Research Article

Finite Time Blowup in a Realistic Food-Chain Model

Rana D. Parshad, 1,2 Hamid Ait Abderrahmane, 2 Ranjit Kumar Upadhyay, 3 and Nitu Kumari 1,4

1 Department of Mathematics and Computer Science, Clarkson University, Potsdam, NY 13699, USA
2 King Abdullah University of Science and Technology, Applied Mathematics and Computational Science, Thuwal 23955-6900, Saudi Arabia
3 Department of Applied Mathematics, Indian School of Mines, Dhanbad, Jharkhand 826004, India
4 School of Basic Sciences, Indian Institute of Technology Mandi, Mandi, Himachal Pradesh 175 001, India

Correspondence should be addressed to Rana D. Parshad; rana.parshad@kaust.edu.sa

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We investigate a realistic three-species food-chain model, with generalist top predator. The model based on a modified version of the Leslie-Gower scheme incorporates mutual interference in all the three populations and generalizes several other known models in the ecological literature. We show that the model exhibits finite time blowup in certain parameter range and for large enough initial data. This result implies that finite time blowup is possible in a large class of such three-species food-chain models. We propose a modification to the model and prove that the modified model has globally existing classical solutions, as well as a global attractor. We reconstruct the attractor using nonlinear time series analysis and show that it possesses rich dynamics, including chaos in certain parameter regime, whilst avoiding blowup in any parameter regime. We also provide estimates on its fractal dimension as well as provide numerical simulations to visualise the spatiotemporal chaos.

1. Introduction

Interaction networks in natural ecosystems can be visualized as consisting of simple units known as food chains or food webs that are made up of a number of species linked by trophic interaction [1]. A food chain model essentially comprises of the predator-prey relationship between interacting species in a given ecosystem [2]. In their seminal work [3], Hastings and Powell for the first time demonstrated that the evolution of species participating in a tritrophic relationship might be chaotic [3]. This led to a great deal of research activity in analyzing the dynamical behavior of food chain models. Upadhyay and Rai [4] provided a new example of a chaotic population system in a simple three-species food chain with Holling type II functional response. This model is different from the Hastings and Powell model, in that it considers a generalist top predator, one that can switch its food source, in the absence of its favorite prey. Letellier and Aziz-Alaoui [5] and Aziz-Alaoui [6] revisited the Upadhyay and Rai model and found that the chaotic dynamics is observed via sequences of period-doubling bifurcation of limit cycles which however suddenly break down and reverse giving rise to a sequence of period-halving bifurcation leading to limit cycles. Upadhyay in [7] next proposed a three-species food-chain model, by incorporating mutual interference, in the original model [4], thus generalizing the models in [3, 4, 8]. Parshad and Upadhyay [9] considered the diffusive form of the model proposed in [7]. Under certain restrictions on the parameter space, they proved global existence of solutions as well as the existence of a finite dimensional global attractor, for the diffusive model. The original three-species model for a generalist top predator [4] and its variants are very rich dynamically and have led to a number of works in the literature [4–9]. They continue to be recently investigated, for rich dynamical behavior, such as turing instability [10]. Despite these rich dynamics, the Parshad and Upadhyay model possesses a drawback, in that there is the possibility
of finite time blowup in the model. Recall in general we say that the \( \| \cdot \|_X \) norm of a solution blows up in finite time if
\[
\lim_{t \to T^{-}} \| u \|_X = +\infty,
\]
where \( X \) is a suitable function space where one looks for the solution \( u \), to the PDE or ODE at hand. Blow-up phenomenon in physical and biological problems has a rich history and has been studied in the context of gas dynamics, chemotaxis, cooperative, and competitive loka-Volterra type systems [11–20]. Understanding blowup in a model can help to determine how robust a system is, as well as providing insight on the characteristics of the corresponding physical or biological phenomena.

As the Parshad and Upadhyay model generalises many of the aforementioned three-species models, our result actually implies finite time blowup in all of those models as well. This, however, has not been reported in the literature. It is well documented in the literature [5] that in the absence of the middle predator the top predator becomes extinct if \( cD_2 < w_3 \). This essentially means that if the middle predator is absent the top predator has to switch its food source. However, it is probably not able to predate efficiently enough, and thus the population cannot be sustained, unless the rate of sexual reproduction is above a certain threshold. If this is not so, that is, \( cD_3 < w_3 \), then the species goes extinct. Also, it grows unboundedly if the inequality reverses. We remark that the situation can be much worse than unbounded growth and that finite time blowup is actually possible.

In the current paper our primary contributions are the following.

(1) We show that the diffusive model of Parshad and Upadhyay [9] can blow up in finite time, for certain parameter regime and large enough initial data, via Theorem 2. This immediately shows the blowup of the diffusive form of the model of Upadhyay and Rai [7]. The temporal model proposed in [4] is also shown to blow up in finite time, in certain parameter regime.

(2) We propose a modification to the diffusive model and show global existence of classical solutions and global attractor, for this model, via Theorems 6 and 14. The modified model is well posed and does not blow up in finite time, for any data or in any parameter regime.

(3) We reconstruct the attractor of the modified model, via nonlinear time series analysis, and show that it also exhibits rich dynamics, including chaos in certain parameter regime, just like the old model, whilst avoiding finite time blowup in any parameter regime. Numerical simulations are also provided to show the blowup in the classical models, as well as spatiotemporal chaos, in the modified model.

