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*Research Article*

# Arc Length Inequality for a Certain Class of Analytic Functions Related to Conic Regions

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In our present investigation, we introduce a subclass of analytic function associated with conic regions which is a form of generalized close-to-convexity. The arc-length inequality for a class of analytic function is well known. We derive this inequality for the newly defined class and also study some of its interesting consequences.

## 1. Introduction

Let $\mathscr{A}$ denote the class of functions $f$:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disc $\mathscr{U} = \{z : |z| < 1\}$. Let $\mathscr{S}$ denote the class of all functions in $\mathscr{A}$ which are univalent. Also let $\mathscr{S}^*$, $\mathscr{C}$, and $\mathscr{K}$ be the well-known subclasses of $\mathscr{A}$ consisting of all functions which are, respectively, of starlike, convex, and close-to-convex.

Kanas and Wisniowska [1, 2] studied the classes of $k$-uniformly convex denoted by $k-\mathscr{UCV}$ and the corresponding class $k-\mathscr{ST}$ related by the Alexandar type relation. Later Acu [3] considered the class $k$-uniformly close-to-convex denoted by $k-\mathscr{UK}$ to be defined as

$$k-\mathscr{UK} = \left\{ f(z) \in \mathscr{A} : \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > k\left|\frac{zf'(z)}{g(z)} - 1\right|, \; g(z) \in k-\mathscr{ST}, z \in E \right\}; \tag{2}$$

for more detail see [4–6].

In [7], the conic domain $\Omega_{k,\gamma}$ with complex order is defined as

$$\Omega_{k,\gamma} = \gamma\Omega_k + (1-\gamma), \quad 0 < \operatorname{Re}\gamma \le k+1, \tag{3}$$

where

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{4}$$

The domain $\Omega_{k,\gamma}$ is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic for $k = 1$, and right half plane when $k = 0$. The functions which play the role of extremal functions for the conic regions of complex order are given as

$$p_{k,\gamma}(z) = \begin{cases} \dfrac{1+(2\gamma-1)z}{1-z}, & k = 0, \\ 1 + \dfrac{2\gamma}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1, \\ 1 + \dfrac{2\gamma}{1-k^2}\sinh^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\operatorname{arctan}h\sqrt{z}\right], & 0 < k < 1, \\ 1 + \dfrac{\gamma}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{u(z)/\sqrt{t}} \dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) & \\ \quad + \dfrac{\gamma}{k^2-1}, & k > 1, \end{cases} \tag{5}$$

where $u(z) = (z-\sqrt{t})/(1-\sqrt{tz})$, $t \in (0,1)$, $z \in E$, and $z$ is chosen such that $k = \cosh(\pi R'(t)/4R(t))$, where $R(t)$ is the

Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, see [1, 2].

Let $P = \{p(z) : p(0) = 1 \text{ and } \operatorname{Re} p(z) > 0,\ z \in E\}$ be the class of functions with positive real part, and let $k - P(\gamma)$ be the subclass of $P$ containing the functions $p(z)$, such that $p(z) \prec p_{k,\gamma}(z)$. Motivated from Noor's work [8], we extend class $k - P(\gamma)$ to class $k - P_m(\gamma)$, $m \geq 2$ which is defined as

$$k - P_m(\gamma) = \left\{ p(z) : \begin{array}{c} p(0) = 1 \text{ and } \exists p_1(z),\ p_2(z) \text{ in } k - P(\gamma) \text{ such that} \\ p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), z \in E \end{array} \right\}. \tag{6}$$

Note that $k - P_2(\gamma) = k - P(\gamma)$ and $0 - P_m(0) = P_m$, the class introduced and studied by Pinchuk [9].

