary 7.** Let \(f(z) \in \mathcal{T}_m\). Then
\[
L_r(f) \leq \mathcal{A}(m)\left(\frac{1}{1-r}\right)^{(m+2)/(1+k)}.
\]
Coefficient Growth Problems. The problem of growth rate and asymptotic behavior of coefficients is well known. In the next results, we study these problems for class $\mathcal{U} \mathcal{T}_m(\gamma)$ by varying different parameters.

**Theorem 8.** Let $f \in k - \mathcal{U} \mathcal{T}_m(\gamma)$ with $k \geq 0$, $m \geq 2$ and $((m+2)/(1+k)) \text{Re} \gamma > 1$. Then

$$|a_n| = O(1)^{\left((m/2)+1\right)/\left((\text{Re} \gamma/(1+k))^{-1}\right)}, \quad (n \to \infty).$$

(28)

The exponent is sharp.

**Proof.** With $z = re^{i\theta}$, Cauchy's theorem gives us

$$m n_a = \frac{1}{2\pi n} \int_0^{2\pi} |z f'(z)| \, d\theta = \frac{1}{2\pi n} L_r (f), \quad z = re^{i\theta}.$$  

(29)

Using Theorem 5 and putting $r = 1 - (1/n)$, we obtain the required result. The sharpness follows from the function $F_0$ defined by the relation (24). \hfill $\square$

**Corollary 9.** Let $f \in k - \mathcal{U} \mathcal{T}_m(1)$, and let it be of the form (1). Then, for $n > 3$, $k \geq 2$, one has

$$|a_n| = O(1)^{\left((m/2)+1\right)/\left((1+k)\right)}.$$  

(30)

For $k = 0$, in the above corollary, we have growth rate of coefficients problem for functions in class $\mathcal{T}_m$, and, for $k = 0$, $m = 2$ gives us the growth rate of coefficients for close-to-convex functions, which are well known.

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**References**


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