Research Article

Asymptotic Dichotomy in a Class of Odd-Order Nonlinear Differential Equations with Impulses

Kunwen Wen and Lijun Pan

Department of Mathematics, Jiaying University, Meizhou, Guangdong 514015, China

Correspondence should be addressed to Kunwen Wen; wenkunwen@126.com

Received 26 September 2013; Accepted 2 December 2013

Academic Editor: Ting Chen

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We investigate the oscillatory and asymptotic behavior of a class of odd-order nonlinear differential equations with impulses. We obtain criteria that ensure every solution is either oscillatory or (nonoscillatory and) zero convergent. We provide several examples to show that impulses play an important role in the asymptotic behaviors of these equations.

1. Introduction

Impulsive effect, likewise, exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments in time, involving such fields as medicine and biology, economics, mechanics, electronics, telecommunications, and so forth. It has been observed that the solutions of quite a few first- or second-order impulsive differential equations are either oscillatory or (nonoscillatory and) zero convergent (see, e.g., [1–10]). For example, Bainov et al. studied the oscillation properties of first-order impulsive differential equations with deviating arguments [3]. Especially in [4], Chen investigated oscillations of second-order nonlinear differential equations with impulses, and he proposed that the impulses may change the oscillatory behavior of an equation. Based on [4], the authors were devoted to oscillations of impulsive differential equations (see, e.g., [5–10]). Such a dichotomy may yield useful information in real problems. The implications of this dichotomy are applied to the deflection of an elastic beam [11]. Thus, it is of interest to see whether similar dichotomies occur in different types of impulsive differential equations.

One such type consists of impulsive differential equations which are important in the simulation of processes with jump conditions. But papers devoted to the study of asymptotic behaviors of higher order equations with impulses are quite rare. For this reason, Wen et al. studied in [12]

the dichotomous properties of the following third-order nonlinear differential equation with impulses:

$$\begin{aligned} & \left(r(t) x''(t) \right)' + f(t, x) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ & x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k)), \quad i = 0, 1, 2; \quad k = 1, 2, \dots, \quad (1) \\ & x^{(i)}(t_0^+) = x_0^{(i)}, \quad i = 0, 1, 2, \end{aligned}$$

where $x^{(0)}(t) = x(t)$, $0 \leq t_0 < t_1 < \dots < t_k < \dots$ such that $\lim_{k \rightarrow \infty} t_k = \infty$. On the other hand, in [13], Chen and Wen investigated the oscillatory and asymptotic behaviors for odd-order nonlinear differential equations with impulses of the form

$$\begin{aligned} & x^{(2n+1)}(t) + p(t)x(t) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ & x^{(i)}(t_k^+) = a_k^{[i]}x^{(i)}(t_k), \quad i = 0, 1, \dots, 2n; \quad k = 1, 2, \dots, \quad (2) \\ & x^{(i)}(t_0^+) = x_0^{(i)}, \quad i = 0, 1, \dots, 2n, \end{aligned}$$

where n is a positive integer and $0 \leq t_0 < t_1 < \dots < t_k < \dots$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$. They obtained some interesting results for assuring that every bounded solution of (2) is either oscillatory or nonoscillatory and zero convergent.

In this paper, we will study a class of odd-order nonlinear differential equations with impulses of the form

$$\begin{aligned} (r(t)x^{(2n)}(t))' + f(t, x) &= 0, \quad t \geq t_0, \quad t \neq t_k, \\ x^{(i)}(t_k^+) &= g_k^{[i]}(x^{(i)}(t_k)), \quad i = 0, 1, \dots, 2n; \quad k = 1, 2, \dots, \\ x^{(i)}(t_0^+) &= x_0^{[i]}, \quad i = 0, 1, \dots, 2n, \end{aligned} \quad (3)$$

where $x^{(0)}(t) = x(t)$, n is a positive integer, and $0 \leq t_0 < t_1 < \dots < t_k < \dots$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$,

$$\begin{aligned} x^{(i)}(t_k) &= \lim_{h \rightarrow 0^-} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k)}{h}, \\ x^{(i)}(t_k^+) &= \lim_{h \rightarrow 0^+} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k^+)}{h}. \end{aligned} \quad (4)$$

By a solution of (3), we mean a real function $x = x(t)$ defined on $[t_0, +\infty)$ such that

- (I) $x^{(i)}(t_0^+) = x_0^{[i]}$ for $i = 0, 1, \dots, 2n$;
- (II) $x^{(i)}(t)$, $i = 0, 1, \dots, 2n$, and $(r(t)x^{(2n)}(t))'$ are continuous on $[t_0, +\infty) \setminus \{t_k\}$; for $i = 0, 1, \dots, 2n$, $x^{(i)}(t_k^+)$ and $x^{(i)}(t_k^-)$ exist, $x^{(i)}(t_k^-) = x^{(i)}(t_k)$ and $x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k))$ for any t_k ;
- (III) $x(t)$ satisfies $(r(t)x^{(2n)}(t))' + f(t, x(t)) = 0$ at each point $t \in [t_0, +\infty) \setminus \{t_k\}$.

A solution of (3) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

We will establish oscillatory and asymptotic results of (3) based on combinations of the following conditions.

- (i) $r(t) > 0$ and $r(t)$ is continuous on $[t_0, +\infty)$; $f(t, x)$ is continuous on $[t_0, +\infty) \times (-\infty, +\infty)$, $xf(t, x) > 0$ for $x \neq 0$, and $f(t, x)/\varphi(x) \geq p(t)$, where $p(t)$ is positive and continuous on $[t_0, +\infty)$ and φ is differentiable in $(-\infty, +\infty)$ such that $\varphi'(x) \geq 0$ for $x \in (-\infty, +\infty)$.
- (ii) For $k = 1, 2, \dots$, $g_k^{[i]}(x)$ are continuous in $(-\infty, +\infty)$ and there exist positive numbers $a_k^{[i]}, b_k^{[i]}$ such that

$$a_k^{[i]} \leq g_k^{[i]}(x)/x \leq b_k^{[i]}, \quad i = 0, 1, 2, \dots, 2n. \quad (5)$$

(iii)

$$\begin{aligned} \int_{t_0}^{+\infty} \prod_{t_0 < t_k < s} \frac{a_k^{[i]}}{b_k^{[i-1]}} ds &= +\infty; \quad i = 1, 2, \dots, 2n-1, \\ \int_{t_0}^{+\infty} \frac{1}{r(s)} \prod_{t_0 < t_k < s} \frac{a_k^{[2n]}}{b_k^{[2n-1]}} ds &= +\infty. \end{aligned} \quad (6)$$

Our plan is the following. We first obtain three theorems (Theorems 1–3) to ensure every solution of (3) is either oscillatory or (nonoscillatory and) zero convergent. We will also illustrate our results with several examples. As applications of our results, we state three corollaries (Corollaries 5–7). These corollaries are new even for the special case (2).

