Research Article

A Study on Ricci Solitons in Kenmotsu Manifolds

C. S. Bagewadi, Gurupadavva Ingalahalli, and S. R. Ashoka

Department of Mathematics, Kuvempu University, Shankaraghatta-577 451, Shimoga, Karnataka, India

Correspondence should be addressed to Gurupadavva Ingalahalli; gurupadavva@gmail.com

Received 27 December 2012; Accepted 16 January 2013

Academic Editors: R. Farnsteiner and V. S. Matveev

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We study and obtain results on Ricci solitons in Kenmotsu manifolds satisfying $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \bar{P} = 0$, and $\bar{P}(\xi, X) \cdot S = 0$, where B and $\bar{P}$ are C-Bochner and pseudo-projective curvature tensor.

1. Introduction

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) . A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field, and λ a real scalar such that

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \quad (1)$$

where S is a Ricci tensor of M and $\mathcal{L}_V$ denotes the Lie derivative operator along the vector field V . The Ricci soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively [1]. In this paper, we prove conditions for Ricci solitons in Kenmotsu manifolds to be shrinking, steady, and expanding.

In 1972, Kenmotsu [2] studied a class of contact Riemannian manifolds satisfying some special conditions and this manifold is known as Kenmotsu manifolds. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kaehler manifold N with warping function $f(t) = se^t$, where s is a nonzero constant. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X, Y) \cdot R = 0$ holds, then the manifold is of negative curvature -1 , where R is the curvature tensor of type $(1, 3)$ and $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the tangent space.

The authors in [3–7] have studied Ricci solitons in contact and Lorentzian manifolds. The authors in [8] have obtained some results on Ricci solitons satisfying $R(\xi, X) \cdot \bar{C} = 0$, $P(\xi, X) \cdot \bar{C} = 0$, $H(\xi, X) \cdot S = 0$ and $\bar{C}(\xi, X) \cdot S = 0$ and now we extend the work to $R(\xi, X) \cdot B = 0$, $B(\xi, X) \cdot S = 0$, $S(\xi, X) \cdot R = 0$, $R(\xi, X) \cdot \bar{P} = 0$ and $\bar{P}(\xi, X) \cdot S = 0$.

2. Preliminaries

An n -dimensional differential manifold M is said to be an almost contact metric manifold [9] if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1, 1)$, a vector field ξ , a 1-form η , and a Riemannian metric g compatible with (ϕ, ξ, η, g) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$\eta \circ \phi = 0, \quad \phi \xi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (2)$$

for all vector fields X, Y on M .

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold [2] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \quad (3)$$

From (3), we have

$$\nabla_X \xi = X - \eta(X)\xi, \quad (4)$$

where ∇ denotes the Riemannian connection of g .

In an n -dimensional Kenmotsu manifold, we have

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (5)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (6)$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad (7)$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \quad (8)$$

where R is the Riemannian curvature tensor.

Let (g, V, λ) be a Ricci soliton in an n -dimensional Kenmotsu manifold M . From (4) we have

$$(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)]. \quad (9)$$

From (1) and (9) we get

$$S(X, Y) = -(\lambda + 1)g(X, Y) + \eta(X)\eta(Y). \quad (10)$$

The above equation yields that

$$QX = -(\lambda + 1)X + \eta(X)\xi, \quad (11)$$

$$S(X, \xi) = -\lambda\eta(X), \quad (12)$$

$$r = -\lambda n - (n - 1), \quad (13)$$

where S is the Ricci tensor, Q is the Ricci operator, and r is the scalar curvature on M .

2.1. Example for 3-Dimensional Kenmotsu Manifolds. We consider 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3; z \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be linearly independent given by

$$E_1 = z \frac{\partial}{\partial x}, \quad E_2 = z \frac{\partial}{\partial y}, \quad E_3 = -z \frac{\partial}{\partial z}. \quad (14)$$

Let g be the Riemannian metric defined by $g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0$, $g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$, where g is given by

$$g = \frac{1}{z^2} (dx \otimes dx + dy \otimes dy + dz \otimes dz). \quad (15)$$

The (ϕ, ξ, η) structure is given by

$$\eta = -\frac{1}{z}dz, \quad \xi = E_3 = -z \frac{\partial}{\partial z}, \quad (16)$$

$$\phi E_1 = -E_2, \quad \phi E_2 = E_1, \quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$, $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$, for any vector fields U, W on M . By definition of Lie bracket, we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2. \quad (17)$$