This paper is organized as follows. In Section 2, we recount the diffusive three-species food-chain model [9], that generalises several other known models in the literature. Finite time blowup results are summarized in Section 3. We propose a modification to the general model in Section 4 and show global existence of classical solutions and global attractor for this model. Various estimates made here concerning the middle predator and prey species are very similar to [21]. We reconstruct the chaotic attractor of the modified model, using nonlinear time series analysis, and estimate its fractal dimension. This, as well as some numerical simulations, is presented in Section 5. Some important conclusions are discussed in Section 6.

2. Food Chain Model System

In this section, we recount the model from [9]. Consider a situation where a prey population \( x_1 \) is predated by individuals of population \( x_2 \). The population \( x_2 \) in turn serves as a favourite food for individuals of population \( x_3 \). This interaction is represented by the following system of a simple prey-specialist predator-generalist predator interaction with the inclusion of spatial spread:

\[
\begin{align*}
\frac{dx_1}{dt} &= d_1 \Delta x_1 + a_1 x_1 - b_1 x_1^2 - w_0 x_1 x_2^m \left( \frac{x_1}{x_1 + D_0} \right)^{m_1} \\
&= g_1(x_1, x_2, x_3) + d_1 \Delta x_1, \\
\frac{dx_2}{dt} &= d_2 \Delta x_2 - a_2 x_2 + w_3 x_2 x_3^m \left( \frac{x_1}{x_1 + D_1} \right)^{m_1} \\
&- w_3 x_3^m \left( \frac{x_3}{x_2 + D_2} \right)^{m_2} \\
&= g_2(x_1, x_2, x_3) + d_2 \Delta x_2, \\
\frac{dx_3}{dt} &= d_3 \Delta x_3 + c x_3^m - w_3 f_1(x_2) x_3^m \\
&= g_3(x_1, x_2, x_3) + d_3 \Delta x_3,
\end{align*}
\]

where \( a_1, a_2, b_1, c, w_0, w_1, w_2, w_3, D_0, D_1, D_2, D_3 \geq 0, f_1(x_2) = 1/(x_2 + D_1) \in C(R_+). \) The parameters \( m_i \) for \( i = 1, 2, 3 \) are mutual interference parameters that model the intraspecific competition among predators when hunting for prey [22–27]. The problem is posed on \( \Omega \subset \mathbb{R}^2 \). \( \Omega \) is bounded, and \( \partial \Omega \) is assumed to be smooth. We consider the Dirichlet boundary conditions:

\[
x_1 = x_2 = x_3 = 0 \quad \text{on} \quad \partial \Omega.
\]
Note, this is easily generalised to higher dimensions. We next present our blow-up result.

**Theorem 2.** Consider the three-species food-chain model as described via (2)–(4). For \( c > (w_3/D_3) \) and initial data \( x_{30}(x) \) large enough, that is

\[
\int_{\Omega} x_{30} (x) W_1 (x) dx > \left( \frac{\lambda_1 d_3}{(c - w_3/D_3)} \right)^{(m_3 - 1)/m_3},
\]

there exists a finite time \( T^* > 0 \), such that for any \( m_3 > 1 \),

\[
\lim_{t \to T^*} \int_{\Omega} x_3 (x, t) W_1 (x) dx = \infty.
\]

Therefore, \( x_3(x, t) \) is a function such that \(-\Delta x_3(x) = \lambda_1 x_1(x)/\Omega, W_1(x) = 0 \) on \( \partial \Omega \). Furthermore, \( x_3(x) > 0 \) and \( \int_{\Omega} W_1(x) dx = 1 \).

**Proof.** Consider the equation for \( x_3 \) rewritten as

\[
\frac{\partial x_3}{\partial t} + w_3 \frac{x_3^3}{x_2 + D_3} = d_3 \Delta x_3 + c x_3^{m_3}.
\]

Since \( x_2 \) is assumed positive, we obtain

\[
\frac{\partial x_3}{\partial t} + w_3 \frac{x_3^3}{D_3} \geq d_3 \Delta x_3 + c x_3^{m_3}.
\]

Thus,

\[
\frac{\partial x_3}{\partial t} \geq d_3 \Delta x_3 + \left( c - \frac{w_3}{D_3} \right) x_3^{m_3}.
\]

Now we multiply (II) by \( W_1 \), consider \( F(t) = \int_{\Omega} x_3(x, t) W_1(x) dx \), and integrate by parts to obtain

\[
F'(t) = \int_{\Omega} \frac{\partial x_3}{\partial t} W_1(x) dx \\
\geq \int_{\Omega} \left( d_3 \Delta x_3 + \left( c - \frac{w_3}{D_3} \right) x_3^{m_3} \right) W_1(x) dx \\
= -d_3 \lambda_1 F(t) + \left( c - \frac{w_3}{D_3} \right) \int_{\Omega} x_3^{m_3} W_1(x) dx.
\]

Via assumptions on the eigenfunction \( W_1 \), it is seen to be a probability measure. That is, we can define a probability measure \( p(x) = W_1(x) ) \) on \( \Omega \) and \( p(x) = 0 \) on \( \mathbb{R}^2 / \Omega \). Now we consider the convex function \( f(x_3) = x_3^{m_3} \) (note that any \( m_3 > 1 \) works as the \( x_3 \) in question is always positive). Thus, the application of Jensen’s inequality yields

\[
\left( c - \frac{w_3}{D_3} \right) \int_{\Omega} x_3^{m_3} W_1(x) dx \\
\geq \left( c - \frac{w_3}{D_3} \right) \left( \int_{\Omega} x_3(x, t) W_1(x) dx \right)^{m_3} \\
= \left( c - \frac{w_3}{D_3} \right) (F(t))^{m_3}.
\]

3. **Finite Time Blowup**

In this section, we show that (2)–(4) blow up in finite time, for certain parameter regime and large enough initial data. In particular, the equation for \( x_3 \) as described via (4) blows up. To this end we employ the first eigenvalue method for blow up, [38]. First recall Jensen’s inequality with a probability measure.

