Research Article

Universality Properties of a Double Series by the Generalized Walsh System

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Received 16 December 2012; Accepted 2 April 2013

Academic Editor: Frédéric Robert

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We consider a question on existence of a double series by the generalized Walsh system, which is universal in weighted $L^1_\mu[0, 1]^2$ spaces. In particular, we construct a weighted function $\mu(x, y)$ and a double series by generalized Walsh system of the form $\sum_{n,k=1}^\infty c_{n,k} \psi_n(x) \psi_k(y)$ with the $\sum_{n,k=1}^\infty |c_{n,k}|^q < \infty$ for all $q > 2$, which is universal in $L^1_\mu[0, 1]^2$ concerning subseries with respect to convergence, in the sense of both spherical and rectangular partial sums.

1. Introduction

Let X be a Banach space.

Definition 1. A series

$$\sum_{k=1}^\infty f_k, \quad f_k \in X \quad (1)$$

is said to be universal in X with respect to rearrangements, if for any $f \in X$ the members of (1) can be rearranged so that the obtained series $\sum_{k=1}^\infty f_{\sigma(k)}$ converges to f by norm of X .

Definition 2. The series (1) is said to be universal (in X) concerning subseries, if for any $f \in X$, it is possible to choose a subseries $\sum_{k=1}^\infty f_{n_k}$ from (1), which converges to the f by norm of X .

Note that for one-dimensional case there are many papers that are devoted to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure (see [1–10]).

Let $a \geq 2$ be a fixed integer and $\omega_a = e^{2\pi i/a}$. Recall the following definitions (see [11]).

The Rademacher system of order a is defined inductively as follows. For $n = 0$ let

$$\varphi_0(x) = \omega_a^k \quad \text{if } x \in \left[\frac{k}{a}, \frac{k+1}{a} \right), \quad k = 0, 1, \dots, a-1, \quad (2)$$

and for $n \geq 1$ let

$$\varphi_n(x+1) = \varphi_n(x) = \varphi_0(a^n x). \quad (3)$$

The generalized Walsh system of order a is defined by

$$\psi_0(x) = 1, \quad (4)$$

and if $n = \alpha_1 a^{n_1} + \dots + \alpha_s a^{n_s}$, where $n_1 > \dots > n_s$, $0 \leq \alpha_j < a$, $j = 1, 2, \dots, s$, then

$$\psi_n(x) = \varphi_{n_1}^{\alpha_1}(x) \cdot \dots \cdot \varphi_{n_s}^{\alpha_s}(x). \quad (5)$$

We denote the generalized Walsh system of order a by Ψ_a . Not that Ψ_2 is the classical Walsh system. The basic properties of the generalized Walsh system of order a have been obtained by Chrestenson, Fine, Watari, Young, Vilenkin, and others (see [11–16]).

In [6–9], the existence of universal one-dimensional series by trigonometric and the classical Walsh system with respect to rearrangements and subseries in some weighted

space $L^1_\mu[0, 1]$. Some results for two-dimensional case for the classical Walsh system were obtained in [10] is proved. In this paper we consider the universality properties of a double series by the generalized Walsh system.

2. Preliminary Notes

Now we list some properties of Ψ_a , $a \geq 2$, which will be useful later.

- (i) Each n th Rademacher function has period a^{-n} and

$$\varphi_n(x) = \text{const} \in \Omega_a = \{1, \omega_a, \omega_a^2, \dots, \omega_a^{a-1}\}, \quad (6)$$

$$\text{if } x \in \Delta_{n+1}^{(k)} = [k/a^{n+1}, (k+1)/a^{n+1}), k = 0, \dots, a^{n+1} - 1, n = 1, 2, \dots$$

- (ii)

$$(\varphi_n(x))^k = (\varphi_n(x))^m, \quad \forall n, k \in \mathcal{N}, m = k \pmod{a}. \quad (7)$$

- (iii) $\psi_n(x)$ is a finite product of the Rademacher functions with values in Ω_a .

