Research Article

Approximate Controllability of a Semilinear Heat Equation

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We apply Rothe's type fixed point theorem to prove the interior approximate controllability of the following semilinear heat equation: $z_t(t, x) = \Delta z(t, x) + 1_\omega u(t, x) + f(t, z(t, x), u(t, x))$ in $(0, \tau] \times \Omega$, $z = 0$, on $(0, \tau) \times \partial\Omega$, $z(0, x) = z_0(x)$, $x \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $z_0 \in L^2(\Omega)$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control u belongs to $L^2(0, \tau; L^2(\Omega))$, and the nonlinear function $f : [0, \tau] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth enough, and there are $a, b, c \in \mathbb{R}$, $R > 0$ and $1/2 \leq \beta < 1$ such that $|f(t, z, u) - az| \leq c|u|^\beta + b$, for all $u, z \in \mathbb{R}$, $|u|, |z| \geq R$. Under this condition, we prove the following statement: for all open nonempty subset ω of Ω , the system is approximately controllable on $[0, \tau]$. Moreover, we could exhibit a sequence of controls steering the nonlinear system from an initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$.

1. Introduction

In this paper, we prove the interior approximate controllability of the following semilinear heat equation:

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x) + f(t, z, u(t, x)), \\ &\text{in } (0, \tau] \times \Omega, \\ z &= 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) &= z_0(x), \quad x \in \Omega, \end{aligned} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $z_0 \in L^2(\Omega)$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control u belong to $L^2(0, \tau; L^2(\Omega))$, and the nonlinear function $f : [0, \tau] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is smooth enough, and there are $a, b, c \in \mathbb{R}$, $R > 0$ and $1/2 \leq \beta < 1$ such that

$$|f(t, z, u) - az| \leq c|u|^\beta + b, \quad \forall u, z \in \mathbb{R}, \quad |u|, |z| \geq R, \quad (2)$$

which implies that

$$|f(t, z, u)| \leq c|u|^\beta + e|z| + b, \quad \forall u, z \in \mathbb{R}, \quad |u|, |z| \geq R. \quad (3)$$

Definition 1 (approximate controllability). The system (1) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z = U = L^2(\Omega)$, $\epsilon > 0$, there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (1) corresponding to u verifies

$$z(0) = z_0, \quad \|z(\tau) - z_1\|_Z < \epsilon; \quad (4)$$

see (Figure 1), where

$$\|z(\tau) - z_1\|_Z = \left(\int_\Omega |z(\tau, x) - z_1(x)|^2 dx \right)^{1/2}. \quad (5)$$

Remark 2. It is clear that exact controllability of the system (1) implies approximate controllability, null controllability, and controllability to trajectories of the system. But it is well known (see [1]) that due to the diffusion effect or the compactness of the semigroup generated by $-\Delta$, the heat equation can never be exactly controllable. We observe also that in the linear case, controllability to trajectories and null controllability are equivalent. Nevertheless, the approximate controllability and the null controllability are in general independent. Therefore, in this paper, we shall concentrate only on the study of the approximate controllability of the system (1).

Now, before proceeding with the introduction of this paper, we should mention the work of other authors and show

FIGURE 1

that ours is different in relation with new perturbation and the technique used. The approximate controllability of the heat equation under nonlinear perturbations $f(z)$ independents of t and u variable

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x) + f(z), \quad \text{in } (0, \tau] \times \Omega, \\ z &= 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) &= z_0(x), \quad x \in \Omega \end{aligned} \quad (6)$$

has been studied by several authors, particularly in [2–4], depending on conditions that impose to the nonlinear term $f(z)$. For instance, in [3, 4], the approximate controllability of the system (6) is proved if $f(z)$ is sublinear at infinity, that is,

$$|f(z)| \leq d|z| + e. \quad (7)$$

Also, in the above reference, they mentioned that when f is superlinear at the infinity, the approximate controllability of the system (6) fails.

Recently, the interior controllability of the semilinear heat equation (1) has been proved in [5–7] under the following condition: there are constants $a, c \in \mathbb{R}$, with $c \neq -1$, such that

$$\sup_{(t, z, u) \in Q_\tau} |f(t, z, u) - az - cu| < \infty, \quad (8)$$

where $Q_\tau = [0, \tau] \times \mathbb{R} \times \mathbb{R}$.