**Lemma 1.** Let \( p(x) \) be a probability measure on \( \mathbb{R} \), and let \( u \) be a nonnegative function. Then the following holds whenever \( f \) is a convex function:

\[
f \left( \int_{\mathbb{R}} u(x) \ p(x) \right) \leq \int_{\mathbb{R}} f(u(x)) \ p(x) .
\]

\[\text{(6)}\]
Thus, substituting the previous equation into (12) yields
\[ F'(t) \geq -d_3 \lambda_1 F(t) + \left( c - \frac{w_3}{D_3} \right) (F(t))^{m_3}. \] (14)

Simple integration in time of the previous equation yields that \( F(t) \) blows up in finite time; that is
\[ \lim_{t \to T^*} F(t) = \int_{t_0}^T \int_{\Omega} x_3(x,t) W_1(x) \, dx = \infty, \] (15)
where
\[ T^* = \frac{1}{d_3 \lambda_1} \log \left( \frac{c - (w_3/D_3)}{(F(0))^{1-m_3}} \right) \] (16)
and \( T^* \leq T^{**} \).

After some algebra, we obtain that for blowup the data has to satisfy
\[ F(0) = \int_{\Omega} x_3(0) W_1(x) \, dx > \left( x_3(0) \right)^{1/(m_3-1)}. \] (17)
This proves the theorem.

Remark 3. Note that the blowup of \( F(t) \) immediately implies that the \( L^\infty \) norm of \( r \) blows up as follows:
\[ \lim_{t \to T^*} F(t) \leq \lim_{t \to T^*} \|r(x,t)\|_{\infty} \int_{\Omega} W_1(x) \, dx = \lim_{t \to T^*} \|r(x,t)\|_{\infty} = \infty. \] (18)

Remark 4. We also show blow-up in the Upadhyay-Rai model proposed in [4], that is, the temporal case with \( m_3 = 1 \), \( m_2 = 1 \), and \( m_1 = 2 \). Also, for the model in [4], the blow-up time is given by
\[ T^{**} = \frac{1}{d_3 \lambda_1} \log \left( \frac{c - (w_3/D_3)}{F(0) - \lambda_1} \right). \] (19)

Remark 5. We note that finite time blowup is also possible in the same weighted \( L^1(\Omega) \) norm, for Neumann boundary conditions. In this case, the eigenfunction \( W_1 \) will have to be differently defined, that is, satisfying Neumann boundary conditions itself.

4. Modification to the Old Model

The aim of the current section is to introduce an improvement to (2)–(4), which is well posed. The key issue in amending the old model, (2)–(4), is to correct (4). Upadhyay and Rai introduced sexual reproduction in (4), by considering \( cx_3^{1-\epsilon} \). Under this assumption, the number of males and females in the population is assumed to be equal, [5]. Thus, mating takes place as \( c \times (1/2)x_3 \times (1/2)x_3 \), and consequently the population grows at rate \( c \). We point out that this assumption may fail in various structured populations, where due to specific demographics, the numbers of females and males are rarely equal. We modify this by considering the mating term to be \( cx_3^{1-\epsilon} = cx_3 \times x_3^{(1-\epsilon)} \), where we restrict \( 0 < \epsilon < 1 \). This modification may be interpreted as modeling a population where the number of males is different from the number of females (even if very slightly so). If the numbers of males and females are in fact equal or close to, one can choose \( 0 < \epsilon \ll 1 \). Next, the mutual interference parameter is incorporated, by raising the reaction terms to the \( m_3 \) power.

The primary role of the previously stated modification is that now the reaction term in (4) is \( cx_3^{m_3} - w_3 f_1(x_3) x_3^{m_3} \), and because we enforce \( m_3 < m_4 \), standard parabolic theory [31] guarantees global existence. Based on the above, we introduce the following three-species food-chain model, with a generalist top predator,
\[ \frac{\partial x_1}{\partial t} = d_1 \Delta x_1 + a_1 x_1 - b_1 x_1^2 - w_0 x_1^2 \left( \frac{x_1}{x_1 + D_0} \right)^{m_1}, \] (20)
\[ \frac{\partial x_2}{\partial t} = d_2 \Delta x_2 - a_2 x_2 + w_1 x_2^2 \left( \frac{x_1}{x_1 + D_0} \right)^{m_2} - w_2 \left( \frac{x_2}{x_2 + D_2} \right)^{m_3}, \] (21)
\[ \frac{\partial x_3}{\partial t} = d_3 \Delta x_3 + cx_3^{m_3} - w_3 f_1(x_3) x_3^{m_3}. \] (22)

The problem is posed on \( \Omega \subset \mathbb{R}^2 \). \( \Omega \) is bounded, and \( \partial \Omega \) is assumed to be smooth. We consider the Dirichlet boundary conditions
\[ x_1 = x_2 = x_3 = 0 \text{ on } \partial \Omega. \] (23)

Note, the proofs of global existence to follow will work with the Neumann boundary conditions as well. The forms of (2)–(3) are kept essentially the same; however, we assume that \( D_2 = D_1 \), as commonly done in numerical simulations, [5]. We also assume \( 0 < m_2 \leq 1 \). This facilitates the estimates to follow. Restricting \( m_1 \) to the range \( 1 < m_1 < m_3 \) has distinct advantages over the old model (2)–(4). Firstly, finite time blowup is prevented, and this is independent of the sign of \( c - w_3/D_3 \). Secondly, in the absence of \( x_2 \), if \( c - w_3/D_3 < 0 \), the dynamics of (2)–(4) is quite boring, and \( x_3 \) decays to 0, [5, 6]. However, if \( 1 < m_3 < m_4 \), even if \( x_2 = 0 \) and \( c - w_3/D_3 < 0 \), \( x_3 \) can still persist. Thus, this formulation plausibly models a true generalist \( x_3 \). One that does not go extinct, even if its favorite food source does.