- (iv)

$$\psi_{a^k+j}(x) = \varphi_k(x) \cdot \psi_j(x), \quad \text{if } 0 \leq j \leq a^k - 1. \quad (8)$$

- (v) $\Psi_a, a \geq 2$ is a complete orthonormal system in $L^2[0, 1]$ and it is a basis in $L^p[0, 1]$ for $p > 1$.

The rectangular and spherical partial sums of the double series

$$\sum_{k,v=1}^{\infty} c_{k,v} \psi_k(x) \psi_v(y), \quad (x, y) \in T = [0, 1]^2 \quad (9)$$

will be denoted by

$$S_{n,m}(x, y) = \sum_{k=1}^n \sum_{v=1}^m c_{k,v} \psi_k(x) \psi_v(y), \quad (10)$$

$$S_R(x, y) = \sum_{v^2+k^2 \leq R^2} c_{k,v} \psi_k(x) \psi_v(y).$$

If $g(x, y)$ is a continuous function on $T = [0, 1]^2$, then we set

$$\|g(x, y)\|_C = \max_{(x,y) \in T} |g(x, y)|. \quad (11)$$

3. Main Results

Let us denote the generalized Walsh system of order a by Ψ_a , $a \geq 2$. These are the main results of the paper.

Theorem 3. *There exists a double series of the form*

$$\sum_{n,k=1}^{\infty} c_{n,k} \psi_n(x) \psi_k(y) \quad \text{with} \quad \sum_{n,k=1}^{\infty} |c_{n,k}|^q < \infty \quad (12)$$

$$\forall q > 2$$

with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x, y)$ satisfying

$$0 < \mu(x, y) \leq 1, \quad |\{(x, y) \in T : \mu(x, y) \neq 1\}| < \varepsilon \quad (13)$$

can be constructed so that the series (12) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence in the sense of both spherical and rectangular partial sums.

Theorem 4. *There exists a double series of the form (12) with the following property: for any number $\varepsilon > 0$ a weighted function $\mu(x, y)$ with (13) can be constructed, so that the series (12) is universal in $L^1_\mu(T)$ concerning rearrangements with respect to convergence in the sense of both spherical and rectangular partial sums.*

Repeating the reasoning of the proof of [17, Lemma 2] we will receive the following lemma.

Lemma 5. *For any given numbers $0 < \varepsilon < 1$, $N_0 > 2$ ($N_0 \in \mathcal{N}$) and a step function*

$$f(x) = \sum_{s=1}^q \gamma_s \cdot \chi_{\Delta_s}(x), \quad (14)$$

where Δ_s is an interval of the form $\Delta_m^{(i)} = [(i-1)/2^m, i/2^m]$, $1 \leq i \leq 2^m$, there exist a measurable set $E \subset [0, 1]$ and a polynomial $P(x)$ of the form

$$P(x) = \sum_{k=N_0}^N c_k \psi_k(x) \quad (15)$$

which satisfy the following conditions:

- (1)

$$P(x) = f(x) \text{ on } E, \quad (16)$$

- (2)

$$|E| > (1 - \varepsilon), \quad (17)$$

- (3)

$$\sum_{k=N_0}^N |c_k|^{2+\varepsilon} < \varepsilon, \quad (18)$$

- (4)

$$\max_{N_0 \leq m < N} \left[\int_E \left| \sum_{k=N_0}^m c_k \psi_k(x) \right| dx \right] < \varepsilon + \int_E |f(x)| dx, \quad (19)$$

for every measurable subset e of E .

Then applying this Lemma we get the next one.