We note that the interior approximate controllability of the linear heat equation

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x), \quad \text{in } (0, \tau] \times \Omega, \\ z &= 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) &= z_0(x), \quad x \in \Omega, \end{aligned} \quad (9)$$

has been studied by several authors, particularly by [8–10]; and in a general fashion in [11].

Now, we shall describe the strategy of this work.

First, we prove that the linear system

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x) + az(t, x), \quad \text{in } (0, \tau] \times \Omega, \\ z &= 0, \quad \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) &= z_0(x), \quad x \in \Omega \end{aligned} \quad (10)$$

is approximately controllable.

After that, we write the system (1) as follows:

$$\begin{aligned} z_t(t, x) &= \Delta z(t, x) + 1_\omega u(t, x) + az + g(t, z, u), \\ &\text{in } (0, \tau] \times \Omega, \end{aligned} \quad (11)$$

$$z = 0, \quad \text{on } (0, \tau) \times \partial\Omega,$$

$$z(0, x) = z_0(x), \quad x \in \Omega,$$

where $g(t, z, u) = f(t, z, u) - az$ is a smooth enough functions satisfying the following conditions: there are $a, b, c \in \mathbb{R}$, $R > 0$ and $1/2 \leq \beta < 1$ such that

$$|g(t, z, u)| \leq c|u|^\beta + b, \quad \forall u, z \in \mathbb{R}, |u|, |z| \geq R. \quad (12)$$

Finally, the approximate controllability of the system (11) follows from the controllability of (10), the compactness of the semigroup generated by the Laplacian operator $-\Delta$, the condition (12) satisfied by the nonlinear term g , and the following results.

Proposition 3. *Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$ and $1 \leq q < r < \infty$. Then, $L_r(\mu) \subset L_q(\mu)$ and*

$$\|f\|_q \leq \mu(X)^{(r-q)/rq} \|f\|_r, \quad f \in L_r(\mu). \quad (13)$$

Proof. The proof of this proposition follows from Theorem I.V.6 from [12] by putting $p = r/q > 1$ and considering the relation

$$\int_X (|f|^q)^p d\mu = \int_X |f|^r d\mu, \quad \forall f \in L_r(\mu). \quad (14) \quad \square$$

Theorem 4 (Rothe's theorem [13, page 129]). *Let $E(\tau)$ be a Hausdorff topological vector space. Let $B \subset E$ be a closed convex subset such that the zero of E is contained in the interior of B .*

Let $\Phi : B \rightarrow E$ be a continuous mapping with $\Phi(B)$ relatively compact in E and $\Phi(\partial B) \subset B$.

Then, there is a point $x^ \in B$ such that $\Phi(x^*) = x^*$.*

The technique we use here to prove the approximate controllability of the linear part of (10) is based on the classical unique continuation for elliptic equations (see [14]) and the following lemma.

Lemma 5 (see Lemma 3.14 from [15, page 62]). *Let $\{\alpha_j\}_{j \geq 1}$ and $\{\beta_{i,j} : i = 1, 2, \dots, m\}_{j \geq 1}$ be two sequences of real numbers such that $\alpha_1 > \alpha_2 > \alpha_3 \dots$. Then,*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, \tau], \quad i = 1, 2, \dots, m. \quad (15)$$

If and only if

$$\beta_{i,j} = 0, \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots, \infty. \quad (16)$$

2. Abstract Formulation of the Problem

In this section, we choose a Hilbert space where system (1) can be written as an abstract differential equation; to this end, we

consider the following results appearing in [15, page 46], [16, page 335], and [17, page 147].

Let us consider the Hilbert space $Z = L^2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \rightarrow \infty$ the eigenvalues of $-\Delta$ with the Dirichlet homogeneous conditions, each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Then, we have the following well-known properties.

- (i) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of $A = -\Delta$.
- (ii) For all $z \in D(A)$, we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \quad (17)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \quad (18)$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z$, $z \in Z$.