2. Main Results

The main results of the paper are as follows.

Theorem 1. Assume that the conditions (i)–(iii) hold. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$, $a_k^{[0]} \geq 1$,

$$\begin{aligned} \sum_{k=1}^{+\infty} (b_k^{[0]} - 1) &< +\infty, \\ \int_{t_0}^{+\infty} \prod_{t_0 < t_k < s} \frac{1}{b_k^{[2n]}} p(s) ds &= +\infty. \end{aligned} \quad (7)$$

Then every solution of (3) is either oscillatory or (nonoscillatory and) zero convergent.

Theorem 2. Assume that the conditions (i)–(iii) hold and that $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$,

$$\begin{aligned} b_k^{[0]} \leq 1, \quad b_k^{[2n]} \leq 1, \quad b_k^{[2n]} \leq \varphi(a_k^{[0]}), \\ \int_{t_0}^{+\infty} p(s) ds = +\infty. \end{aligned} \quad (8)$$

Then every solution of (3) is either oscillatory or (nonoscillatory and) zero convergent.

Theorem 3. Assume that the conditions (i)–(iii) hold and that $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. Suppose further that $b_k^{[2n]} \leq a_k^{[0]}$, $\{\prod_{k=1}^n b_k^{[0]}\}$ is bounded,

$$\begin{aligned} \sum_{k=1}^{+\infty} \max \{ |a_k^{[0]} - 1|, |b_k^{[0]} - 1| \} &< +\infty, \\ \sum_{k=1}^{+\infty} |b_k^{[2n]} - 1| &< +\infty, \\ \int_{t_0}^{+\infty} p(s) ds &= +\infty. \end{aligned} \quad (9)$$

Then every solution of (3) is either oscillatory or (nonoscillatory and) zero convergent.

Remark 4. When $n = 1$, (3) reduces to (1). Our Theorems 1–3 are Theorem 2.1, Theorem 2.3, Theorem 2.4 in [12], respectively. So our results generalize and contain results in [12].

Next, for (2), we will also be able to obtain some new results. It is easy to see that (2) has the form of (3) by setting $r(t) = 1$, $f(t, x(t)) = p(t)x(t)$ and $g_k^{[i]}(x^{(i)}(t_k)) = a_k^{[i]}x^{(i)}(t_k)$ for $i = 0, 1, \dots, 2n$ and $k = 1, 2, \dots$. Let

$$\begin{aligned} a_k^{[i]} &= b_k^{[i]}, \quad i = 0, 1, \dots, 2n; \quad k = 1, 2, \dots, \varphi(x) \\ &= x. \end{aligned} \quad (10)$$

Our Theorems 1–3 can directly lead us to the following corollaries for (2).

Corollary 5. Assume that the conditions (i)–(iii) hold. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$, $a_k^{[0]} \geq 1$,

$$\sum_{k=1}^{+\infty} (a_k^{[0]} - 1) < +\infty, \quad (11)$$

$$\int_{t_0}^{+\infty} \prod_{t_0 < t_k < s} \frac{1}{a_k^{[2n]}} p(s) ds = +\infty.$$

Then every solution of (2) is either oscillatory or (nonoscillatory and) zero convergent.

Corollary 6. Assume that the conditions (i)–(iii) hold. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$,

$$a_k^{[0]} \leq 1, \quad a_k^{[2n]} \leq 1, \quad a_k^{[2n]} \leq a_k^{[0]}, \quad (12)$$

$$\int_{t_0}^{+\infty} p(s) ds = +\infty.$$

Then every solution of (2) is either oscillatory or (nonoscillatory and) zero convergent.

Corollary 7. Assume that the conditions (i)–(iii) hold. Suppose further that $a_k^{[2n]} \leq a_k^{[0]}$, $\{\prod_{k=1}^n a_k^{[0]}\}$ is bounded,

$$\sum_{k=1}^{+\infty} |a_k^{[0]} - 1| < +\infty,$$

$$\sum_{k=1}^{+\infty} |a_k^{[2n]} - 1| < +\infty, \quad (13)$$

$$\int_{t_0}^{+\infty} p(s) ds = +\infty.$$

Then every solution of (2) is either oscillatory or (nonoscillatory and) zero convergent.

Remark 8. We note that the above corollaries for ensuring every solution of (2) to be either oscillatory or tend to zero with fixed sign eventually is also new for (2).

Example 9. Consider the equation

$$x'''(t) + \frac{3}{8}tx^3(t) = 0, \quad t \geq \frac{1}{2}, \quad t \neq k,$$

$$x^{(i)}(k^+) = \left(1 + \frac{1}{k^2}\right)x^{(i)}(k), \quad i = 0, 1, 2; \quad k = 1, 2, \dots,$$

$$x\left(\frac{1}{2}\right) = x_0, \quad x'\left(\frac{1}{2}\right) = x_0^{[1]}, \quad x''\left(\frac{1}{2}\right) = x_0^{[2]}, \quad (14)$$

where $a_k^{[i]} = b_k^{[i]} = (1 + (1/k^2)) \geq 1$ for $i = 0, 1, 2$; $r(t) = 1$, $p(t) = (3/8)t$, $t_k = k$, and $\varphi(x) = x^3$. It is not difficult to see that conditions (i)–(iii) are satisfied. Furthermore,

$$\sum_{k=1}^{+\infty} \max\{|a_k^{[0]} - 1|, |b_k^{[0]} - 1|\} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty,$$

$$\int_{t_0}^{+\infty} \prod_{t_0 < t_k < s} \frac{1}{b_k^{[2]}} p(s) ds = \int_{1/2}^{+\infty} \frac{3}{8} \prod_{1/2 < t_k < s} \frac{k^2}{k^2 + 1} s ds = +\infty. \quad (15)$$

Thus by Theorem 1, every solution of (14) is either oscillatory or (nonoscillatory and) zero convergent.