Lemma 6. For any numbers $\gamma \neq 0$, $0 < \delta < 1$, $N > 1$ and for any square $\Delta = \Delta_1 \times \Delta_2 \subset T$, there exists a measurable set $E \subset T$ and a polynomial $P(x, y)$ of the form

$$P(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \quad (20)$$

with the following properties:

(1)

$$|E| > 1 - \delta, \quad (21)$$

(2)

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\delta} < \delta, \quad (22)$$

(3)

$$P(x, y) = \gamma \cdot \chi_\Delta(x, y) \quad \text{for } (x, y) \in E, \quad (23)$$

(4)

$$\begin{aligned} & \max_{N \leq \bar{n}, \bar{m} \leq M} \left[\iint_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ & + \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[\iint_e \left| \sum_{2N^2 \leq k^2 + s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ & \leq 16 \cdot |\gamma| \cdot |\Delta|, \end{aligned} \quad (24)$$

for every measurable subset e of E .

Proof. We apply Lemma 5, setting

$$f(x) = \gamma \cdot \chi_{\Delta_1}(x), \quad N_0 = N, \quad \varepsilon = \frac{\delta}{2}. \quad (25)$$

Then we can define a measurable set $E_1 \subset [0, 1]$ and a polynomial $P_1(x)$ of the form

$$P_1(x) = \sum_{k=N}^{N_1} a_k \psi_k(x) \quad (26)$$

which satisfy the following conditions:

(1⁰)

$$P_1(x) = \gamma \cdot \chi_{\Delta_1}(x) \quad \text{for } x \in E_1, \quad (27)$$

(2⁰)

$$|E_1| > 1 - \frac{\delta}{2}, \quad (28)$$

(3⁰)

$$\sum_{k=N}^{N_1} |a_k|^{2+\delta} < \delta, \quad (29)$$

(4⁰)

$$\max_{N \leq \bar{n} \leq N_1} \left[\int_{e_1} \left| \sum_{k=N}^{\bar{n}} a_k \psi_k(x) \right| dx \right] \leq 2 \cdot |\gamma| \cdot |\Delta_1|, \quad (30)$$

for every measurable subset e_1 of E_1 .
Set

$$M_0 = 2 \cdot (N_1^2 + 1) \quad (31)$$

and apply Lemma 5 again, setting

$$f(y) = \chi_{\Delta_2}(y), \quad N_0 = M_0, \quad \varepsilon = \frac{\delta}{2}. \quad (32)$$

Then we can define a measurable set $E_2 \subset [0, 1]$ and a polynomial $P_2(y)$ of the form

$$P_2(y) = \sum_{s=M_0}^M b_s \psi_s(y), \quad (33)$$

which satisfy the following conditions:

(1⁰⁰)

$$P_2(y) = \chi_{\Delta_2}(y) \quad \text{for } y \in E_2, \quad (34)$$

(2⁰⁰)

$$|E_2| > 1 - \frac{\delta}{2}, \quad (35)$$

(3⁰⁰)

$$\sum_{s=M_0}^M |b_s|^{2+\delta} < \delta, \quad (36)$$

(4⁰⁰)

$$\max_{M_0 \leq \bar{m} \leq M} \left[\int_{e_2} \left| \sum_{s=M_0}^{\bar{m}} b_s \psi_s(y) \right| dy \right] \leq 2 \cdot |\Delta_2|, \quad (37)$$

for every measurable subset e_2 of E_2 .
Set

$$E = E_1 \times E_2,$$

$$P(x, y) = P_1(x) \cdot P_2(y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \quad (38)$$

where

$$\begin{aligned} c_{k,s} &= a_k \cdot b_s, \quad \text{if } N \leq k \leq N_1, \quad M_0 \leq s \leq M, \\ c_{k,s} &= 0, \quad \text{for other } k, s. \end{aligned} \quad (39)$$

By $(1^0)-(3^0)$, $(1^{00})-(3^{00})$, and (38), (39), we obtain

$$|E| > 1 - \delta, \quad (1^0)$$

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\delta} = \sum_{k=N}^{N_1} |a_k|^{2+\delta} \cdot \sum_{s=M_0}^M |b_s|^{2+\delta} < \delta, \quad (40)$$

$$P(x, y) = \gamma \cdot \chi_{\Delta}(x, y) \quad \text{for } (x, y) \in E.$$

Thus, the statements (1)–(3) of Lemma 6 are satisfied. Now we will check the fulfillment of statement (4).