- (iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z, \quad \|T(t)\| \leq e^{-\lambda_1 t}, \quad t \geq 0. \quad (19)$$

Consequently, systems (1), (10), and (11) can be written, respectively, as an abstract differential equations in Z :

$$z' = -Az + B_\omega u + f^e(t, z, u), \quad z \in Z, \quad t \geq 0, \quad (20)$$

$$z' = -Az + B_\omega u + az, \quad z \in Z, \quad t \geq 0, \quad (21)$$

$$z' = -Az + B_\omega u + az + g^e(t, z, u), \quad z \in Z, \quad t \geq 0, \quad (22)$$

where $u \in L^2([0, \tau]; U)$, $U = Z$, $B_\omega : U \rightarrow Z$, $B_\omega u = 1_\omega u$ is a bounded linear operator, $f^e : [0, \tau] \times Z \times U \rightarrow Z$ is defined by $f^e(t, z, u)(x) = f(t, z(x), u(x))$, for all $x \in \Omega$, and $g^e(t, z, u) = f^e(t, z, u) - az$. On the other hand, from condition (12), we get the following estimate.

Proposition 6. Under condition (12), the function $g^e : [0, \tau] \times Z \times U \rightarrow Z$ defined by $g^e(t, z, u)(x) = g(t, z(x), u(x))$, for all $x \in \Omega$, satisfies for all $u, z \in Z = L^2(\Omega)$:

$$\begin{aligned} \|g^e(t, z, u)\|_Z &\leq 2c\mu(\Omega)^{\beta/(2\beta-1/2)} \|u\|_Z^\beta + 2b\sqrt{\mu(\Omega)}, \\ \|z\|, \|u\| &\geq R\sqrt{\mu(\Omega)}. \end{aligned} \quad (23)$$

Proof

$$\begin{aligned} \|g^e(t, z, u)\|_Z &= \left(\int_\Omega |g(t, z(x), u(x))|^2 dx \right)^{1/2} \\ &\leq \left(\int_\Omega (c|u(x)|^\beta + b)^2 dx \right)^{1/2} \\ &\leq \left(\int_\Omega (4c^2|u(x)|^{2\beta} + 4b^2) dx \right)^{1/2} \\ &\leq 2c \left\{ \left(\int_\Omega |u(x)|^{2\beta} dx \right)^{1/2\beta} \right\}^\beta + 2b\sqrt{\mu(\Omega)} \\ &= 2c\|u\|_{L^{2\beta}(\Omega)}^\beta + 2b\sqrt{\mu(\Omega)}. \end{aligned} \quad (24)$$

Now, since $1/2 \leq \beta < 1 \Leftrightarrow 1 \leq 2\beta < 2$, applying Proposition 3, we obtain that

$$\|g^e(t, z, u)\|_Z \leq 2c\mu(\Omega)^{(1-\beta)/\beta} \|u\|_Z^\beta + 2b\sqrt{\mu(\Omega)}. \quad (25) \quad \square$$

3. Interior Controllability of the Linear Equation

In this section, we shall prove the interior approximate controllability of the linear system (21). To this end, we note that, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$, the initial value problem

$$\begin{aligned} z' &= -Az + B_\omega u(t) + az(t), \quad z \in Z, \\ z(0) &= z_0, \end{aligned} \quad (26)$$

where the control function u belongs to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = e^{at} T(t) z_0 + \int_0^t e^{a(t-s)} T(t-s) B_\omega u(s) ds, \quad t \in [0, \tau]. \quad (27)$$

Definition 7. For system (21), we define the following concept: the controllability map (for $\tau > 0$) $G_a : L^2(0, \tau; U) \rightarrow Z$ is given by

$$G_a u = \int_0^\tau e^{a(\tau-s)} T(\tau-s) B_\omega u(s) ds, \quad (28)$$

whose adjoint operator $G_a^* : Z \rightarrow L^2(0, \tau; Z)$ is given by

$$(G_a^* z)(s) = B_\omega^* e^{a(\tau-s)} T^*(\tau-s) z, \quad \forall s \in [0, \tau], \quad \forall z \in Z. \quad (29)$$

The following lemma holds in general for a linear bounded operator $G : W \rightarrow Z$ between Hilbert spaces W and Z .

Lemma 8 (see [11, 15, 18]). Equation (21) is approximately controllable on $[0, \tau]$ if and only if one of the following statements holds.