We define the following class:

$$k - \mathcal{UT}_m(\gamma) = \left\{ f(z) \in \mathcal{A} : \frac{f'(z)}{g'(z)} \in P, g(z) \in k - \mathcal{UV}_m(\gamma), z \in E \right\}, \tag{7}$$

where

$$k - \mathcal{UV}_m(\gamma) = \left\{ f(z) \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in k - P_m(\gamma), z \in E \right\}. \tag{8}$$

Geometrically, a function $f(z) \in \mathcal{UV}_m(\gamma)$ means that the functional $1 + (zf''(z)/f'(z))$ takes all the values in conic domain $\Omega_{k,\gamma}$ and its boundary rotation is at most $m\pi$. We note that class $k$–$\mathcal{UT}_m(\gamma)$ coincides with already known classes of analytic functions by choosing special values for the involved parameters. For example, for $k = 0, \gamma = 1$, we have the class $\mathcal{T}_m$ introduced and studied by Noor [10], and further along with this by taking $m = 2$, we obtain the well-known class $\mathcal{K}$ of close-to-convex functions. The purpose of this paper is to investigate some interesting properties of class $k - \mathcal{UT}_m(\gamma)$. For this, we require the following results.

**Lemma 1.** *A function $f \in k - \mathcal{UV}_m(\gamma)$ if and only if*

(i) $f'(z) = [f_1'(z)]^{\gamma/(1+k)}$, $f_1(z) \in \mathcal{V}_m$,

(ii) *there exist two normalized starlike functions $s_1(z)$ and $s_2(z)$ such that*

$$f'(z) = \left[ \frac{(s_1(z)/z)^{((k/4)+(1/2))}}{(s_2(z)/z)^{((k/4)-(1/2))}} \right]^{\gamma/(1+k)}. \tag{9}$$

The above lemma can be proved by using the similar procedure as in [11]; also see [8].

**Lemma 2** (see [12]). *Let $h \in P$ with $z = re^{i\theta}$. Then,*

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \leq \frac{1 + 3r^2}{1 - r^2}. \tag{10}$$

## 2. Some Properties of the Class $k$–$\mathcal{UT}_m(\gamma)$

In this section, we provide some of the interesting properties of class $k - \mathcal{UT}_m(\gamma)$ such as radius of convexity problem, arc length, and growth rate of its coefficients. The following theorem is readily seen when we proceed on similar lines as in [13].

**Theorem 3.** *The function $f(z) \in k - \mathcal{UT}_m(\gamma)$ if and only if*

$$f'(z) = \frac{(f_1(z))^{((m/4)+(1/2))\gamma}}{(f_2(z))^{((m/4)-(1/2))\gamma}}, \tag{11}$$

*where $f_1(z)$ and $f_2(z)$ are close-to-convex functions.*

**Theorem 4.** *Let $f \in k - \mathcal{UT}_m(\rho, \gamma)$ in $E$. Then, $f \in C$ for $|z| < r_0$, where*

$$r_0 = \frac{2(1+k)}{m\gamma + 2k + 2 + \sqrt{[m\gamma + 2k + 2]^2 - 4(1+k)(2\gamma - k - 1)}}. \tag{12}$$

*This result is sharp.*

*Proof.* We can write

$$f'(z) = g'(z) h(z), \quad g(z) \in k - \mathcal{UV}_m(\gamma),\ h(z) \in P. \tag{13}$$

Using Lemma 1(ii), we get

$$f'(z) = \left[ \frac{(s_1(z)/z)^{((m/4)+(1/2))}}{(s_2(z)/z)^{((m/4)-(1/2))}} \right]^{\gamma/(1+k)} h(z), \tag{14}$$

where $s_1$ and $s_2$ are starlike functions. Logarithmic differentiation of (14) gives us

$$\frac{zf''(z)}{f'(z)} = \frac{\gamma}{1+k} \left[ -1 + \left(\frac{m}{4} + \frac{1}{2}\right) \frac{zs_1'(z)}{s_1(z)} - \left(\frac{m}{4} - \frac{1}{2}\right) \frac{zs_2'(z)}{s_2(z)} \right] + \frac{zh'(z)}{h(z)}. \tag{15}$$