Let $N^2 + M_0^2 < R^2 < N_1^2 + M^2$, then for some $m_0 > M_0$ we have $m_0 < R < m_0 + 1$ and from (31) it follows that $R^2 - N_1^2 > (m_0 - 1)^2$.

Consequently taking relations (4^0) , (4^{00}) , and (38), (39) for any measurable set $e \subset E$ ($e = e_1 \times e_2$, $e_1 \subset E_1$, $e_2 \subset E_2$) we obtain

$$\begin{aligned} & \left| \iint_e \sum_{N^2+M^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) dx dy \right| \\ & \leq \iint_e \left| \sum_{k=N}^{N_1} \sum_{s=M_0}^{m_0-1} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \\ & \quad + \max_{N < n \leq N_1} \left[\iint_e \left| \sum_{k=N}^n c_{k,m_0} \psi_k(x) \cdot \psi_{m_0}(y) \right| dx dy \right] \\ & \leq \left[\iint_{e_1} \left| \sum_{k=N}^{N_1} a_k \psi_k(x) \right| dx \right] \cdot \left[\iint_{e_2} \left| \sum_{s=M_0}^{m_0-1} b_s \psi_s(y) \right| dy \right] \\ & \quad + |b_{m_0}| \cdot \left[\iint_{e_2} |\psi_{m_0}(y)| dy \right] \\ & \quad \cdot \max_{N < n \leq N_1} \left[\iint_{e_1} \left| \sum_{k=N}^n a_k \psi_k(x) \right| dx \right] \\ & \leq 12 \cdot |\gamma| \cdot |\Delta|. \end{aligned} \quad (41)$$

Similarly, for $N \leq \bar{n} \leq N_1$, $M_0 \leq \bar{m} \leq M$, we get

$$\iint_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \leq 4 \cdot |\gamma| \cdot |\Delta|. \quad (42)$$

Lemma 6 is proved. $\square$

Lemma 7. For any numbers $\varepsilon > 0$, $N > 1$ and a step function

$$f(x, y) = \sum_{\nu=1}^{\nu_0} \gamma_{\nu} \cdot \chi_{\Delta_{\nu}}(x, y), \quad (43)$$

there exists a measurable set $E \subset T$ and a polynomial $P(x, y)$ of the form

$$P(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \quad (44)$$

which satisfy the following conditions:

$$(1^0)$$

$$P(x, y) = f(x, y) \quad \text{for } (x, y) \in E, \quad (45)$$

$$(2^0)$$

$$|E| > 1 - \varepsilon, \quad (46)$$

$$(3^0)$$

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\varepsilon} < \varepsilon, \quad (47)$$

$$(4^0)$$

$$\begin{aligned} & \max_{N \leq \bar{n}, \bar{m} < M} \left[\iint_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\ & \quad + \max_{\sqrt{2}N \leq R \leq \sqrt{2}M} \left[\iint_e \left| \sum_{2N^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \right. \right. \\ & \quad \left. \left. \cdot \psi_s(y) \right| dx dy \right] \\ & \leq 2 \cdot \iint_e |f(x, y)| dx dy + \varepsilon, \end{aligned} \quad (48)$$

for every measurable subset e of E .

Proof. Without any loss of generality, we assume that

$$\max_{1 \leq \nu \leq \nu_0} (|\gamma_{\nu}| \cdot |\Delta_{\nu}|) < \frac{\varepsilon}{32}, \quad (49)$$

Δ_{ν} , $1 \leq \nu \leq \nu_0$ are the constancy rectangular domain of $f(x, y)$, that is, where the function $f(x, y)$ is constant.