- (a) $\overline{\text{Rang}(G_a)} = Z$.
- (b) $\text{Ker}(G_a^*) = \{0\}$.
- (c) $\langle G_a G_a^* z, z \rangle > 0, z \neq 0$ in Z .
- (d) $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + G_a G_a^*)^{-1} z = 0$.
- (e) $B_\omega^* e^{at} T^*(t) z = 0$, for all $t \in [0, \tau]$, $\Rightarrow z = 0$.
- (f) For all $z \in Z$, we have $G u_\alpha = z - \alpha(\alpha I + G_a G_a^*)^{-1} z$, where

$$u_\alpha = G_a^* (\alpha I + G_a G_a^*)^{-1} z, \quad \alpha \in (0, 1]. \quad (30)$$

So, $\lim_{\alpha \rightarrow 0} G_a u_\alpha = z$ and the error $E_\alpha z$ of this approximation is given by

$$E_\alpha z = \alpha(\alpha I + G_a G_a^*)^{-1} z, \quad \alpha \in (0, 1]. \quad (31)$$

Remark 9. Lemma 8 implies that the family of linear operators $\Gamma_\alpha : Z \rightarrow L^2(0, \tau; U)$, defined for $0 < \alpha \leq 1$ by

$$\Gamma_\alpha z = B_\omega^* e^{a(\cdot)} T^*(\cdot) (\alpha I + G_a G_a^*)^{-1} z = G_a^* (\alpha I + G_a G_a^*)^{-1} z, \quad (32)$$

is an approximate inverse for the right of the operator G_a in the sense that

$$\lim_{\alpha \rightarrow 0} G_a \Gamma_\alpha = I. \quad (33)$$

Proposition 10 (see [7]). If $\overline{\text{Rang}(G_a)} = Z$, then

$$\sup_{\alpha > 0} \|\alpha(\alpha I + G_a G_a^*)^{-1}\| \leq 1. \quad (34)$$

Remark 11. The proof of the following theorem follows from foregoing characterization of dense range linear operators and the classical unique continuation for elliptic equations (see [14]), and it is similar to the one given in Theorem 4.1 in [6].

Theorem 12. System (21) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (21) from initial state z_0 to an ϵ neighborhood of the final state z_1 at time $\tau > 0$ is given by

$$u_\alpha(t) = B_\omega^* e^{a(\tau-t)} T^*(\tau-t) (\alpha I + G_a G_a^*)^{-1} \times (z_1 - e^{a\tau} T(\tau) z_0), \quad (35)$$

and the error of this approximation E_α is given by

$$E_\alpha z = \alpha(\alpha I + G_a G_a^*)^{-1} (z_1 - e^{a\tau} T(\tau) z_0). \quad (36)$$

4. Controllability of the Semilinear System

In this section, we shall prove the main result of this paper, the interior approximate controllability of the semilinear nD heat equation given by (1), which is equivalent to prove the approximate controllability of the system (22). To this end, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$, the initial value problem

$$\begin{aligned} z' &= -Az + B_\omega u + az + g^e(t, z, u), \quad z \in Z, \quad t \geq 0 \\ z(0) &= z_0 \end{aligned} \quad (37)$$

admits only one mild solution given by

$$\begin{aligned} z_u(t) &= e^{at} T(t) z_0 + \int_0^t e^{a(t-s)} T(t-s) B_\omega u(s) ds \\ &\quad + \int_0^t e^{a(t-s)} T(t-s) g^e(s, z_u(s), u(s)) ds, \end{aligned} \quad (38)$$

$$t \in [0, \tau].$$

Definition 13. For the system (22), we define the following concept: the nonlinear controllability map (for $\tau > 0$) $G_g : L^2(0, \tau; U) \rightarrow Z$ is given by

$$\begin{aligned} G_g u &= \int_0^\tau e^{a(\tau-s)} T(\tau-s) B_\omega u(s) ds \\ &\quad + \int_0^\tau e^{a(\tau-s)} T(\tau-s) g^e(s, z_u(s), u(s)) ds \\ &= G_a(u) + H(u), \end{aligned} \quad (39)$$

where $H : L^2(0, \tau; U) \rightarrow Z$ is the nonlinear operator given by

$$\begin{aligned} H(u) &= \int_0^\tau e^{a(\tau-s)} T(\tau-s) g^e(s, z_u(s), u(s)) ds, \\ u &\in L^2(0, \tau; U). \end{aligned} \quad (40)$$

The following lemma is trivial.

Lemma 14. Equation (22) is approximately controllable on $[0, \tau]$ if and only if $\overline{\text{Rang}(G_g)} = Z$.