4.1. Preliminaries. The system (20)–(22) is an example of a good semilinear parabolic system, and the standard theory for parabolic systems [31] ensures the existence of a non-negative local solution. Next, comparison arguments can be used to derive global existence for initial data in \( L^\infty(\Omega) \). We assume that \( x_1, x_2, \) and \( x_3 \) are nonnegative because they are densities. This assumption ensures that \( x_1 + D_0 > 0, x_1 + D_1 > 0, x_2 + D_2 > 0, \) and \( x_2 + D_3 > 0 \). The nonlinearities are thus well defined. Standard parabolic theory guarantees the uniform boundedness of \( x_3 \). We state the following theorem.
Theorem 6. Consider the three-species food-chain model described by (20)–(22). For any positive initial data \((x_{10}, x_{20}, x_{30}) \in L^2(\Omega)\) and \(1 < m_3 < m_2 < m_1 \leq 1\), there exists a global classical solution \((x_1, x_2, x_3)\) to the system.

Remark 7. In this section, we demonstrate a stronger version of global existence. That is, we derive uniform \(L^\infty(\Omega)\) bounds on the solution with the initial data only in \(L^2(\Omega)\).

The opposite signed nonlinearities here cause problems in making direct \(H^1\) estimates and proving the existence of bounded absorbing sets for the \(x_2\) variable. To circumvent this difficulty, we resort to grouping estimates via appropriate addition of the equations at hand. This technique is very similar to what we adopted in [21]. This condition plays a key role in the dissipation process. If it is satisfied, there is often a global attractor. We will also show the existence of a global attractor for the reaction diffusion system (20)–(22).

Remark 8. Note, in [9] in order to proceed, we need to assume \(c \leq w_2/(h+\nu)\). Thus, our results in [9] are only true in the parameter regime, such that \(\nu < (w_2/h) - D_2\). Furthermore in [9], strong \(L^\infty\) boundedness requirements on the solutions are assumed. Here, we enforce no such requirement, and the methods of analysis are quite different from [9].

Definition 9. Consider a semiflow, \(S(t)\), acting on a reflexive Banach space, \(H\). Then, the global attractor, \(\mathcal{A}\) \(\subset H\), for this semiflow is an object that has the following properties:

(i) \(\mathcal{A}\) is compact in \(H\),
(ii) \(\mathcal{A}\) is invariant; that is, \(S(t) \mathcal{A} = \mathcal{A}\), \(t \geq 0\),
(iii) if \(B\) is bounded in \(M\), then \(\text{dist}_M(S(t)B, \mathcal{A}) \to 0, t \to \infty\).

Next, various preliminaries are presented, detailing the phase spaces of interest and recalling certain standard theories. Let us define our phase spaces of interest as

\[
H = \left[ L^2(\Omega) \right]^3, \quad V = \left[ H^1(\Omega) \right]^3.
\]

To prove the existence of a global attractor, we are required to show [32]

(i) the existence of a bounded absorbing set in the phase space;
(ii) the asymptotic compactness property of the semiflow in question.

These are defined next.

Definition 10 (bounded absorbing set). A bounded set, \(\mathcal{B}\), in a reflexive Banach space, \(H\), is called a bounded absorbing set if, for each bounded subset \(U\) of \(H\), there is a time, \(T = T(U)\), such that \(S(t)U \subset \mathcal{B}\) for all \(t > T\). The number \(T = T(U)\) is referred to as the compactification time for \(S(t)U\). This is essentially the time after which the semiflow compactifies.

Definition 11 (asymptotic compactness). The semiflow \(\{S(t)\}_{t \geq 0} : H \to H\) associated with a dynamical system is said to be asymptotically compact in \(H\) if, for any \(\{u_n(0)\}_{n=1}^\infty\) bounded in \(H\) and a sequence of times \(t_n \to \infty\), \(S(t_n)u_n(0)\) possesses a convergent subsequence in \(H\).

4.2 Global Existence and Existence of Absorbing Sets in \(L^2(\Omega)\).

We begin by multiplying (20) by \(x_1\) and integrating by parts over \(\Omega\). This yields

\[
\frac{1}{2} \frac{d}{dt} \|x_1\|^2 = -d_1 \|\nabla x_1\|^2 + a_1 \|x_1\|^2 - b_1 \|x_1\|^3 - w_0 \int \frac{x_1^{m_1+1} x_2^{m_2}}{(x_1 + D_0)^{m_1}} dx.
\]

Next, using the compact Sobolev embedding, \(H^1(\Omega) \hookrightarrow L^2(\Omega)\), and the positivity of \(x_1\) and \(x_2\), we find that

\[
\frac{d}{dt} \|x_1\|^2 + C_1 \|x_1\|^2 \leq C,
\]

where \(C\) depends explicitly on \(b_1\) and \(a_1\).