Given an integer $1 \leq \nu \leq \nu_0$, by applying Lemma 6 with $\delta = \varepsilon/16\nu_0$, we find that there exists a measurable set $E_{\nu} \subset T$ and a polynomial $P_{\nu}(x, y)$ of the form

$$P_{\nu}(x, y) = \sum_{k,s=N_{\nu}}^{M_{\nu}} c_{k,s}^{(\nu)} \psi_k(x) \cdot \psi_s(y) \quad (50)$$

with the following properties:

$$|E_{\nu}| > 1 - \frac{\varepsilon}{2\nu}, \quad (51)$$

$$\sum_{k,s=N_{\nu}}^{M_{\nu}} |c_{k,s}^{(\nu)}|^{2+\varepsilon} < \frac{\varepsilon}{\nu_0}, \quad (52)$$

$$P_{\nu}(x, y) = \gamma_{\nu} \cdot \chi_{\Delta_{\nu}}(x, y) \quad \text{for } (x, y) \in E_{\nu}, \quad (53)$$

$$\begin{aligned}
& \max_{N_\nu \leq \bar{n}, \bar{m} \leq M_\nu} \left[\iint_e \left| \sum_{k,s=N_\nu}^{\bar{n}, \bar{m}} c_{k,s}^{(\nu)} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\
& + \max_{\sqrt{2}N_\nu \leq R \leq \sqrt{2}M_\nu} \left[\iint_e \left| \sum_{2N_\nu^2 \leq k^2+s^2 \leq R^2} c_{k,s}^{(\nu)} \psi_k(x) \right. \right. \\
& \quad \left. \left. \cdot \psi_s(y) \right| dx dy \right] \\
& \leq 16 \cdot |\gamma_\nu| \cdot |\Delta_\nu| < \frac{\varepsilon}{2},
\end{aligned} \tag{54}$$

for every measurable subset e of E_ν (see (49)).

Then we can take

$$N_1 = N, \quad n_\nu = M_{\nu-1} + 1, \quad 1 \leq \nu \leq \nu_0. \tag{55}$$

Set

$$E = \bigcap_{\nu=1}^{\nu_0} E_\nu, \tag{56}$$

$$P(x, y) = \sum_{\nu=1}^{\nu_0} P_\nu(x, y) = \sum_{k,s=N}^M c_{k,s} \psi_k(x) \cdot \psi_s(y), \tag{57}$$

$$M = M_{\nu_0},$$

where

$$c_{k,s} = c_{k,s}^{(\nu)}, \quad \text{for } N_\nu \leq k, s \leq M_\nu, \quad 1 \leq \nu \leq \nu_0, \tag{58}$$

$$c_{k,s} = 0, \quad \text{for other } k, s. \tag{59}$$

From (51)–(53) and (56)–(58) we obtain

$$\begin{aligned}
P(x, y) &= f(x, y) \quad \text{for } (x, y) \in E, \\
|E| &> 1 - \varepsilon,
\end{aligned} \tag{60}$$

$$\sum_{k,s=N}^M |c_{k,s}|^{2+\varepsilon} < \sum_{\nu=1}^{\nu_0} \left[\sum_{k,s=N_\nu}^{M_\nu} |c_{k,s}^{(\nu)}|^{2+\varepsilon} \right] < \varepsilon.$$

Then, let $R \in [\sqrt{2}N, \sqrt{2}M]$, then for some ν' , $1 \leq \nu' \leq \nu_0$ we have $\sqrt{2}N_{\nu'} \leq R \leq \sqrt{2}N_{\nu'+1}$, consequently from (57) and (58) we have

$$\begin{aligned}
& \sum_{2N^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \\
&= \sum_{\nu=1}^{\nu'-1} P_\nu(x, y) + \sum_{2N_{\nu'}^2 \leq k^2+s^2 \leq R^2} c_{k,s}^{(\nu')} \psi_k(x) \cdot \psi_s(y).
\end{aligned} \tag{61}$$

In view of the conditions (51)–(54) and the equality $P(x, y) = f(x, y)$ on E , for any measurable set $e \subset E$ we obtain

$$\begin{aligned}
& \iint_e \left| \sum_{2N^2 \leq k^2+s^2 \leq R^2} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \\
& \leq \iint_e \left| \sum_{\nu=1}^{\nu'-1} P_\nu(x, y) \right| dx dy \\
& + \iint_e \left| \sum_{2N_{\nu'}^2 \leq k^2+s^2 \leq R^2} c_{k,s}^{(\nu')} \psi_k(x) \cdot \psi_s(y) \right| dx dy \\
& \leq \iint_e |f(x, y)| dx dy + \frac{\varepsilon}{2}.
\end{aligned} \tag{62}$$