Let ∇ be the Levi-Civita connection; with respect to above metric g is given by Koszula formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) \\ &\quad - Z(g(X, Y)) - g(X, [Y, Z]) \\ &\quad - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned} \quad (18)$$

and by virtue of it we have

$$\begin{aligned} \nabla_{E_1} E_3 &= E_1, & \nabla_{E_2} E_3 &= E_2, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_1} E_2 &= 0, & \nabla_{E_2} E_2 &= -E_3, & \nabla_{E_3} E_2 &= 0, \\ \nabla_{E_1} E_1 &= -E_3, & \nabla_{E_2} E_1 &= 0, & \nabla_{E_3} E_1 &= 0. \end{aligned} \quad (19)$$

Clearly (19) shows that (ϕ, ξ, η, g) satisfies (2), (3), and (4). Thus M is a Kenmotsu manifold.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (20)$$

With the help of (19) and (20), it can be easily verified that

$$\begin{aligned} R(E_1, E_2)E_2 &= -E_1, & R(E_1, E_3)E_3 &= -E_1, \\ R(E_1, E_1)E_1 &= 0, & R(E_2, E_1)E_1 &= -E_2, \\ R(E_2, E_3)E_3 &= -E_2, & R(E_2, E_2)E_2 &= 0, \\ R(E_3, E_1)E_1 &= -E_3, & R(E_3, E_2)E_2 &= -E_3, \\ R(E_3, E_3)E_3 &= 0. \end{aligned} \quad (21)$$

From the above expression of the curvature tensor we obtain

$$\begin{aligned} S(E_1, E_1) &= g(R(E_1, E_2)E_2, E_1) \\ &\quad + g(R(E_1, E_3)E_3, E_1) = -2. \end{aligned} \quad (22)$$

Similarly we have

$$S(E_2, E_2) = S(E_3, E_3) = -2, \quad (23)$$

$$(\mathcal{L}_\xi g)(E_i, E_i) = 2[g(E_i, E_i) - \eta(E_i)\eta(E_i)].$$

Now by $X = Y = E_i$, in (1), where $i = 1, 2, 3$ and by virtue of above equations we get the value of λ which is strictly greater than 0. Thus this is an example of expanding Ricci solitons in Kenmotsu manifolds.

3. Ricci Soliton in a Kenmotsu Manifold Satisfying $R(\xi, X) \cdot B = 0$

Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [10]. A geometric meaning of the Bochner curvature tensor is given by Blair in [11] by using the Boothby-Wang's fibration. In 1969, Matsumoto and Chūman [12] constructed the notion of C-Bochner curvature tensor in a Sasakian manifold and studied its several properties.

The C-Bochner curvature tensor [13] B in M is defined by

$$\begin{aligned}
 B(X, Y)Z &= R(X, Y)Z + \frac{1}{n+3} \\
 &\times [g(X, Z)QY - S(Y, Z)X \\
 &\quad - g(Y, Z)QX + S(X, Z)Y \\
 &\quad + g(\phi X, Z)Q\phi Y - S(\phi Y, Z)\phi X \\
 &\quad - g(\phi Y, Z)Q\phi X + S(\phi X, Z)\phi Y \\
 &\quad + 2S(\phi X, Y)\phi Z + 2g(\phi X, Y)Q\phi Z \\
 &\quad + \eta(Y)\eta(Z)QX - \eta(Y)S(X, Z)\xi \\
 &\quad + \eta(X)S(Y, Z)\xi - \eta(X)\eta(Z)QY] \\
 &- \frac{D+n-1}{n+3} [g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X \\
 &\quad + 2g(\phi X, Y)\phi Z] \\
 &+ \frac{D}{n+3} [\eta(Y)g(X, Z)\xi - \eta(Y)\eta(Z)X \\
 &\quad + \eta(X)\eta(Z)Y - \eta(X)g(Y, Z)\xi] \\
 &- \frac{D-4}{n+3} [g(X, Z)Y - g(Y, Z)X],
 \end{aligned} \tag{24}$$

where $D = (r+n-1)/(n+1)$.