Definition 15. The following equation shall be called the controllability equation associated with the nonlinear equation (22):

$$\begin{aligned} u_\alpha &= \Gamma_\alpha (z - H(u_\alpha)) = G_a^* (\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha)), \\ (0 < \alpha \leq 1). \end{aligned} \quad (41)$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear nD heat equation (1)

Theorem 16. The system (22) is approximately controllable on $[0, \tau]$. Moreover, a sequence of controls steering the system (22) from initial state z_0 to an ϵ -neighborhood of the final state z_1 at time $\tau > 0$ is given by

$$\begin{aligned} u_\alpha(t) &= B_\omega^* e^{a(\tau-t)} T^*(\tau-t) (\alpha I + G_a G_a^*)^{-1} \\ &\quad \times (z_1 - e^{a\tau} T(\tau) z_0 - H(u_\alpha)), \end{aligned} \quad (42)$$

and the error of this approximation $E_\alpha z$ is given by

$$E_\alpha z = \alpha(\alpha I + G_a G_a^*)^{-1} (z_1 - e^{a\tau} T(\tau) z_0 - H(u_\alpha)). \quad (43)$$

Proof. For each $z \in Z$ fixed, we shall consider the following family of nonlinear operators $K_\alpha : L^2(0, \tau; U) \rightarrow L^2(0, \tau; U)$ given by

$$K_\alpha(u) = \Gamma_\alpha(z - H(u)) = G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u)), \quad (0 < \alpha \leq 1). \quad (44)$$

First, we shall prove that for all $\alpha \in (0, 1]$ the operator K_α has a fixed point u_α . In fact, since g^e is smooth and satisfies (12) and the semigroup $\{T(t)\}_{t \geq 0}$ given by (19) is compact (see [19, 20]), then using the result from [1], we obtain that the operator H is compact, which implies that the operator K_α is compact.

Moreover,

$$\overline{\lim}_{\|u\|_{L_2} \rightarrow \infty} \frac{\|K_\alpha(u)\|_{L_2}}{\|u\|_{L_2}} = 0. \quad (45)$$

In fact, putting $C = 2c\mu(\Omega)^{\beta/(2\beta-1/2)}$ and $B = 2b\sqrt{\mu(\Omega)}$, we get from Proposition 6 that

$$\|g^e(t, z, u)\|_Z \leq C\|u\|_Z^\beta + B, \quad \|z\|, \|u\| \geq R\sqrt{\mu(\Omega)}, \quad (46)$$

and from the definition of the operator $H(u)$, Proposition 3, and (19) we have, for $u \in L^2(0, \tau; U)$, the following estimate:

$$\begin{aligned} \|H(u)\| &\leq \int_0^\tau e^{(a+\lambda_1)(\tau-s)} \|g^e(s, z_u(s), u(s))\| ds \\ &\leq \left(\int_0^\tau e^{2(a+\lambda_1)(\tau-s)} ds \right)^{1/2} \\ &\quad \times \left(\int_0^\tau \|g^e(s, z_u(s), u(s))\|^2 ds \right)^{1/2} \\ &= N \left(\int_0^\tau \|g^e(s, z_u(s), u(s))\|^2 ds \right)^{1/2} \\ &\leq N \left(\int_0^\tau (C\|u(s)\|^\beta + B)^2 ds \right)^{1/2} \\ &\leq N \left(\int_0^\tau (4C^2\|u(s)\|^{2\beta} + 4B^2) ds \right)^{1/2} \\ &\leq 2NC \left(\int_0^\tau \|u(s)\|^{2\beta} ds \right)^{1/2} + 2NB\sqrt{\tau} \\ &\leq 2NC \left\{ \left(\int_0^\tau \|u(s)\|^{2\beta} ds \right)^{1/2\beta} \right\}^\beta + 2NB\sqrt{\tau} \\ &= 2CN\|u\|_{L_{2\beta}}^\beta + 2NB\sqrt{\tau}. \end{aligned} \quad (47)$$

Now, since $1/2 \leq \beta < 1 \Leftrightarrow 1 \leq 2\beta < 2$, applying Proposition 3, we obtain that

$$\|H(u)\| \leq 2CN\tau^{(1-\beta)/\beta} \|u\|_{L_2}^\beta + 2NB\sqrt{\tau}. \quad (48)$$

Therefore,

$$\overline{\lim}_{\|u\|_{L_2} \rightarrow \infty} \frac{\|H(u)\|_{L_2}}{\|u\|_{L_2}} = 0. \quad (49)$$