Thus, application of Gronwall’s Lemma gives us the following estimate:

\[
\|x_1\|^2 \leq C e^{-C_1 t} \|x_1(0)\|^2 + \frac{C}{C_1}.
\]

Therefore, there exists time \(t_1\) given explicitly by

\[
t_1 = \max \left(0, \frac{\ln \left(\frac{\|x_1(0)\|^2}{C} \right)}{C_1} \right),
\]

such that, for all \(t \geq t_1\), the following estimate holds uniformly:

\[
\|x_1\|^2 \leq 1 + \frac{C_2}{C_1}.
\]

We now make a local in time estimate for \(\nabla x_1\). By integrating (26) in the time interval \([t_1, t_1 + 1]\), we obtain

\[
\int_{t_1}^{t_1 + 1} \|\nabla x_1\|^2 dt \leq \|x_1(t_1)\|^2 + \int_{t_1}^{t_1 + 1} C_2 dt \leq C.
\]

Thus, via a mean value theorem for integrals, there exists a time \(t_2 \in [t_1 + 1, t_1 + 1]\) such that the following estimate holds:

\[
\|\nabla x_1(t_2)\|^2 \leq C.
\]

We now move on to showing the existence of an absorbing set for \(x_2\) in \(L^2(\Omega)\). By multiplying (20) by \(w_1\) and (21) by \(w_0\),
adding the two together, and setting \( w = w_1 x_1 + w_0 x_2 \), we obtain

\[
\frac{d\omega}{dt} = d_1 \Delta \omega + (d_1 - d_2) w_1 \Delta x_1 + w_1 a_1 x_1
\]

\[
- w_1 b_1 x_1^2 - w_0 a_2 x_2 - w_0 b_2 x_2^2.
\]

By multiplying (33) by \( w \), integrating by parts over \( \Omega \), and using Hölder’s inequality followed by Young’s inequality, keeping in mind the positivity of the solutions, we find that

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + d_2 \|\nabla w\|^2
\]

\[
\leq w_1 a_1 \int_\Omega x_1 w \, dx + \int_\Omega (d_2 - d_1) w \nabla x_1 \cdot \nabla w \, dx
\]

\[
\leq C \|w\|^2 + C_2 \|x_1\|^2 + C_1 \|\nabla x_1\|^2 + d_2 \|\nabla w\|^2.
\]

(34)

Thus, by using the compact Sobolev embedding, \( H^1(\Omega) \hookrightarrow L^2(\Omega) \), we obtain

\[
\frac{d}{dt} \|w\|^2 + C \|w\|^2 \leq C_1 \|\nabla x_1\|^2 + C_2 \|x_1\|^2.
\]

(35)

Next, we multiply the previous equation by \( e^{C t} \) and integrate in time from 0 to \( t \) to find

\[
\|w\|^2 \leq e^{-C t} \|w(0)\|^2 + \int_0^t e^{-(t-\tau)} \|\nabla x_1\|^2 \, d\tau + C_1 (x_{10}, t),
\]

(36)

where \( C_1(x_{10}, t) = e^{-C t} \int_0^t e^{-C \tau} \|x_1(\tau)\|^2 \, d\tau + (C/C_1) \). This follows via (30).

We now make the following estimate via time integration of (26), after multiplying through by \( e^{C t} \):

\[
\int_0^t e^{C \tau} \|\nabla x_1\|^2 \, d\tau
\]

\[
\leq C e^{C t} \frac{1}{2} \int_0^t \left( e^{C d} \|x_1\|^2 \right) \, d\tau
\]

\[
= C e^{C t} \frac{1}{2} \left[ e^{C t} \|x_1(t)\|^2 - \|x_{10}\|^2 - \int_0^t e^{C \tau} \|x_1(\tau)\|^2 \, d\tau \right]
\]

\[
\leq C e^{C t} + \|x_{10}\|^2 + \int_0^t e^{C \tau} \|x_1(\tau)\|^2 \, d\tau
\]

\[
\leq C e^{C t} + \|x_{10}\|^2.
\]

(37)

This follows via (30).

We now substitute the above into (36) and use the form of \( C_1(x_{10}, t) \) to obtain

\[
\|w\|^2 \leq e^{-C t} \|w(0)\|^2 + C_1 (x_{10}, t) + C e^{-C t} (C e^{C t} + \|x_{10}\|^2).
\]

(38)

Therefore, there exists time \( t_3 \), given explicitly by

\[
t_3 = \max \left( 0, \frac{\ln \left( \frac{\|w_1\|^2 + \|w_0\|^2}{C} \right)}{C} \right),
\]

(39)

such that, for all \( t \geq t_3 \), the following estimate holds uniformly:

\[
\|x_2\|^2 \leq \frac{1}{C} \|w\|^2 \leq 1 + C + \frac{C}{C_1}.
\]

(40)

Therefore,

\[
\lim_{t \to \infty} \sup \|x_2\|^2 \leq \frac{1}{C} \lim_{t \to \infty} \sup \|w\|^2 \leq C,
\]

(41)

where \( C \) is a constant that is independent of the time and the initial data.

We next derive local in time estimates for \( \nabla x_2 \). We proceed by multiplying (21) by \( x_1 \) and integrating by parts over \( \Omega \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|x_2\|^2 = -d_2 \|\nabla x_2\|^2 - d_1 \|x_2\|^2
\]

\[
+ w_1 \int_\Omega \left( \frac{x_1}{(x_1 + D_2)} \right) x_2^{m_2+1} \, dx
\]

\[
- w_2 \int_\Omega \frac{x_2^{m_2+1}}{(x_2 + D_2)^{m_2}} x_2 \, dx.
\]

(42)

We now use Hölder’s inequality, positivity of solutions, the embedding \( L^2 \hookrightarrow L^{m_2+1} \) (via assumption on \( m_2 \)), and the earlier estimate via (40) to yield

\[
\frac{d}{dt} \|x_2\|^2 + d_2 \|\nabla x_2\|^2 \leq \|x_2\|^{m_2} \|x_2\|^2 \leq C_1.
\]

(43)

By integrating this inequality in the time interval \([t, t + 1] \), for \( t \geq t_3 \), we obtain

\[
\int_t^{t+1} \|\nabla x_2\|^2 \, ds \leq \|x_2\|^{t+1} \int_t^{t+1} C_1 |\Omega| \, ds \leq C.
\]

(44)

The existence of an absorbing set for \( x_3 \) in \( L^2(\Omega) \) follows easily via the form of (22).