Taking $Z = \xi$ in (24) and using (6), (10), (11), we get

$$B(X, Y)\xi = \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] [\eta(X)Y - \eta(Y)X]. \tag{25}$$

Similarly using (5), (10), (11), (12) in (24), we get

$$\begin{aligned}
 \eta(B(X, Y)Z) &= \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] \\
 &\times [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)].
 \end{aligned} \tag{26}$$

We assume that the condition $R(\xi, X) \cdot B = 0$, then we have

$$\begin{aligned}
 R(\xi, X)B(Y, Z)W - B(R(\xi, X)Y, Z)W \\
 - B(Y, R(\xi, X)Z)W - B(Y, Z)R(\xi, X)W = 0.
 \end{aligned} \tag{27}$$

Using (7) in (27), we get

$$\begin{aligned}
 \eta(B(Y, Z)W)X - g(B(Y, Z)W, X)\xi + g(X, Y)B(\xi, Z)W \\
 - \eta(Y)B(X, Z)W + g(X, Z)B(Y, \xi)W \\
 - \eta(Z)B(Y, X)W + g(X, W)B(Y, Z)\xi \\
 - \eta(W)B(Y, Z)X = 0.
 \end{aligned} \tag{28}$$

By taking an inner product with ξ , we have

$$\begin{aligned}
 \eta(B(Y, Z)W)\eta(X) - g(B(Y, Z)W, X) \\
 + g(X, Y)\eta(B(\xi, Z)W) - \eta(Y)\eta(B(X, Z)W) \\
 + g(X, Z)\eta(B(Y, \xi)W) - \eta(Z)\eta(B(Y, X)W) \\
 + g(X, W)\eta(B(Y, Z)\xi) - \eta(W)\eta(B(Y, Z)X) = 0.
 \end{aligned} \tag{29}$$

By using (25), (26) in (29), we have

$$\begin{aligned}
 \left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] [g(Y, W)g(Z, X) - g(Z, W)g(Y, X)] \\
 - g(B(Y, Z)W, X) = 0.
 \end{aligned} \tag{30}$$

In view of (24) in (30), then we have

$$\begin{aligned}
 &\left[1 - \frac{\lambda}{n+3} + \frac{4}{n+3}\right] [g(Y, W)g(Z, X) - g(Z, W)g(Y, X)] \\
 &- g(R(Y, Z)W, X) \\
 &- \frac{1}{n+3} [g(Y, W)S(Z, X) - S(Z, W)g(Y, X) \\
 &\quad - g(Z, W)S(Y, X) + S(Y, W)g(Z, X) \\
 &\quad + g(\phi Y, W)S(\phi Z, X) - S(\phi Z, W)g(\phi Y, X) \\
 &\quad - g(\phi Z, W)S(\phi Y, X) + S(\phi Y, W)g(\phi Z, X) \\
 &\quad + 2S(\phi Y, Z)g(X, \phi W) + 2g(\phi Y, Z)S(X, \phi W) \\
 &\quad + \eta(W)\eta(Z)S(Y, X) - \eta(X)\eta(Z)S(Y, W) \\
 &\quad + \eta(Y)\eta(X)S(Z, W) - \eta(W)\eta(Y)S(Z, X)] \\
 &- \frac{D}{n+3} [\eta(X)\eta(Z)g(Y, W) - \eta(W)\eta(Z)g(Y, X) \\
 &\quad + \eta(W)\eta(Y)g(Z, X) - \eta(Y)\eta(X)g(Z, W)] \\
 &+ \frac{D+n-1}{n+3} [g(\phi Y, W)g(\phi Z, X) \\
 &\quad - g(\phi Z, W)g(\phi Y, X) \\
 &\quad + 2g(\phi Y, Z)g(X, \phi W)] \\
 &+ \frac{D-4}{n+3} [g(Y, W)g(Z, X) - g(Z, W)g(Y, X)] = 0.
 \end{aligned} \tag{31}$$

Taking $X = Y = e_i$ in (31) and summing over $i = 1, 2, \dots, n$. By virtue of (10), (11), (12), and on simplification, we get

$$\begin{aligned} S(Z, W) &= \left[\frac{-(n+4)\lambda - 2n - 3}{n+3} + \frac{-n^2 - 6n + 8}{n+3} - \frac{r}{n+3} \right] g(W, Z) \\ &+ \left[\frac{(\lambda+1)(n+4) - 2 + n}{n+3} + \frac{r + 4(n-1)}{n+3} \right] \eta(W) \eta(Z). \end{aligned} \quad (32)$$

Putting $Z = W = \xi$ in (32) and by virtue of (10) and (13), we have

$$\lambda = (n-1). \quad (33)$$

Therefore, λ positive that is, the Ricci soliton in Kenmotsu manifold is expanding.