Consequently,

$$\overline{\lim}_{\|u\|_{L_2} \rightarrow \infty} \frac{\|K_\alpha(u)\|}{\|u\|_{L_2}} = 0. \quad (50)$$

Then, from condition (45), we obtain that, for a fixed $0 < \rho < 1$, there exists $R_\alpha > 0$ big enough such that

$$\|K_\alpha(u)\|_{L_2} \leq \rho\|u\|_{L_2}, \quad \|u\|_{L_2} = R_\alpha. \quad (51)$$

Hence, if we denote by $B(0, R_\alpha)$ the ball of center zero and radius $R_\alpha > 0$, we get that $K_\alpha(\partial B(0, R_\alpha)) \subset B(0, R_\alpha)$. Since K_α is compact and maps the sphere $\partial B(0, R_\alpha)$ into the interior of the ball $B(0, R_\alpha)$, we can apply Roth's fixed point Theorem 4 to ensure the existence of a fixed point $u_\alpha \in L_2(0, \tau; U)$ such that

$$\begin{aligned} u_\alpha &= K_\alpha u_\alpha = \Gamma_\alpha(z - H(u_\alpha)) \\ &= G^*(\alpha I + GG^*)^{-1}(z - H(u_\alpha)), \quad (0 < \alpha \leq 1). \end{aligned} \quad (52)$$

Claim. The family of fixed point $\{u_\alpha\}_{0 < \alpha \leq 1}$ is bounded.

In fact, for the purpose of contradiction, let us assume the contrary. Then, there exists a subsequence $\{u_{\alpha_n}\}_{n \geq 1} \subset \{u_\alpha\}_{0 < \alpha \leq 1}$ such that

$$\lim_{n \rightarrow \infty} \|u_{\alpha_n}\|_{L_2} = \infty. \quad (53)$$

On the other hand, we have that $u_{\alpha_n} = K_\alpha u_{\alpha_n}$. So,

$$\|u_{\alpha_n}\|_{L_2} = \|K_\alpha u_{\alpha_n}\|_{L_2} \iff \frac{\|K_\alpha(u_{\alpha_n})\|_{L_2}}{\|u_{\alpha_n}\|_{L_2}} = 1. \quad (54)$$

Hence,

$$\overline{\lim}_{n \rightarrow \infty} \frac{\|K_\alpha(u_{\alpha_n})\|_{L_2}}{\|u_{\alpha_n}\|_{L_2}} = 1, \quad (55)$$

which is evidently a contradiction. Then, the claim is true and there exists $\gamma > 0$ such that

$$\|u_\alpha\|_{L_2} \leq \gamma, \quad (0 < \alpha \leq 1). \quad (56)$$

Therefore, without loss of generality, we can assume that the sequence $H(u_\alpha)$ converges to $y \in Z$. So, if

$$u_\alpha = \Gamma_\alpha(z - H(u)) = G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \quad (57)$$

then

$$\begin{aligned} G_a u_\alpha &= G_a \Gamma_\alpha(z - H(u)) \\ &= G_a G_a^*(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \\ &= (\alpha I + G_a G_a^* - \alpha I)(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)) \\ &= z - H(u_\alpha) - \alpha(\alpha I + G_a G_a^*)^{-1}(z - H(u_\alpha)). \end{aligned} \quad (58)$$

Hence,

$$G_a u_\alpha + H(u_\alpha) = z - \alpha(\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha)). \quad (59)$$

To conclude the proof of this theorem, it is enough to prove that

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha))\} = 0. \quad (60)$$

From Lemma 8 (d), we get that

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha))\} \\ &= -\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + G_a G_a^*)^{-1} H(u_\alpha)\} \\ &= -\lim_{\alpha \rightarrow 0} -\alpha(\alpha I + G_a G_a^*)^{-1} y \\ &\quad - \lim_{\alpha \rightarrow 0} -\alpha(\alpha I + G_a G_a^*)^{-1} (H(u_\alpha) - y) \\ &= \lim_{\alpha \rightarrow 0} -\alpha(\alpha I + G_a G_a^*)^{-1} (H(u_\alpha) - y). \end{aligned} \quad (61)$$