Example 10. Consider the equation

$$\left((2 - \sin t)g(t)x^{(4)}(t)\right)' + tx^3(t) = 0, \quad t \geq \frac{1}{2}, \quad t \neq k,$$

$$x^{(i)}(k^+) = \left(1 + \frac{1}{k^2}\right)x^{(i)}(k),$$

$$x^{(4)}(k^+) = \left(1 - \frac{1}{k^2}\right)x^{(4)}(k), \quad i = 0, 1, 2, 3; \quad k = 1, 2, \dots,$$

$$x^{(j)}\left(\frac{1}{2}\right) = x_0^{[j]}, \quad j = 0, 1, 2, 3, 4, \quad (16)$$

where $a_k^{[i]} = b_k^{[i]} = (1 + (1/k^2))$ for $i = 0, 1, 2, 3$, $a_k^{[4]} = b_k^{[4]} = (1 - (1/k^2))$; $p(t) = t$, $t_k = k$, $\varphi(x) = x^3$, and

$$r(t) = (2 - \sin t)g(t), \quad (17)$$

where

$$g(t) = \left|t - k - \frac{1}{2}\right| + 1, \quad t \in [k, k+1), \quad k = 1, 2, \dots \quad (18)$$

Here, we do not assume that $r(t)$ is bounded, monotonic, or differentiable. It is not difficult to see that conditions (i)–(iii) are satisfied. Furthermore,

$$b_k^{[4]} = \left(1 - \frac{1}{k^2}\right) \leq \left(1 + \frac{1}{k^2}\right) = a_k^{[0]},$$

$$\sum_{k=1}^{+\infty} \max\{|a_k^{[0]} - 1|, |b_k^{[0]} - 1|\}$$

$$= \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty, \quad (19)$$

$$\sum_{k=1}^{+\infty} |b_k^{[4]} - 1| = \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty$$

$$\int_{t_0}^{+\infty} p(s) ds = \int_{1/2}^{+\infty} \frac{1}{s} ds = +\infty.$$

Thus by Theorem 3, every solution of (16) is either oscillatory or (nonoscillatory and) zero convergent.

Example 11. Consider the equation

$$\begin{aligned} (t^{-1}x^{(4)}(t))' + (t^{-1} - t^{-2})x(t) &= 0, \quad t \geq \frac{1}{2}, \quad t \neq k, \\ x(k^+) &= x(k), \quad x'(k^+) = x'(k), \\ x^{(i)}(k^+) &= \frac{k}{k+1}x^{(i)}(k), \quad i = 2, 3, 4; \quad k = 1, 2, \dots, \\ x^{(j)}\left(\frac{1}{2}\right) &= x_0^{[j]}, \quad j = 0, 1, 2, 3, 4, \end{aligned} \quad (20)$$

where $a_k^{[0]} = b_k^{[0]} = a_k^{[1]} = b_k^{[1]} = 1, a_k^{[i]} = b_k^{[i]} = k/(k+1)$ for $i = 2, 3, 4$; $p(t) = t^{-1} - t^{-2}$, $r(t) = t^{-1}$, $t_k = k$; $\varphi(x) = x$. It is not difficult to see that conditions (i)–(iii) are satisfied. Furthermore,

$$\begin{aligned} b_k^{[2n]} &= \frac{k}{k+1} \leq 1 = \varphi(a_k^{[0]}), \\ \int_{t_0}^{+\infty} p(s) ds &= \int_{1/2}^{+\infty} \left(\frac{1}{s} - \frac{1}{s^2}\right) ds = +\infty. \end{aligned} \quad (21)$$

Thus, by Theorem 2, every solution of (20) is either oscillatory or (nonoscillatory and) zero convergent. But the ordinary differential equation

$$(t^{-1}x^{(4)}(t))' + (t^{-1} - t^{-2})x(t) = 0, \quad (22)$$

has a solution $x(t) = -e^t$ which tends to $-\infty$ as $t \rightarrow +\infty$. This example shows that impulses play an important role in oscillatory and asymptotic behaviors of equations.

3. Proofs

To prove our Theorems, we need the following Lemmas.

Lemma 12 (Lakshmikantham et al. [1]). Assume that

(H₀) $m \in PC'(R^+, R)$ and $m(t)$ is left-continuous at t_k for $k = 1, 2, \dots$

(H₁) for $t_k, k = 1, 2, \dots$ and $t \geq t_0$,

$$\begin{aligned} m'(t) &\leq p(t)m(t) + q(t), \quad t \neq t_k, \\ m(t_k^+) &\leq d_k m(t_k) + b_k, \end{aligned} \quad (23)$$

where $p, q \in PC(R^+, R)$, $d_k \geq 0$, and b_k are real constants. Then for $t \geq t_0$,

$$\begin{aligned} m(t) &\leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s) ds\right) \\ &+ \sum_{t_0 < t_k < t} \left(\prod_{t_k < t_j < t} d_j \exp\left(\int_{t_k}^t p(s) ds\right) \right) b_k \\ &+ \int_{t_0}^t \left(\prod_{s < t_k < t} d_k \right) \exp\left(\int_s^t p(\sigma) d\sigma\right) q(s) ds. \end{aligned} \quad (24)$$

Remark 13. If the inequalities in (23) are reversed, then the inequality in (24) should be reversed as well.

Lemma 14. Suppose that conditions (i)–(iii) hold and $x(t)$ is a solution of (3). One has the following statements:

- (a) if there exists some $T \geq t_0$ such that $x^{(2n)}(t) > 0$ and $(r(t)x^{(2n)}(t))' \geq 0$ for $t \geq T$, then there exists some $T_1 \geq T$ such that $x^{(2n-1)}(t) > 0$ for $t \geq T_1$;
- (b) if there exists $i \in \{1, 2, \dots, 2n-1\}$ and some $T \geq t_0$ such that $x^{(i)}(t) > 0$ and $x^{(i+1)}(t) \geq 0$ for $t \geq T$, then there exists some $T_1 \geq T$ such that $x^{(i-1)}(t) > 0$ for $t \geq T_1$.