Hence we state the following theorem:

Theorem 1. *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X) \cdot B = 0$ is expanding.*

4. Ricci Soliton in a Kenmotsu Manifolds Satisfying $B(\xi, X) \cdot S = 0$

The condition $B(\xi, X) \cdot S = 0$ implies that

$$S(B(\xi, X)Y, Z) + S(Y, B(\xi, X)Z) = 0. \quad (34)$$

By using (10) in (34), we have

$$\begin{aligned} \eta(Z) \eta(B(\xi, X)Y) - (\lambda+1) g(B(\xi, X)Y, Z) \\ - (\lambda+1) g(Y, B(\xi, X)Z) + \eta(Y) \eta(B(\xi, X)Z) = 0, \end{aligned} \quad (35)$$

the above equation implies that

$$\begin{aligned} [\eta(Z) \eta(B(\xi, X)Y) + \eta(Y) \eta(B(\xi, X)Z)] \\ = (\lambda+1) [g(B(\xi, X)Y, Z) + g(Y, B(\xi, X)Z)]. \end{aligned} \quad (36)$$

By using (24) and (26) in (36), we have

$$\begin{aligned} 2\eta(X) \eta(Y) \eta(Z) \left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)} \right] \\ - \left[1 - \frac{\lambda}{(n+3)} + \frac{4}{(n+3)} \right] \\ \times [g(X, Z) \eta(Y) + g(X, Y) \eta(Z)] = 0. \end{aligned} \quad (37)$$

Put $X = Y = \xi$ in (37) then the equation is identically satisfied and we do not get the value for λ . So, we proceed as follows: Taking $X = Y = e_i$ in (37) and summing over $i = 1, 2, \dots, n$ and by virtue of (13) and $\eta(Z) \neq 0$ conditions, we obtain

$$\lambda = n+7. \quad (38)$$

Therefore, λ is positive that is Ricci soliton in Kenmotsu manifolds satisfying $B(\xi, X) \cdot S = 0$ is expanding.

Hence we can state the following theorem.

Theorem 2. *A Ricci soliton in a Kenmotsu manifold satisfying $B(\xi, X) \cdot S = 0$ is expanding.*

5. Ricci Soliton in a Kenmotsu Manifold Satisfying $S(\xi, X) \cdot R = 0$

Using the following equations:

$$\begin{aligned} S((X, \xi) \cdot R)(U, V)W &= ((X \wedge_S \xi) \cdot R)(U, V)W = (X \wedge_S \xi) R(U, V)W \\ &+ R((X \wedge_S \xi)U, V)W + R(U, (X \wedge_S \xi)V)W \\ &+ R(U, V)(X \wedge_S \xi)W, \end{aligned} \quad (39)$$

where the endomorphism $X \wedge_S Y$ is defined by

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y, \quad (40)$$

we have

$$\begin{aligned} S((X, \xi) \cdot R)(U, V)W &= S(\xi, R(U, V)W)X - S(X, R(U, V)W)\xi \\ &+ S(\xi, U)R(X, V)W - S(X, U)R(\xi, V)W \\ &+ S(\xi, V)R(U, X)W - S(X, V)R(U, \xi)W \\ &+ S(\xi, W)R(U, V)X - S(X, W)R(U, V)\xi. \end{aligned} \quad (41)$$

By using the condition $S(\xi, X) \cdot R = 0$, and by virtue of (10), (12), we have

$$\begin{aligned} -\lambda \eta(R(U, V)W)X &- [- (\lambda+1) g(X, R(U, V)W) + \eta(X) \eta(R(U, V)W)] \xi \\ &- \lambda \eta(U)R(X, V)W \\ &- [- (\lambda+1) g(X, U) + \eta(X) \eta(U)] R(\xi, V)W \\ &- \lambda \eta(V)R(U, X)W \\ &- [- (\lambda+1) g(X, V) + \eta(X) \eta(V)] R(U, \xi)W \\ &- \lambda \eta(W)R(U, V)X \\ &- [- (\lambda+1) g(X, W) + \eta(X) \eta(W)] R(U, V)\xi = 0. \end{aligned} \quad (42)$$