Standard estimates imply the existence of a time \( t_5 \) given explicitly by

\[
t_5 = \max \left( 0, \frac{\ln \left( \|x_3(0)\|^{t+1} \right)}{C} \right),
\]

(45)

such that, for all \( t \geq t_5 \), the following estimates hold uniformly:

\[
\|x_3\|^2 \leq 1 + \frac{C_2}{C},
\]

\[
\int_t^{t+1} \|\nabla x_2\|^2 \, ds \leq C.
\]

(46)

We can now state the following lemma.
Lemma 12. Consider \((x_1, x_2, x_3)\) which are solutions to the diffusive three-species food-chain model described via (20)–(22), with \((x_{10}, x_{20}, x_{30})\) \(\in L^2(\Omega)\), \(1 < m_3 < m_4\), and \(0 < m_2 \leq 1\). There exists a time \(t^* = \max(t_1, t_2, t_3)\) and a constant \(C\) independent of time and the initial data and dependent only on the parameters in (20)–(22), such that, for any \(t > t^*\), the following uniform estimates hold:

\[
\|x_1\|_2^2 \leq C, \quad \|x_2\|_2^2 \leq C, \quad \|x_3\|_2^2 \leq C.
\] (47)

We now make the \(H^1\) estimate. We take the inner product of (20) with \(-\Delta x_1\) and integrate by parts over \(\Omega\). Thus, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla x_1\|_2^2 + d_1 \|\Delta x_1\|_2^2 + 2b_1 \int_\Omega x_1 |\nabla x_1|^2 dx = a_1 \|\nabla x_1\|_2^2 + w_0 \int_\Omega \left( \frac{x_1}{x_1 + D_0} \right)^{m_1} (-\Delta x_1)(x_2)^{m_2} dx.
\] (48)

We apply the Cauchy-Schwartz inequality, the Cauchy with epsilon, and Young's inequalities on the last term, along with the embedding \(L^2 \hookrightarrow L^{2m_2}\) (via assumption on \(m_2\)) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla x_1\|_2^2 + d_1 \|\Delta x_1\|_2^2 + 2b_1 \int_\Omega x_1 |\nabla x_1|^2 dx \leq a_1 \|\nabla x_1\|_2^2 + \frac{d_1}{2} \|\Delta x_1\|_2^2 + C_1 \left( \frac{x_1}{x_1 + D_0} \right)^{m_1} \int_\Omega |x_2|^2 dx.
\] (49)

This yields

\[
\frac{1}{2} \frac{d}{dt} \|\nabla x_1\|_2^2 \leq C \|\nabla x_1\|_2^2 + C_1 \|x_2\|_2^2.
\] (50)

We now recall the following lemma.

Lemma 13 (the uniform Gronwall lemma). Let \(\beta, \zeta,\) and \(h\) be nonnegative functions in \(L^1_{\infty}[0, \infty; R]\). Assume that \(\beta\) is absolutely continuous on \((0, \infty)\) and the following differential inequality is satisfied:

\[
\frac{d\beta}{dt} \leq \zeta \beta + h, \quad \text{for } t > 0.
\] (51)

If there exists a finite time \(t_1 > 0\) and some \(q > 0\) such that

\[
\int_t^{t+q} \zeta(r) dr \leq A, \quad \int_t^{t+q} \beta(r) dr \leq B, \quad \int_t^{t+q} h(r) dr \leq C,
\] (52)

for any \(t > t_1\), where \(A, B,\) and \(C\) are some positive constants, then

\[
\beta(t) \leq \left( \frac{B}{q} + C \right) e^{A}, \quad \text{for any } t > t_1 + q.
\] (53)

We apply the previous lemma to (50) with

\[
\zeta(t) = C, \quad \beta(t) = \|\nabla x_1\|_2^2, \quad h(t) = C_1 \|x_2\|_2^2,
\] (54)

and \(t_1 = t^*, q = 1\), and we use the estimates via (31) and Lemma 12 to obtain

\[
\|\nabla x_1(t + 1)\|_2^2 \leq C, \quad t > t^*.
\] (55)

The \(H^1\) estimates on the other components, \(x_2\) and \(x_3\), are obtained similarly.

Recall that in \(R^2\), we have via the Sobolev embedding that

\[
H^2(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for all } p,
\]

\[
\limsup_{t \to \infty} \|x_1(t)\|_p \leq C,
\] (56)

and similarly, for \(x_2\) and \(x_3\), we have

\[
\limsup_{t \to \infty} \|x_2(t)\|_p \leq C, \quad \limsup_{t \to \infty} \|x_3(t)\|_p \leq C.
\] (57)

Using the previous equation it is shown that the reaction terms \(g_1, g_2,\) and \(g_3\) are in \(L^p\), for \(p > 1\). In general, \(p > N/2\) is required [33], but in the current scenario \(N = 2\); thus the condition reduces to \(p > 1\). The estimates trivially follow from Hölder’s inequality, applied on the reaction terms along with (56) and (57). This proves the global existence result.