This implies that

$$1+\frac{zf''(z)}{f'(z)} = \frac{1+k-\gamma}{1+k}+\frac{\gamma}{1+k}\left[\left(\frac{m}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)} - \left(\frac{m}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)}\right]+\frac{zh'(z)}{h(z)}. \tag{16}$$

Now using distortion results for the class $P$, we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geq \frac{1+k-\gamma}{1+k}+\frac{\gamma}{1+k}\left[\left(\frac{m}{4}+\frac{1}{2}\right)\frac{1-r}{1+r} - \left(\frac{m}{4}-\frac{1}{2}\right)\frac{1+r}{1-r}\right]-\frac{2r}{1-r^2}\\ &= \frac{(1+k-\gamma)\left(1-r^2\right)+\gamma\left[1+r^2-mr\right]-2r(1+k)}{(1+k)\left(1-r^2\right)}.\end{aligned} \tag{17}$$

The right hand side of (17) is positive for $|z|<r_0$, where $r_0$ is given by (12). The sharpness can be viewed from the function $f_0 \in k-\mathcal{UT}_m(\gamma)$, given by

$$f_0'(z) = \frac{(1+z)^{(\gamma/(k+1))(((m/2)-1)+1)}}{(1-z)^{(\gamma/(k+1))(((m/2)+1)+1)}}, \quad z\in E. \tag{18}$$

We note the following interesting special cases:

(i) For $\gamma = 1$, we have the radius of convexity for class $k-\mathcal{UT}_m$.

(ii) For $\gamma = 1$ and $k = 0$, we have the radius of convexity for class $\mathcal{T}_m$, proved by Noor [10].

(iii) For $\gamma = 1$, $k = 0$ and $m = 2$, we have radius of convexity for close-to-convex functions which is well known.

□

**Theorem 5.** *Let $f \in k-\mathcal{UT}_m(\gamma)$ with $k\geq 0$, $m\geq 2$, and $((m+2)/(1+k))\operatorname{Re}\gamma > 1$. Then,*

$$L_r(f) \leq A(k,\gamma,m)\left(\frac{1}{1-r}\right)^{(1/2)((m+2)/(1+k))\operatorname{Re}\gamma}. \tag{19}$$

*The exponent $(1/2)((m+2)/(1+k))\operatorname{Re}\gamma$ is sharp.*

*Proof.* Let $f\in k-\mathcal{UT}_m(\rho,\gamma)$. Then, there exists $g(z)\in k-\mathcal{UV}_m(\gamma)$ such that

$$f'(z) = g'(z)h(z), \quad h\in P. \tag{20}$$

From the definition of $k-\mathcal{UV}_m(\gamma)$, one can deduce that $g(z)\in k-\mathcal{UV}_m(1)$ implies that $g(z)\in\mathcal{V}_m(k/(k+1))$.

Now using (20), Lemma 1(ii), and distortion theorems for starlike functions, we have

$$\begin{aligned}L_r(f) &= \int_0^{2\pi}\left|zg'(z)h(z)\right|d\theta,\\ g(z)\in k-\mathcal{UV}_m(\gamma),\ h(z)\in P\\ &= \int_0^{2\pi}\left|z\frac{(s_1(z)/z)^{((m/4)+(1/2))(1/(1+k))\gamma}}{(s_2(z)/z)^{((m/4)-(1/2))(1/(1+k))\gamma}}\right||h(z)|\,d\theta\\ &= \int_0^{2\pi}\left|z^{1-\gamma(1/(k+1))}\frac{(s_1(z))^{((m/4)+(1/2))(1/(1+k))\gamma}}{(s_2(z))^{((m/4)-(1/2))(1/(1+k))\gamma}}\right| \times|h(z)|\,d\theta\\ &\leq \frac{2^{((m/2)-1)(1/(k+1))\operatorname{Re}\gamma}}{r^{((m/4)+(1/2))(1/(k+1))\operatorname{Re}\gamma-1}} \times\int_0^{2\pi}|s_1(z)|^{(1/4)((m+2)/(1+k))\operatorname{Re}\gamma}|h(z)|\,d\theta.\end{aligned} \tag{21}$$