By taking an inner product with ξ and by virtue of (5), (6), (7), and (8), we have

$$\begin{aligned} -(\lambda+1) \eta(X) [g(U, W) \eta(V) - g(V, W) \eta(U)] \\ + \lambda [g(V, W) \eta(X) \eta(U) - g(U, W) \eta(X) \eta(V) \\ - g(U, X) \eta(V) \eta(W) + g(V, X) \eta(U) \eta(W)] \\ + (\lambda+1) g(X, R(U, V)W) \\ + (\lambda+1) g(X, U) [\eta(W) \eta(V) - g(V, W)] \\ + (\lambda+1) g(X, V) [g(U, W) - \eta(W) \eta(U)] \\ + g(V, W) \eta(X) \eta(U) - g(U, W) \eta(X) \eta(V) = 0. \end{aligned} \quad (43)$$

Taking $X = U = e_i$ and summing over $i = 1, 2, \dots, n$, we obtain

$$\begin{aligned} & 2(\lambda + 1)[g(V, W) - \eta(W)\eta(V)] \\ & + (\lambda + 1)S(V, W) - (\lambda + 1)(n - 1)g(V, W) \\ & + (n - 1)\eta(V)\eta(W) = 0. \end{aligned} \quad (44)$$

Taking $V = W = \xi$ in (44) and by virtue of (12), (13), we obtain

$$-\lambda(\lambda + n) = 0. \quad (45)$$

This implies either

$$\lambda = 0 \quad \text{or} \quad \lambda = -n. \quad (46)$$

Therefore for any $\lambda = 0$ or $\lambda = -n$ the Ricci soliton in Kenmotsu manifolds satisfying $S(\xi, X) \cdot R = 0$ is either steady or shrinking.

Hence we can state the following theorem.

Theorem 3. *A Ricci soliton in a Kenmotsu manifold satisfying $S(\xi, X) \cdot R = 0$ is either steady or shrinking.*

6. Ricci Soliton in a Kenmotsu Manifolds

Satisfying $R(\xi, X) \cdot \bar{P} = 0$

The Pseudo-projective curvature tensor $\bar{P}$ is defined by

$$\begin{aligned} \bar{P}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] \\ &- \frac{r}{n} \left(\frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (47)$$

where $a, b \neq 0$ are constants. Taking $Z = \xi$ in (47) and using (6), (10), (11), we get

$$\bar{P}(X, Y)\xi = \left[a + b\lambda + \frac{r}{n} \left(\frac{a}{n-1} + b \right) \right] [\eta(X)Y - \eta(Y)X]. \quad (48)$$

Similarly using (5), (10), (11), (12) in (47), we get

$$\begin{aligned} \eta(\bar{P}(X, Y)Z) &= \left[a + b(\lambda + 1) + \frac{r}{n} \left(\frac{a}{n-1} + b \right) \right] \\ &\times [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \end{aligned} \quad (49)$$

We assume that the condition $R(\xi, X) \cdot \bar{P} = 0$, then we have

$$\begin{aligned} R(\xi, X)\bar{P}(U, V)W - \bar{P}(R(\xi, X)U, V)W \\ - \bar{P}(U, R(\xi, X)V)W - \bar{P}(U, V)R(\xi, X)W = 0. \end{aligned} \quad (50)$$

Using (7) in (50), we find

$$\begin{aligned} & \eta(\bar{P}(U, V)W)X - g(X, \bar{P}(U, V)W)\xi \\ & - \eta(U)\bar{P}(X, V)W + g(X, U)\bar{P}(\xi, V)W \\ & - \eta(V)\bar{P}(U, X)W + g(X, V)\bar{P}(U, \xi)W \\ & - \eta(W)\bar{P}(U, V)X + g(X, W)\bar{P}(U, V)\xi = 0. \end{aligned} \quad (51)$$

By taking an inner product with ξ then we get

$$\begin{aligned} & \eta(\bar{P}(U, V)W)\eta(X) - g(X, \bar{P}(U, V)W) \\ & - \eta(U)\eta(\bar{P}(X, V)W) + g(X, U)\eta(\bar{P}(\xi, V)W) \\ & - \eta(V)\eta(\bar{P}(U, X)W) + g(X, V)\eta(\bar{P}(U, \xi)W) \\ & - \eta(W)\eta(\bar{P}(U, V)X) + g(X, W)\eta(\bar{P}(U, V)\xi) = 0. \end{aligned} \quad (52)$$