4.3. Existence of Global Attractor in \(L^2(\Omega)\). We can now use the standard theory [32] to prove that the semiflow \(\{S(t)\}_{t \geq 0}\) is a global attractor for the solution semigroup \(\{S(t)\}_{t \geq 0}\) generated by this model.

Theorem 14. Consider the three-species food-chain model described by (20)–(22), \(1 < m_3 < m_4\), and \(0 < m_2 \leq 1\). There exists a global attractor \(\mathcal{A}\) in \(H\), for the solution semiflow \(\{S(t)\}_{t \geq 0}\) generated by this model.

Proof. We have shown that the system is well posed via Theorem 6. Thus, there exists a well-defined semiflow \(\{S(t)\}_{t \geq 0} : H \to H\). Lemma 12 establishes the existence of bounded absorbing sets in \(H\). Thus, given a sequence \(\{x_n(0)\}_{n=1}^{\infty}\) that is bounded in \(L^2(\Omega)\), we know that, for \(t > t^*\),

\[
S(t)(x_n(0)) \subset B \subset H^1(\Omega).
\] (58)

Here, \(B\) is the bounded absorbing set in \(H^1(\Omega)\). Now, \(t_n > t^*\), for any large enough \(n\). Therefore, for \(t_n\), we have

\[
S(t_n)(x_n(0)) \subset B \subset H^1(\Omega).
\] (59)

This implies that we have the following uniform bound, via (57):

\[
\|S(t_n)(x_n(0))\|_{L^2(\Omega)} \leq C_1.
\] (60)

By standard functional analysis theory [32], a subsequence, still labelled as \(S(t_n)(x_n(0))\), exists such that

\[
S(t_n)(x_n(0)) \rightharpoonup x_1 \quad \text{in } H^1(\Omega),
\] (61)
Table 1: Parameters used in simulations.

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which implies, via the compact Sobolev embedding of $V \hookrightarrow H$, that

$$ S(t_n)(x_{in}(0)) \rightarrow x_1 \text{ in } L^2(\Omega). \quad (62) $$

This yields the asymptotic compactness of the semigroup $\{S(t)\}_{t \geq 0}$ in $H$. Similar analysis is possible for components $x_2$ and $x_3$. We have the existence of an absorbing set via Lemma 12 and the asymptotic compactness via (62). These in conjunction yield the existence of the global attractor, via standard methods [32].

5. Numerical Simulation and Attractor Reconstruction

We now carry out numerical simulations of (2)–(4). Our goal is to firstly show spatiotemporal chaos, even if $m_3 < m_4 = 2$. This is indeed the case in certain parameter regime; see Figure 3. We also want to numerically validate the finite time blow-up results in 1D as well as in 2D. For 1D simulation, the system is simulated in MATLAB (R2010) via the PDEPE solver for partial differential equation. The algorithm essentially uses a central difference in space and an implicit time stepping method. All the simulations performed have been refined several times on spatial grids with 300, 600, 900, and 1200 points on the domain of length $L = \pi$. These refinements lead to the same general shape and structure of the figures. Table 1 lists the parameter values used in the simulations.

In order to explore the dynamics of the model in 2D, we use a finite difference method. A forward difference scheme is used for the reaction terms. For the diffusion terms, a standard five-point explicit finite difference scheme is used. The numerical simulation is carried out at different time levels for two dimensional spatial model system. The system of equations is numerically solved over $200 \times 200$ mesh points, on a domain of size $L_x \times L_y$, with spatial resolution $\Delta x = \Delta y = 1$ and time step $\Delta t = 0.1$. The initial condition used is a small perturbation about $(2.1, 2.9, 1.9)$, and the boundary conditions used are the no flux Neumann conditions. Values for $L_x$ and $L_y$ are taken to be 200 each. Note this is fine, as the blow-up results hold for Neumann boundary conditions as well. See Figure 1 to visualise the blowup in 1D, and Figure 2 to visualise the blowup in 2D.

Our next goal is to quantify the chaotic dynamics present in the system. The dynamics of the three-species model described via (2)–(4) is depicted through the reconstruction of the trajectories in phase space with the aide of delay-time method. This method, proposed by Takens and Mane [34] ensures equivalence between the topological properties of actual and reconstructed attractors. Numerical simulations were conducted for three values of the mutual interference parameter $m_1$. That is $m_1 = 2$, $m_1 = 1.9$, and $m_1 = 1.2$. In all cases, $m_i$ was set equal to 2. The system of PDE (2)–(4) was solved again via the MATLAB routine PDEPE. We then fixed certain points in space, and recorded the trajectory of the solution in time, yielding a time series. Very interesting behavior was observed; this consists of suppression of chaos. In fact for a value of $m_1 = 2$, chaos is found when $c = 0.03$ in Table 1 in Upadhyay et al. [35]; see Figure 4(a).