On the other hand, from Proposition 10, we get that

$$\|\alpha(\alpha I + G_a G_a^*)^{-1} (H(u_\alpha) - y)\| \leq \|H(u_\alpha) - y\|. \quad (62)$$

Therefore, since $H(u_\alpha)$ converges to y , we get that

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + G_a G_a^*)^{-1} (H(u_\alpha) - y)\} = 0. \quad (63)$$

Consequently,

$$\lim_{\alpha \rightarrow 0} \{-\alpha(\alpha I + G_a G_a^*)^{-1} (z - H(u_\alpha))\} = 0. \quad (64)$$

So, by putting $z = z_1 - e^{a\tau} T(\tau) z_0$ and using (38), we obtain the nice result:

$$\begin{aligned} z_1 = \lim_{\alpha \rightarrow 0^+} & \left\{ e^{a\tau} T(\tau) z_0 + \int_0^\tau e^{a(\tau-s)} T(\tau-s) B_\omega u_\alpha(s) ds \right. \\ & \left. + \int_0^\tau e^{a(\tau-s)} T(\tau-s) g^e(s, z_{u_\alpha}(s), u_\alpha(s)) ds \right\}. \end{aligned} \quad (65)$$

□

5. Final Remark

Our technique is simple and can be applied to those systems involving compact semigroups like some control system governed by diffusion processes. For example, the Benjamin-Bona-Mahony equation, the strongly damped wave equations, beam equations, and so forth.

Example 17. The original Benjamin-Bona-Mahony equation is a nonlinear one in [21] the authors proved the approximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of

the nonlinear BBM equation. So, our next work is concerned on the controllability of nonlinear BBM equation:

$$\begin{aligned} z_t - a\Delta z_t - b\Delta z &= 1_\omega u(t, x) + f(t, z, u(t)), \\ t &\in (0, \tau), \quad x \in \Omega, \end{aligned} \quad (66)$$

$$z(t, x) = 0, \quad t \geq 0, \quad x \in \partial\Omega,$$

where $a \geq 0$ and $b > 0$ are constants, Ω is a domain in $\mathbb{R}^N$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$, and $f(t, z, u(t))$ is a nonlinear perturbation.

Example 18. We believe that this technique can be applied to prove the interior controllability of the strongly damped wave equation with Dirichlet boundary conditions:

$$\begin{aligned} w_{tt} + \eta(-\Delta)^{1/2} w_t + \gamma(-\Delta) w &= 1_\omega u(t, x) + f(t, z, u(t)), \\ &\text{in } (0, \tau) \times \Omega, \end{aligned}$$

$$w = 0, \quad \text{in } (0, \tau) \times \partial\Omega,$$

$$w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), \quad \text{in } \Omega, \quad (67)$$

in the space $Z_{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$, and η, γ are positive numbers.

Example 19. Another example where this technique may be applied is partial differential equations modeling the structural damped vibrations of a string or a beam:

$$\begin{aligned} y_{tt} - 2\beta\Delta y_t + \Delta^2 y &= 1_\omega u(t, x) + f(t, z, u(t)), \\ &\text{on } (0, \tau) \times \Omega, \end{aligned} \quad (68)$$

$$y = \Delta y = 0, \quad \text{on } (0, \tau) \times \partial\Omega,$$

$$y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), \quad \text{in } \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$, ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, \tau; L^2(\Omega))$, and $y_0 \in H^2(\Omega) \cap H_0^1$, $y_1 \in L^2(\Omega)$.

Moreover, our result can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space Z :

$$\begin{aligned} \dot{z} &= -Az + Bu(t) + f^e(t, z, u), \quad z \in Z, \quad t \in (0, \tau), \\ z(0) &= z_0, \end{aligned} \quad (69)$$

where $A : D(A) \subset Z \rightarrow Z$ is an unbounded linear operator in Z with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}, \quad (70)$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots \lambda_n \rightarrow \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A . The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t \geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \quad (71)$$

The control $u \in L^2(0, \tau; U)$, with $U = Z$, $B : Z \rightarrow Z$ is a linear and bounded operator (linear and continuous), and the function $f^e : [0, \tau] \times Z \times U \rightarrow Z$ is smooth enough, and

$$\begin{aligned} \|f^e(t, z, u) - az\|_Z &\leq c\|u\|_Z^\beta + b, \\ \forall u, z \in Z, \quad \|u\|, \|z\| &\geq R. \end{aligned} \quad (72)$$

In this case, the characteristic function set is a particular operator B , and the following theorem is a generalization of Theorem 16.