By using (48), (49) in (52), we have

$$\begin{aligned} & -g(X, \bar{P}(U, V)W) + \left[a + b(\lambda + 1) + \frac{r}{n} \left[\frac{a}{n-1} + b \right] \right] \\ & \times [g(X, V)g(U, W) - g(X, U)g(V, W)] = 0. \end{aligned} \quad (53)$$

In view of (47) in (53), we have

$$\begin{aligned} & -ag(X, R(U, V)W) \\ & - b[(\lambda + 1)\{g(V, X)g(U, W) - g(V, W)g(U, X)\} \\ & + \eta(V)\eta(W)g(U, X) \\ & - g(V, X)\eta(U)\eta(W)] \\ & + [a + b(\lambda + 1)] \\ & \times [g(X, V)g(U, W) - g(X, U)g(V, W)] = 0. \end{aligned} \quad (54)$$

Taking $X = U = e_i$ in (54) and summing over $i = 1, 2, \dots, n$, and on simplification, we get

$$aS(V, W) = -a(n - 1)g(V, W) - b(n - 1)\eta(V)\eta(W). \quad (55)$$

Putting $V = W = \xi$ in (55) and by virtue of (12), (13), we get the following equation:

$$\lambda = \frac{(n - 1)(a + b)}{a}. \quad (56)$$

Since $(a + b)/a \neq 0$ implies that $\lambda > 0$, that is, the Ricci soliton in Kenmotsu manifold satisfying $R(\xi, X) \cdot \bar{P} = 0$ is expanding, hence we state the following theorem.

Theorem 4. *A Ricci soliton in a Kenmotsu manifold satisfying $R(\xi, X) \cdot \bar{P} = 0$ is expanding.*

7. Ricci Soliton in a Kenmotsu Manifolds Satisfying $\bar{P}(\xi, X) \cdot S = 0$

The condition $\bar{P}(\xi, X) \cdot S = 0$ implies that

$$S(\bar{P}(\xi, X)Y, Z) + S(Y, \bar{P}(\xi, X)Z) = 0. \quad (57)$$

By using (10) in (57), we have

$$\begin{aligned} \eta(Z) \eta(\bar{P}(\xi, X)Y) - (\lambda + 1) g(\bar{P}(\xi, X)Y, Z) \\ - (\lambda + 1) g(Y, \bar{P}(\xi, X)Z) + \eta(Y) \eta(\bar{P}(\xi, X)Z) = 0, \end{aligned} \quad (58)$$

that is,

$$\begin{aligned} [\eta(Z) \eta(\bar{P}(\xi, X)Y) + \eta(Y) \eta(\bar{P}(\xi, X)Z)] \\ = (\lambda + 1) [g(\bar{P}(\xi, X)Y, Z) + g(Y, \bar{P}(\xi, X)Z)]. \end{aligned} \quad (59)$$

By using (47) and (48) in (59), we have

$$\begin{aligned} \left[a + \frac{r}{n} \left[\frac{a}{n-1} + b \right] \right] \\ \times [2\eta(X) \eta(Y) \eta(Z) \\ - g(X, Z) \eta(Y) - g(X, Y) \eta(Z)] = 0. \end{aligned} \quad (60)$$

Put $X = Y = \xi$ in (60); then the equation is identically satisfied and we do not get the value for λ . So, we proceed as follows: taking $X = Y = e_i$, summing over $i = 1, 2, \dots, n$, and by virtue of (13) and $\eta(Z) \neq 0$ conditions we obtain

$$\lambda = \frac{(n-1)^2(a-b)}{n[a+b(n-1)]}. \quad (61)$$

Therefore, if $a = b$ in (61) then $\lambda = 0$; that is, Ricci soliton in Kenmotsu manifolds satisfying $\bar{P}(\xi, X) \cdot S = 0$ is steady. If $a \neq b$ then either $\lambda > 0$ for $a > b$ or $\lambda < 0$ for $a < b$, that is, the Ricci soliton in Kenmotsu manifold satisfying $\bar{P}(\xi, X) \cdot S = 0$ is expanding or shrinking.

Hence we can state the following theorem.

Theorem 5. A Ricci soliton in a Kenmotsu manifolds satisfying $\bar{P}(\xi, X) \cdot S = 0$ is steady for $a = b$, expanding for $a > b$ and shrinking for $a < b$.

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