We next calculated the dimension of this attractor, using a combination of singular-value decomposition and the Grassberger-Procaccia algorithm, [36]. We essentially want to embed the time series in an $m$ dimensional embedding space of embedded vectors, which take the typical form

$$ y_s = (x(t_0 + (s - 1)l), x(t_0 + (s - 1)l + \tau), \ldots), \quad (63) $$

Here, $\tau$ is called the “delay-time;” $l$ is the sampling interval, and $\omega = (m - 1)\tau$ is the “window length” which represents the time spanned by each embedding vector. This algorithm entails choosing $m$ and the delay, so that the window is sufficiently large. We then perform a singular value decomposition and calculate the correlation dimension, from the correlation integral. We continue to refine the embedding dimension till we have maximized the straight line parts in the log-log plot of the correlation integrals. The dimension of this attractor is approximately equal to 2.3 while the value of the largest Lyapunov exponent, estimated with the method proposed by Lai and Chen [37], is equal to 2.23. This clearly indicates that the dynamics is chaotic. Decreasing slightly the value of $m_1$ to 1.9, the dynamics remained chaotic and the dimension of the attractor remained almost unchanged, that is, equal to 2.3 while the largest Lyapunov exponent had decreased to 0.26; see Figure 4(b). This tells us that the divergence of close trajectories becomes now very slow. The important point here is that the proposed modification to the Parshad-Upadhyay model and subsequently the classical Upadhyay-Rai models still exhibits chaos in certain parameter regime. That is for $1.9 < m_1 < m_4 = 2$. Note since $w_0 < w_1$, this parameter set is not very realistic biologically. However, when we decrease further the value of $m_1$ to $m_3 = 1.2$, the dynamics of the system changes radically and becomes oscillatory with single period; this is depicted by a limit cycle.
in phase space; see Figure 4(c). From this numerical analysis, it seems plausible that the mutual interference parameter $m_3$ plays a crucial role in the dynamics of the three-species model. Setting a value of the parameter $m_3$ ($m_3 = 1.2$), very different from the classical one $m_3 = 2$, suppresses the chaos predicted by the model when $m_3 = 2$. However, keeping the parameters close to 2, that is $1.9 < m_3 < 2$, still retains chaotic dynamics, whilst eliminating the possibility of finite time blowup.

6. Discussion

Various nonlinear effects, like structural instability, dissipative structures, dynamical chaos, and finite time blowup, do exist in food chain models. However, the problem of detecting these phenomena in real ecosystems is far from being well understood. Various studies have suggested that the biology of the top generalist predator has an important role to play as far as the dynamics of food-chain models is concerned [10, 38]. The nature of interactions between the populations and communities may be responsible for chaos evading its capture in the wild. In this work, we have shown that a large class of three-species food-chain models [4, 7, 9] can blowup in finite time, for certain parameter regime and large enough initial data. We reiterate that the situation can be worse than unbounded growth if $c > w_3/D_3$, as reported earlier [5].

We propose a possible correction based on the analysis of the way mating is modeled in (4). That is, if we restrict $m_3$ to the range $1 < m_3 < m_4$, then finite time blowup is prevented, and this is independent of the sign of $c - w_3/D_3$. However, if $c - w_3/D_3 < 0$ and $x_3 = 0$, the top predator can still persist, for $1 < m_3 < m_4$. Thus, this formulation plausibly models a true generalist predator $x_3$, one that does not go extinct, even if its favorite food source does. Finally, by varying $m_3$ in the range $1 < m_3 < m_4$, one can adapt the new model to populations where the numbers of males and females vary. If the numbers are very close, we take $m_3 = 2 - \epsilon$, $\epsilon \ll 1$. One could also model a population where the males far outnumber the females or vice versa. Here one might take $m_3 = 1.5$, saying we have roughly 10 females per 100 males in a population or vice versa. Thus, making the case for the model in structured populations, where such demographics may be dominant. Also, simulations performed by varying the mutual interference parameter $m_3$ indicate clearly that there is a bifurcation point $m_3^*$, in the $m_3$ parameter space, such that, chaotic dynamics is seen for $m_3 > m_3^*$ and oscillatory dynamics for $m_3 < m_3^*$. It would be interesting to obtain the actual value of the mating parameter $m_3$ from real field data. Comparing this “real value” to $m_3^*$ might then be a means to again answer the long standing question of why chaos has evaded its capture in the wild. Lastly, we remark that the blow-up result obtained via Theorem 2 is sufficient but not
Figure 3: The densities of the $x_1$ and $x_2$ species are shown as contour plots in the $x-t$-plane. The corresponding parameters are listed in Table 1. The long-time dynamics (we run up to time $t = 3000$) is seen to be that of spatiotemporal chaos. This shows that one can have interesting dynamics even in the case that $m_1 < m_4$, whilst avoiding finite time blowup.

Figure 4: The figures above show the dependence of the dynamics on $m_3$. For $m_3 = 2$, one sees chaotic dynamics with the largest Lyapunov exponent equal to 2.23. Reducing $m_3$ to 1.9 brings the largest lyapunov exponent equal to 0.26. Further reducing it to 1.2 brings the dynamics to a limit cycle.
necessary. It is not optimal in terms of data. That is, it might still be possible for blowup to occur if the data does not meet the largeness condition

$$\int_{\Omega} x_{30}(x) W_1(x) \, dx > \left( \lambda_1 d_3 \frac{1}{(c - \omega_3/D_3)} \right)^{1/(m_1-1)}.$$  \hspace{1cm} (64)

For the classical diffusive Upadhyay-Rai model [4], we show blowup numerically; see Figure 1. Here, we have precisely all the parameters previously stated, including the first eigenvalue and eigenfunction of the Dirichlet laplacian. We see that, for the parameters chosen to demonstrate blowup in Figure 1,

$$\int_{\Omega} x_{30}(x) W_1(x) \, dx = 0.8 < 1.5 = \left( \lambda_1 d_3 \frac{1}{(c - \omega_3/D_3)} \right).$$  \hspace{1cm} (65)

Thus blowup takes place, even for data smaller than that required by the largeness condition. It would thus be very interesting to try and sharpen this largeness requirement on the data, perhaps via employing certain other techniques available in the blow-up literature, than the version of the first eigenvalue method that we have resorted to currently.

References