Proof. First of all, we will prove the result of (a) to be true. Without loss of generality, we may assume that $x^{(2n)}(t) > 0$ and $(r(t)x^{(2n)}(t))' \geq 0$ for $t \geq t_0$. We assert that there exists some j such that $x^{(2n-1)}(t_j) > 0$ for $t_j \geq t_0$. If this is not true, then for any $t_k > t_0$, we have $x^{(2n-1)}(t_k) \leq 0$. Since $x^{(2n-1)}(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that $x^{(2n-1)}(t) \leq 0$ for $t \geq t_0$. Since $r(t)x^{(2n)}(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that for $t \in (t_1, t_2]$,

$$r(t)x^{(2n)}(t) \geq r(t_1)x^{(2n)}(t_1^+); \quad (25)$$

that is,

$$x^{(2n)}(t) \geq \frac{r(t_1)}{r(t)} x^{(2n)}(t_1^+). \quad (26)$$

In particular,

$$x^{(2n)}(t_2) \geq \frac{r(t_1)}{r(t_2)} x^{(2n)}(t_1^+). \quad (27)$$

Similarly, for $t \in (t_2, t_3]$, we have

$$\begin{aligned} x^{(2n)}(t) &\geq \frac{r(t_2)}{r(t)} x^{(2n)}(t_2^+) \\ &\geq \frac{r(t_2)}{r(t)} a_2^{(2n)} x^{(2n)}(t_2) \\ &\geq \frac{r(t_1)}{r(t)} a_2^{(2n)} x^{(2n)}(t_1^+). \end{aligned} \quad (28)$$

By induction, we know that

$$x^{(2n)}(t) \geq \frac{r(t_1)}{r(t)} \prod_{t_1 < t_k < t} a_k^{(2n)} x^{(2n)}(t_1^+), \quad t \neq t_k. \quad (29)$$

From condition (ii), we have

$$x^{(2n-1)}(t_k^+) \geq b_k^{[2n-1]} x^{(2n-1)}(t_k), \quad t > t_1, \quad k = 2, 3, \dots \quad (30)$$

It follows from (29), (30), and Lemma 12, that for $t \geq t_1$,

$$x^{(2n-1)}(t) \geq \prod_{t_1 < t_k < t} b_k^{[2n-1]} \left\{ x^{(2n-1)}(t_1^+) + x^{(2n)}(t_1^+) r(t_1) \right. \\ \left. \times \int_{t_1}^t \frac{1}{r(s)} \prod_{t_1 < t_k < s} b_k^{[2n-1]} ds \right\}. \quad (31)$$

Note that $a_k^{[i]} > 0$, $b_k^{[i]} > 0$, and the second equality of condition (iii) hold. Thus, we get $x^{(2n-1)}(t) > 0$ for all sufficiently large t , which is contrary to $x^{(2n-1)}(t) \leq 0$ for $t \geq t_0$. So there exists some j such that $t_j \geq t_0$ and $x^{(2n-1)}(t_j) > 0$. Since $x^{(2n-1)}(t)$ is increasing on intervals of the form $(t_{j+\lambda}, t_{j+\lambda+1}]$ for each $\lambda = 0, 1, 2, \dots$, then for $t \in (t_{j+\lambda}, t_{j+\lambda+1}]$, we have

$$x^{(2n-1)}(t) \geq \prod_{t_j \leq t_k < t} a_k^{[2n-1]} x^{(2n-1)}(t_j) > 0. \quad (32)$$

The assertion (a) is thus proved.

Next, we will prove the result of (b) to be true. Without loss of generality, we may assume that $x^{(i)}(t) > 0$ and $x^{(i+1)}(t) \geq 0$ for $t \geq t_0$. We assert there exists some j such that $x^{(i-1)}(t_j) > 0$ for $t_j \geq t_0$. If this is not true, then for any $t_k > t_0$, we have $x^{(i-1)}(t_k) \leq 0$. Since $x^{(i-1)}(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that $x^{(i-1)}(t) \leq 0$ for $t \geq t_0$. By $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$, and $t \in (t_k, t_{k+1}]$, we have that $x^{(i)}(t)$ is nondecreasing on $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have

$$x^{(i)}(t) \geq x^{(i)}(t_1^+). \quad (33)$$

In particular,

$$x^{(i)}(t_2) \geq x^{(i)}(t_1^+). \quad (34)$$

Similarly, for $t \in (t_2, t_3]$, we have

$$x^{(i)}(t) \geq x^{(i)}(t_2^+) \geq a_2^{(i)} x^{(i)}(t_2) \geq a_2^{(i)} x^{(i)}(t_1^+). \quad (35)$$

By induction, we know that

$$x^{(i)}(t) \geq \prod_{t_1 < t_k < t} a_k^{(i)} x^{(i)}(t_1^+), \quad t > t_1, \quad t \neq t_k. \quad (36)$$

From condition (ii), we have

$$x^{(i-1)}(t_k^+) \geq b_k^{(i-1)} x^{(i-1)}(t_k), \quad k = 2, 3, \dots \quad (37)$$

Set $u(t) = -x^{(i-1)}(t)$. Then from (36) and (37), we see that

$$u'(t) \leq - \prod_{t_1 < t_k < t} a_k^{(i)} x^{(i)}(t_1^+), \quad t > t_1, \quad t \neq t_k, \quad (38)$$

$$u(t_k^+) \leq b_k^{(i-1)} u(t_k), \quad k = 2, 3, \dots$$

It follows from Lemma 12 that

$$u(t) \leq u(t_1^+) \prod_{t_1 < t_k < t} b_k^{(i-1)} - x^{(i)}(t_1^+) \int_{t_1}^t \prod_{s < t_k < t} b_k^{(i-1)} \prod_{t_1 < t_k < s} a_k^{(i)} ds \\ = \prod_{t_1 < t_k < t} b_k^{(i-1)} \left\{ u(t_1^+) - x^{(i)}(t_1^+) \int_{t_1}^t \prod_{t_1 < t_k < s} \frac{a_k^{(i)}}{b_k^{(i-1)}} ds \right\}; \quad (39)$$

that is,

$$x^{(i-1)}(t) \geq \prod_{t_1 < t_k < t} b_k^{(i-1)}, \\ \times \left\{ x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+) \int_{t_1}^t \prod_{t_1 < t_k < s} \frac{a_k^{(i)}}{b_k^{(i-1)}} ds \right\}. \quad (40)$$