By using Hölder's inequality, this gives

$$\begin{aligned}L_r(f) &\leq \frac{2^{((m/2)-1)(1/(k+1))\operatorname{Re}\gamma}}{r^{((m/4)+(1/2))(1/(k+1))\operatorname{Re}\gamma-1}} \times\left(\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{(1/2)((m+2)/(1+k))\operatorname{Re}\gamma}d\theta\right)^{1/2}\\ &\times\left(\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2d\theta\right)^{1/2}.\end{aligned} \tag{22}$$

Since $((m+2)/(1+k))\operatorname{Re}\gamma > 1$, therefore subordination for starlike functions and Lemma 2 give us

$$L_r(f) \leq A(k,\gamma,m)\left(\frac{1}{1-r}\right)^{(1/2)((m+2)/(1+k))\operatorname{Re}\gamma}. \tag{23}$$

The function $F_0(z)\in k-\mathcal{UT}_m(\gamma)$ is defined by

$$F_0'(z) = G_0'(z)h_0(z), \tag{24}$$

where

$$G_0'(z) = \frac{(1+z)^{((m/2)-1)(1/(1+k))\gamma}}{(1-z)^{((m/2)+1)(1/(1+k))\gamma}}, \qquad h_0(z) = \frac{1+z}{1-z} \tag{25}$$

shows that the exponent is sharp. □

Some special choices in the above theorem give us the following interesting results.

**Corollary 6.** *Let $f\in k-\mathcal{UT}_2(1)$. Then*

$$L_r(f) \leq A(k)\left(\frac{1}{1-r}\right)^{2/(1+k)}. \tag{26}$$

**Corollary 7.** *Let $f(z)\in\mathcal{T}_m$. Then*

$$L_r(f) \leq A(m)\left(\frac{1}{1-r}\right)^{(m/2)+1}. \tag{27}$$

*Coefficient Growth Problems.* The problem of growth rate and asymptotic behavior of coefficients is well known. In the next results, we study these problems for class $k-\mathcal{UT}_m(\gamma)$ by varying different parameters.

**Theorem 8.** *Let $f \in k-\mathcal{UT}_m(\gamma)$ with $k \geq 0$, $m \geq 2$ and $((m+2)/(1+k))\operatorname{Re}\gamma > 1$. Then*

$$|a_n| = O(1)\, n^{((m/2)+1)(\operatorname{Re}\gamma/(1+k))-1}, \quad (n \longrightarrow \infty). \tag{28}$$

*The exponent is sharp.*

*Proof.* With $z = re^{i\theta}$, Cauchy's theorem gives us

$$na_n = \frac{1}{2\pi r^n}\int_0^{2\pi} |zf'(z)|\, d\theta = \frac{1}{2\pi r^n} L_r(f), \quad z = re^{i\theta}. \tag{29}$$

Using Theorem 5 and putting $r = 1-(1/n)$, we obtain the required result. The sharpness follows from the function $F_0$ defined by the relation (24). □

**Corollary 9.** *Let $f \in k-\mathcal{UT}_m(1)$, and let it be of the form (1). Then, for $n > 3$, $k \geq 2$, one has*

$$|a_n| = O(1)\, n^{((m/2)+1)(1/(1+k))-1}. \tag{30}$$

*For $k = 0$, in the above corollary, we have growth rate of coefficients problem for functions in class $\mathcal{T}_m$, and, for $k = 0$, $m = 2$ gives us the growth rate of coefficients for close-to-convex functions, which are well known.*

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