Theorem 20. *If vectors $B^* \phi_{j,k}$ are linearly independent in Z , then the system (69) is approximately controllable on $[0, \tau]$.*

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References

- [1] D. Barcenas, H. Leiva, and Z. Sivoli, "A broad class of evolution equations are approximately controllable, but never exactly controllable," *IMA Journal of Mathematical Control and Information*, vol. 22, no. 3, pp. 310–320, 2005.
- [2] J. I. Díaz, J. Henry, and A. M. Ramos, "On the approximate controllability of some semilinear parabolic boundary-value problems," *Applied Mathematics and Optimization*, vol. 37, no. 1, pp. 71–97, 1998.
- [3] E. Fernandez-Cara, "Remark on approximate and null controllability of semilinear parabolic equations," vol. 4, pp. 73–81, Proceedings of the Controle et Equations aux Derivees Partielles (ESAIM '98), 1998.
- [4] E. Fernández-Cara and E. Zuazua, "Controllability for blowing up semilinear parabolic equations," *Comptes Rendus de l'Academie des Sciences I*, vol. 330, no. 3, pp. 199–204, 2000.
- [5] H. Leiva, N. Merentes, and J. L. Sanchez, "Interior controllability of the nD semilinear heat equation," *African Diaspora Journal of Mathematics*, vol. 12, no. 2, pp. 1–12, 2011, Special vol. in Honor of Profs. C. Corduneanu, A. Fink, and S. Zaidman.
- [6] H. Leiva, N. Merentes, and J. Sanchez, "Approximate controllability of semilinear reaction diffusion," *Mathematical Control and Related Fields*, vol. 2, no. 2, 2012.
- [7] H. Leiva, N. Merentes, and J. Sanchez, "A characterization of semilinear dense range operators and applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 729093, 11 pages, 2013.
- [8] X. Zhang, "A remark on null exact controllability of the heat equation," *SIAM Journal on Control and Optimization*, vol. 40, no. 1, pp. 39–53, 2002.
- [9] E. Zuazua, "Controllability of a system of linear thermoelasticity," *Journal de Mathématiques Pures et Appliquées*, vol. 74, pp. 291–315, 1995.
- [10] E. Zuazua, "Controllability of partial differential equations and its semi-discrete approximations," *Discrete and Continuous Dynamical Systems*, vol. 8, no. 2, pp. 469–513, 2002.
- [11] H. Leiva and Y. Quintana, "Interior controllability of a broad class of reaction diffusion equations," *Mathematical Problems in Engineering*, vol. 2009, Article ID 708516, 2009.
- [12] H. Brezis, *Analisis Funcional, Teoria y Aplicaciones*, Alianza Universitaria Textos, Masson, Paris, 1983; cast.: Alinza Editorial, S. A., Madrid, Spain, 1984.
- [13] G. Isac, "On Rothe's fixed point theorem in general topological vector space," *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 12, no. 2, pp. 127–134, 2004.
- [14] M. H. Protter, "Unique continuation for elliptic equations," *Transaction of the American Mathematical Society*, vol. 95, no. 1, 1960.
- [15] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems*, vol. 8 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 1978.
- [16] C. E. Lawrence, *Partial Differential Equations*, vol. 19 of *Graduate Studies in Mathematics*, American Mathematical Society, 1999.
- [17] C. Kesavan, *Topics in: Functional Analysis and Applications*, John Wiley & Sons, 1989.
- [18] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, vol. 21 of *Text in Applied Mathematics*, Springer, New York, NY, USA, 1995.
- [19] D. Bárcenas, H. Leiva, and W. Urbina, "Controllability of the Ornstein-Uhlenbeck equation," *IMA Journal of Mathematical Control and Information*, vol. 23, no. 1, pp. 1–9, 2006.
- [20] D. Barcenas, H. Leiva, Y. Quintana, and W. Urbina, "Controllability of Laguerre and Jacobi equations," *International Journal of Control*, vol. 80, no. 8, pp. 1307–1315, 2007.
- [21] H. Leiva, N. Merentes, and J. L. Sanchez, "Interior controllability of the Benjamin-Bona-Mahony equation," *Journal of Mathematics and Applications*, no. 33, pp. 51–59, 2010.