Note that $a_k^{(i)} > 0$, $b_k^{(i)} > 0$, and the first equality of condition (iii) holds. Thus, we get $x^{(i-1)}(t) > 0$ for all sufficiently large t . The relation $x^{(i-1)}(t) \leq 0$ leads to a contradiction. So there exists some j such that $t_j > t_0$ and $x^{(i-1)}(t_j) > 0$. Then

$$x^{(i-1)}(t_j^+) \geq a_j^{(i-1)} x^{(i-1)}(t_j) > 0. \quad (41)$$

Since $x^{(i)}(t) > 0$, then for every positive integer m , we have that $x^{(i-1)}(t)$ is increasing on $(t_{j+m-1}, t_{j+m}]$. For $(t_j, t_{j+1}]$, we have

$$x^{(i-1)}(t) \geq x^{(i-1)}(t_j^+) > 0. \quad (42)$$

In particular,

$$x^{(i-1)}(t_{j+1}) \geq x^{(i-1)}(t_j^+) > 0. \quad (43)$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x^{(i-1)}(t) \geq x^{(i-1)}(t_{j+1}^+) \geq a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) > 0. \quad (44)$$

By induction, for $(t_{j+m-1}, t_{j+m}]$, we have

$$x^{(i-1)}(t) > 0. \quad (45)$$

Summing up the above discussion, we know that there exists some $T_1 \geq T$ such that

$$x^{(i-1)}(t) > 0, \quad t \geq T_1. \quad (46)$$

The proof of Lemma 14 is complete. $\square$

Remark 15. We may prove in similar manners the following statements.

(a') If we replace the condition (a) in Lemma 14 " $x^{(2n)}(t) > 0$ and $(r(t)x^{(2n)}(t))' \geq 0$ for $t \geq T$ " with " $x^{(2n)}(t) < 0$ and $(r(t)x^{(2n)}(t))' \leq 0$ for $t \geq T$ ", then under conditions (i)–(iii), there exists some $T_1 \geq T$ such that $x^{(2n-1)}(t) < 0$ for $t \geq T_1$.

- (b') If we replace the condition (b) in Lemma 14 " $x^{(i)}(t) > 0$ and $x^{(i+1)}(t) \geq 0$ for $t \geq T$ " with " $x^{(i)}(t) < 0$ and $x^{(i+1)}(t) \leq 0$ for $t \geq T$ ", then under conditions (i)–(iii), there exists some $T_1 \geq T$ such that $x^{(i-1)}(t) < 0$ for $t \geq T_1$.

Lemma 16. Suppose conditions (i)–(iii) hold. Let $x = x(t)$ be a solution of (3).

- (a) If there exists some $T \geq t_0$ such that $x(t) > 0$ and $(r(t)x^{(2n)}(t))' < 0$ for $t \geq T$, then $x^{(2n)}(t) > 0$ for all sufficiently large t .
- (b) If there exists $i \in \{1, 2, \dots, 2n\}$ and some $T \geq t_0$ such that $x(t) > 0$ and $x^{(i)}(t) < 0$ for $t \geq T$, then $x^{(i-1)}(t) > 0$ for all sufficiently large t .

Proof. First of all, we will prove the result of (a) to be true. We assert that $x^{(2n)}(t) > 0$ for any $t_k \geq T$. If this is not true, then there exists some $t_j \geq T$ such that $x^{(2n)}(t_j) \leq 0$. Since $r(t) > 0$ and $r(t)x^{(2n)}(t)$ are strictly decreasing on $(t_{j+m}, t_{j+m+1}]$ for $m = 0, 1, 2, \dots$ and for $t \in (t_j, t_{j+1}]$, we have

$$r(t)x^{(2n)}(t) < r(t_j)x^{(2n)}(t_j^+) \leq a_j^{(2n)}r(t_j)x^{(2n)}(t_j) \leq 0. \quad (47)$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$\begin{aligned} r(t)x^{(2n)}(t) &< r(t_{j+1})x^{(2n)}(t_{j+1}^+) \\ &\leq a_j^{(2n)}a_{j+1}^{(2n)}r(t_j)x^{(2n)}(t_j) \leq 0. \end{aligned} \quad (48)$$

We can easily prove that, for any positive integer $n \geq 2$ and $t \in (t_{j+n}, t_{j+n+1}]$, we have

$$r(t)x^{(2n)}(t) < a_j^{(2n)}a_{j+1}^{(2n)} \dots a_{j+n}^{(2n)}r(t_j)x^{(2n)}(t_j) \leq 0. \quad (49)$$

Hence, $x^{(2n)}(t) < 0$ for $t \geq t_{j+1}$. By the result (a) of Remark 13, for sufficiently large t , we have $x^{(2n-1)}(t) < 0$. Using the result (b) of Remark 13 repeatedly, for all sufficiently large t , we get $x(t) < 0$, which is contrary to $x(t) > 0$ for $t \geq T$. Hence we have $x^{(2n)}(t_k) > 0$ for any $t_k \geq T$. So we get $x^{(2n)}(t) > 0$ for all sufficiently large t .

Next, we will prove the result of (b) to be true. We assert that $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. If this is not true, then there exists some $t_j \geq T$ such that $x^{(i-1)}(t_j) < 0$. Since $x^{(i-1)}(t)$ is strictly monotony decreasing on $(t_{j+n}, t_{j+n+1}]$ for $n = 0, 1, 2, \dots$ and for $t \in (t_j, t_{j+1}]$, we have

$$x^{(i-1)}(t) < x^{(i-1)}(t_j^+) \leq a_j^{(i-1)}x^{(i-1)}(t_j) \leq 0. \quad (50)$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x^{(i-1)}(t) < x^{(i-1)}(t_{j+1}^+) \leq a_j^{(i-1)}a_{j+1}^{(i-1)}x^{(i-1)}(t_j) \leq 0. \quad (51)$$

We can easily prove that for any positive integer $n \geq 2$ and $t \in (t_{j+n}, t_{j+n+1}]$, we have

$$x^{(i-1)}(t) < a_j^{(i-1)} \dots a_{j+n}^{(i-1)}x^{(i-1)}(t_j) \leq 0. \quad (52)$$

Hence, $x^{(i-1)}(t) < 0$ for $t \geq t_{j+1}$. By the result (b') of Remark 8, for sufficiently large t , we have $x^{(i-2)}(t) < 0$. Similarly, by using the result (b') of Remark 8 again, we can conclude that for all sufficiently large t , $x(t) < 0$. That is contrary with $x(t) > 0$ for $t \geq T$. Hence, we have $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. So we get $x^{(i-1)}(t) > 0$ for all sufficiently large t .

The proof of Lemma 16 is complete. $\square$

Lemma 17. Suppose conditions (i)–(iii) hold. Let $x = x(t)$ be a solution of (3). Suppose that $T \geq t_0$ and $x(t) > 0$ for $t \geq T$. Then there exists some $T' \geq T$ and $l \in \{0, 2, 4, \dots, 2n\}$ such that for $t \geq T'$,

$$\begin{aligned} x^{(i)}(t) &> 0, \quad i = 0, 1, \dots, l; \\ (-1)^{(i-l)}x^{(i)}(t) &> 0, \quad i = l+1, \dots, 2n; \end{aligned} \quad (53)$$

$$(r(t)x^{(2n)}(t))' < 0.$$

Proof. Let $x(t) > 0$ for $t \geq T \geq t_0$. By (3) and condition (i), we have

$$(r(t)x^{(2n)}(t))' = -f(t, x) \leq -p(t)\varphi(x) < 0. \quad (54)$$

By the result (a) of Lemma 16, there exists some $T_1 \geq T$ such that $x^{(2n)}(t) > 0$ for $t \geq T_1$. Without loss of generality, we may let $x^{(2n)}(t) > 0$ for $t \geq t_0$. Then $x^{(2n-1)}(t)$ is strictly increasing on $(t_k, t_{k+1}]$. If for any t_k , $x^{(2n-1)}(t_k) < 0$, then $x^{(2n-1)}(t) < 0$ for $t \geq T_1$. If there exists some t_j such that $x^{(2n-1)}(t_j) \geq 0$, since $x^{(2n-1)}(t)$ is strictly monotony increasing and $a_k^{[2n-1]} > 0$, then $x^{(2n-1)}(t) > 0$ for $t > t_j$. Thus, there exists some $T_2 \geq T_1$ such that $x^{(2n-1)}(t) > 0$ for $t \geq T_2$. So one of the following statements holds:

$$(A_1) \quad x^{(2n)}(t) > 0, \quad x^{(2n-1)}(t) > 0, \quad t \geq T_2; \quad (55)$$

$$(B_1) \quad x^{(2n)}(t) > 0, \quad x^{(2n-1)}(t) < 0, \quad t \geq T_2. \quad (56)$$

If (55) holds, then by the result (b) of Lemma 14, $x^{(2n-2)}(t) > 0$ for all sufficiently large t . Using the result (b) of Lemma 14 repeatedly, for all sufficiently large t , we can conclude that

$$x^{(2n)}(t) > 0, \quad x^{(2n-1)}(t) > 0, \dots, x'(t) > 0, \quad x(t) > 0. \quad (57)$$

If (56) holds, by Lemma 16, we have $x^{(2n-3)}(t) < 0$ for all sufficiently large t . Similarly, there exists some $T_3 \geq T_2$ such that one of the following statements holds:

$$(A_2) \quad x^{(2n-2)}(t) > 0, \quad x^{(2n-3)}(t) > 0, \quad t \geq T_3; \quad (58)$$

$$(B_2) \quad x^{(2n-2)}(t) > 0, \quad x^{(2n-3)}(t) < 0, \quad t \geq T_3.$$

Repeating the discussion above, we can eventually get that there exist some $T' \geq T$ and $l \in \{0, 2, 4, \dots, 2n\}$ such that for $t \geq T'$,

$$\begin{aligned} x^{(i)}(t) &> 0, \quad i = 0, 1, \dots, l; \\ (-1)^{(i-1)} x^{(i)}(t) &> 0, \quad i = l+1, \dots, 2n; \\ (r(t) x^{(2n)}(t))' &< 0. \end{aligned} \quad (59)$$

The proof of Lemma 17 is complete. $\square$

Lemma 18 (see [13]). Suppose for $t > 0$, $x = x(t)$ is continuous at $t \neq t_k$, left-continuous at $t = t_k$, and $\lim_{t \rightarrow t_k^+} x(t)$ exists for $k = 1, 2, \dots$. Further, assume that

- (H₂) there exists $\bar{t} \in R^+$, such that $x(t) > 0 (< 0)$ for $t \geq \bar{t}$;
- (H₃) $x(t)$ is nonincreasing (resp., nondecreasing) on $(t_k, t_{k+1}]$ for $k = 1, 2, \dots$;
- (H₄) $\sum_{k=1}^{+\infty} [x(t_k^+) - x(t_k)]$ is convergent.

Then $\lim_{t \rightarrow \infty} x(t) = r$ exists and $r \geq 0$ (resp. ≤ 0).

We now turn to the proof of Theorem 1. Without loss of generality, we may assume that $k_0 = 1$. If (3) has a nonoscillatory solution $x = x(t)$, we may assume that $x(t) > 0$ for $t \geq t_0$. By Lemma 17, there exists a $T \geq t_0$ and $l \in \{0, 2, 4, \dots, 2n\}$ such that for $t \geq T$, (53) holds. Next, we rewrite (53) as

$$\begin{aligned} x'(t) &< 0, \dots, x^{(2n)}(t) > 0, \\ (r(t) x^{(2n)}(t))' &< 0 \text{ for } l = 0; \end{aligned} \quad (60)$$

$$\begin{aligned} x'(t) &> 0, \dots, x^{(l)}(t) > 0, \quad (-1)^{(i)} x^{(i)}(t) > 0, \\ (r(t) x^{(2n)}(t))' &< 0 \text{ for } l \geq 2. \end{aligned} \quad (61)$$

If (60) holds, then we see that the conditions (H₂) and (H₃) of Lemma 18 are satisfied. Furthermore, note that $\sum_{k=1}^{+\infty} (b_k^{[0]} - 1) < +\infty$ and $b_k^{[0]} \geq a_k^{[0]} \geq 1$, then one has

$$\prod_{k=1}^{+\infty} b_k^{[0]} < +\infty. \quad (62)$$

Since $x'(t) < 0$ and $t \geq T$, one obtains for any $t_k > T$,

$$x(t_k) \leq \prod_{T < t_j < t_k} b_j^{[0]} x(T^+). \quad (63)$$

By (62) and (63), we know $\{x(t_k)\}$ is bounded. Thus, there exists $M > 0$ such that $|x(t_k)| \leq M$. It follows from condition (ii) that

$$|x(t_k^+) - x(t_k)| \leq |b_k^{[0]} - 1| |x(t_k)| \leq M (b_k^{[0]} - 1). \quad (64)$$

From (64) and note that $\sum_{k=1}^{+\infty} (b_k^{[0]} - 1)$ is convergent, we know that $\sum_{k=1}^{+\infty} [x(t_k^+) - x(t_k)]$ is convergent. Therefore, the condition (H₄) of Lemma 18 is also satisfied. By Lemma 18, we have $\lim_{t \rightarrow +\infty} x(t) = r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq T$, such that for any $t \geq T_1$, $x(t) > r/2 > 0$.

Note that $\varphi'(x) \geq 0$; hence we may see that $\varphi(x(t)) \geq \varphi(r/2)$ for $t \geq T_1$. Let $\Phi(t) = r(t)x^{(2n)}(t)$. Then $\Phi(t) > 0$ for $t \geq T_1$. By conditions (i), (ii), and (3), one has for $t \geq T_1$,

$$\begin{aligned} \Phi'(t) &= -f(t, x) \leq -p(t) \varphi(x(t)) \\ &\leq -\varphi\left(\frac{r}{2}\right) p(t), \quad t \neq t_k, \\ \Phi(t_k^+) &= r(t_k^+) x^{(2n)}(t_k^+) \leq b_k^{[2n]} r(t_k) x^{(2n)}(t_k) \\ &\leq b_k^{[2n]} \Phi(t_k), \quad t_k \geq T_1. \end{aligned} \quad (65)$$

From (65) and Lemma 12, one gets for $t \geq T_1$,

$$\begin{aligned} \Phi(t) &\leq \Phi(T_1^+) \prod_{T_1 < t_k < t} b_k^{[2n]} - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t \prod_{s < t_k < s} b_k^{[2n]} p(s) ds \\ &= \prod_{T_1 < t_k < t} b_k^{[2n]} \left(\Phi(T_1^+) - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t \prod_{T_1 < t_k < s} \frac{1}{b_k^{[2n]}} p(s) ds \right). \end{aligned} \quad (66)$$

It is easy to see from (7) and (66) that $\Phi(t) < 0$ for sufficiently large t . This is contrary to $\Phi(t) > 0$ for $t \geq T_1$. Thus, $r = 0$; that is, $\lim_{t \rightarrow +\infty} x(t) = 0$.

If (61) holds, then let $\Psi(t) = r(t)x^{(2n)}(t)/\varphi(x(t))$. Then $\Psi(t) > 0$ for $t \geq T$. By (3) and condition (i), one has for $t \geq T$,

$$\begin{aligned} \Psi'(t) &= \frac{-f(t, x(t))}{\varphi(x(t))} - \frac{r(t) x^{(2n)}(t) \varphi'(x(t)) x'(t)}{\varphi^2(x(t))} \\ &\leq \frac{-f(t, x(t))}{\varphi(x(t))} \leq -p(t), \quad t \neq t_k. \end{aligned} \quad (67)$$

From the conditions (i) and (ii) and note that $a_k^{[0]} \geq 1$, we know that

$$\begin{aligned} \Psi(t_k^+) &= \frac{r(t_k^+) x^{(2n)}(t_k^+)}{\varphi(x(t_k^+))} \leq \frac{r(t_k) b_k^{[2n]} x^{(2n)}(t_k)}{\varphi(a_k^{[0]} x(t_k))} \\ &\leq b_k^{[2n]} \frac{r(t_k) x^{(2n)}(t_k)}{\varphi(x(t_k))} = b_k^{[2n]} \Psi(t_k). \end{aligned} \quad (68)$$

From (67), (68), and Lemma 12, one gets for $t \geq T$,

$$\begin{aligned} \Psi(t) &\leq \Psi(T^+) \prod_{T < t_k < t} b_k^{[2n]} - \int_T^t \prod_{u < t_k < u} b_k^{[2n]} p(u) du \\ &= \prod_{T < t_k < t} b_k^{[2n]} \left\{ \Psi(T^+) - \int_T^t \prod_{T < t_k < u} \frac{1}{b_k^{[2n]}} p(u) du \right\}. \end{aligned} \quad (69)$$

It is easy to see from (7) and (69) that $\Psi(t) < 0$ for sufficiently large t . This is contrary to $\Psi(t) > 0$ for $t \geq T$. Thus, every solution of (3) is oscillatory. The proof of Theorem 1 is complete.

We now give the proof of Theorem 2. Without loss of generality, we may assume that $k_0 = 1$. If (3) has a nonoscillatory solution $x = x(t)$, we may assume that $x(t) > 0$ for $t \geq t_0$. By Lemma 17, there exists a $T \geq t_0$ and $l \in \{0, 2, 4, \dots, 2n\}$ such that for $t \geq T$, (53) holds. Next, we may rewrite (53) as (60) and (61).

If (60) holds, then note that $b_k^{[0]} \leq 1$. Since $x(t)$ is decreasing on $(t_{j+l}, t_{j+l+1}]$ for $t_j \geq T$ and each $l = 0, 1, 2, \dots$, one has for $t \in (t_{j+l}, t_{j+l+1}]$,

$$\begin{aligned} x(t) &< x(t_{j+l}^+) \leq \dots \leq x(t_{j+1}) \\ &< x(t_j^+) \leq b_j^{[0]} x(t_j) \leq x(t_j). \end{aligned} \quad (70)$$

So, since $x(t)$ is decreasing and bounded on $(t_j, +\infty)$, we know that $x(t)$ is convergent as $t \rightarrow +\infty$. Let $\lim_{t \rightarrow +\infty} x(t) = r$. Then $r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq T$ such that for $t \geq T_1$, $x(t) > r/2 > 0$. Since $\varphi'(x) \geq 0$, one has $\varphi(x(t)) \geq \varphi(r/2)$. By (3) and condition (i), one has for $t \geq T$,

$$\begin{aligned} (r(t) x^{(2n)}(t))' &= -f(t, x) \leq -p(t) \varphi(x(t)) \\ &\leq -\varphi\left(\frac{r}{2}\right) p(t) < 0, \quad t \neq t_k. \end{aligned} \quad (71)$$

From condition (ii) and noting that $b_k^{[2n]} \leq 1$, one has

$$\begin{aligned} r(t_k^+) x^{(2n)}(t_k^+) &\leq b_k^{[2n]} r(t_k) x^{(2n)}(t_k) \\ &\leq r(t_k) x^{(2n)}(t_k), \quad t_k \geq T_1. \end{aligned} \quad (72)$$

By (71), (72), and Lemma 12, one has for $t \geq T_1$,

$$r(t) x^{(2n)}(t) \leq r(T_1) x^{(2n)}(T_1^+) - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t p(s) ds. \quad (73)$$

It is easy to see from (8) and (73) that $x^{(2n)}(t) < 0$ for sufficiently large t . This is contrary to $x^{(2n)}(t) > 0$ for $t \geq T$. Thus $r = 0$, that is, $\lim_{t \rightarrow +\infty} x(t) = 0$;

If (61) holds, then let $\Psi(t) = r(t)x^{(2n)}(t)/\varphi(x(t))$. We see that $\Psi(t) > 0$ for $t \geq T$. By (3) and the condition (i), one gets for $t \geq T$

$$\begin{aligned} \Psi'(t) &= -\frac{f(t, x(t))}{\varphi(x(t))} - \frac{r(t) x^{(2n)}(t) \varphi'(x(t)) x'(t)}{\varphi^2(x(t))} \\ &\leq -\frac{f(t, x(t))}{\varphi(x(t))} \leq -p(t), \quad t \neq t_k. \end{aligned} \quad (74)$$

From the conditions (i) and (ii), we know that

$$\begin{aligned} \Psi(t_k^+) &= \frac{r(t_k) x^{(2n)}(t_k^+)}{\varphi(x(t_k^+))} \leq \frac{r(t_k) b_k^{[2n]} x^{(2n)}(t_k)}{\varphi(a_k^{[0]} x(t_k))} \\ &\leq \frac{b_k^{[2n]} r(t_k) x^{(2n)}(t_k)}{\varphi(a_k^{[0]}) \varphi(x(t_k))} \leq \Psi(t_k). \end{aligned} \quad (75)$$

From (74), (75), and Lemma 12, one gets for $t \geq T$,

$$\Psi(t) \leq \Psi(T^+) - \int_T^t p(s) ds. \quad (76)$$

It is easy to see from (8) and (76) that $\Psi(t) < 0$ for sufficiently large t . This is contrary to $\Psi(t) > 0$ for $t \geq T$. Thus, every solution of (3) is oscillatory. The proof of Theorem 2 is complete.

Finally, we give the proof of Theorem 3. Without loss of generality, we may assume that $k_0 = 1$. If (3) has a nonoscillatory solution $x = x(t)$, we may assume that $x(t) > 0$ for $t \geq t_0$. By Lemma 17, there exists a $T \geq t_0$ and $l \in \{0, 2, 4, \dots, 2n\}$ such that for $t \geq T$, (53) hold. Next, we rewrite (53) as (60) and (61).

If (60) holds, then we see that the conditions (H_2) and (H_3) of Lemma 18 are satisfied. Furthermore, since $x'(t) < 0$ for $t \geq T$, we see that there exists some $t_i \geq T$ such that for any $t_k > t_i$,

$$x(t_k) \leq \prod_{t_i < t_j < t_k} b_j^{[0]} x(t_i^+). \quad (77)$$

Since $\{\prod_{k=1}^n b_k^{[0]}\}$ is bounded, in view of (77), we know that $\{x(t_k)\}$ is bounded. Thus, there exists $M_1 > 0$, such that $|x(t_k)| \leq M_1$. It follows from the condition (ii) that

$$\begin{aligned} |x(t_k^+) - x(t_k)| &\leq \max\{|a_k^{[0]} - 1|, |b_k^{[0]} - 1|\} |x(t_k)| \\ &\leq M_1 \max\{|a_k^{[0]} - 1|, |b_k^{[0]} - 1|\}. \end{aligned} \quad (78)$$

By (78), we know that $\sum_{k=1}^{+\infty} [x(t_k^+) - x(t_k)]$ is convergent. Therefore, the condition (H_4) of Lemma 18 is also satisfied. By Lemma 18, we know that $\lim_{t \rightarrow +\infty} x(t) = r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq T$, such that for $t \geq T_1$, $x(t) > r/2 > 0$. Since $\varphi'(x) \geq 0$, we see that $\varphi(x(t)) \geq \varphi(r/2)$ for $t \geq T_1$. By $b_k^{[2n]} \leq a_k^{[0]}$, the boundedness of $\{\prod_{k=1}^n b_k^{[0]}\}$ and the condition (ii), we know that $\{\prod_{k=1}^n b_k^{[2n]}\}$ is bounded. Note that $(r(t)x^{(2n)}(t))' < 0$ for $t \geq T_1$. So we may also see that there exists $M_2 > 0$ such that $|r(t_k)x^{(2n)}(t_k)| \leq M_2$. Therefore, one has

$$|(b_k^{[2n]} - 1) r(t_k) x^{(2n)}(t_k)| \leq M_2 |b_k^{[2n]} - 1|. \quad (79)$$

By (3) and the condition (i), one has

$$(r(t) x^{(2n)}(t))' = -f(t, x) \leq -p(t) \varphi(x(t)) < 0, \quad t \neq t_k. \quad (80)$$

Integrating (80) from T_1 to t , it follows from (79) and the condition (ii) that

$$\begin{aligned}
 & r(t) x^{(2n)}(t) - r(T_1) x^{(2n)}(T_1^+) \\
 & \leq \sum_{T_1 < t_k < t} r(t_k) [x^{(2n)}(t_k^+) - x^{(2n)}(t_k)] \\
 & \quad - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t p(s) ds \\
 & \leq \sum_{T_1 < t_k < t} |b_k^{[2n]} - 1| r(t_k) x^{(2n)}(t_k) \\
 & \quad - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t p(s) ds \\
 & \leq \sum_{T_1 < t_k < t} M_2 |b_k^{[2n]} - 1| - \varphi\left(\frac{r}{2}\right) \int_{T_1}^t p(s) ds.
 \end{aligned} \tag{81}$$

Since $\sum_{k=1}^{+\infty} |b_k^{[2n]} - 1|$ is convergent, it is easy to see from (9) and (81) that $x^{(2n)}(t) < 0$ for sufficiently large t . This is contrary to $x^{(2n)}(t) > 0$ for $t \geq T$. Thus $r = 0$; that is, $\lim_{t \rightarrow +\infty} x(t) = 0$.

If (61) holds, then let $\Psi(t) = r(t)x^{(2n)}(t)/\varphi(x(t))$. We see that $\Psi(t) > 0$ for $t \geq T$. Similar to the proof of (76), we also obtain

$$\Psi(t) \leq \Psi(T^+) - \int_T^t p(s) ds. \tag{82}$$

It is easy to see from (9) and (82) that $\Psi(t) < 0$ for sufficiently large t . This is contrary to $\Psi(t) > 0$ for $t \geq T$. Thus, every solution of (3) is oscillatory. The proof of Theorem 3 is complete.

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