Similarly, for any $e \subset E$ we have

$$\begin{aligned}
& \max_{N \leq \bar{n}, \bar{m} \leq M} \left[\iint_e \left| \sum_{k,s=N}^{\bar{n}, \bar{m}} c_{k,s} \psi_k(x) \cdot \psi_s(y) \right| dx dy \right] \\
& \leq \iint_e |f(x, y)| dx dy + \frac{\varepsilon}{2}.
\end{aligned} \tag{63}$$

Lemma 7 is proved. $\square$

4. Proofs of the Theorems

Theorem 3 is proved similarly [10, Theorem 3], but for maintenance of integrity of this paper, here we will give the proof.

Proof of Theorem 3. Let

$$\{f_s(x, y)\}_{s=1}^\infty, \quad (x, y) \in T \tag{64}$$

be a sequence of all step functions, values, and constancy interval endpoints which are rational numbers. Applying Lemma 7 consecutively, we can find a sequence $\{E_s\}_{s=1}^\infty$ of sets and a sequence of polynomials

$$P_s(x, y) = \sum_{k,\nu=N_{s-1}}^{N_s-1} c_{k,\nu}^{(s)} \psi_k(x) \psi_\nu(y), \tag{65}$$

$$1 = N_0 < N_1 < \dots < N_s < \dots, \quad s = 1, 2, \dots,$$

which satisfy the following conditions:

$$P_s(x, y) = f_s(x, y), \quad (x, y) \in E_s, \tag{66}$$

$$|E_s| > 1 - 2^{-(s+1)}, \quad E_s \subset T, \tag{67}$$

$$\sum_{k,\nu=N_{s-1}}^{N_s-1} |c_{k,\nu}^{(s)}|^{2+2^{-2s}} < 2^{-2s}, \tag{68}$$

$$\begin{aligned}
& \max_{N_{s-1} \leq \bar{n}, \bar{m} < N_s} \left[\iint_E \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| dx dy \right] \\
& + \max_{\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s} \left[\iint_E \left| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} \psi_k(x) \right. \right. \\
& \quad \left. \left. \cdot \psi_\nu(y) \right| dx dy \right] \\
& \leq 2 \cdot \iint_E |f_s(x, y)| dx dy + 2^{-2(s+1)},
\end{aligned} \tag{69}$$

for every measurable subset e of E_s .

Denote

$$\begin{aligned}
& \sum_{k, \nu=1}^{\infty} c_{k, \nu} \psi_k(x) \psi_\nu(y) \\
& = \sum_{s=1}^{\infty} \left[\sum_{k, \nu=N_{s-1}}^{N_s-1} c_{k, \nu}^{(s)} \psi_k(x) \psi_\nu(y) \right],
\end{aligned} \tag{70}$$

where

$$c_{k, \nu} = c_{k, \nu}^{(s)} \quad \text{for } N_{s-1} \leq k, \nu < N_s, \quad s = 1, 2, \dots \tag{71}$$

For an arbitrary number $\varepsilon > 0$ we set

$$\begin{aligned}
\Omega_n &= \bigcap_{s=n}^{\infty} E_s, \quad n = 1, 2, \dots, \\
E &= \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \quad n_0 = [\log_{1/1} \varepsilon] + 1, \\
B &= \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \cup \left(\bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right).
\end{aligned} \tag{72}$$

It is obvious (see (67) and (72)) that $|B| = 1$ and $|E| > 1 - \varepsilon$.

We define a function $\mu(x, y)$ in the following way:

$$\mu(x, y) = \begin{cases} 1, & \text{for } (x, y) \in E \cup (T \setminus B); \\ \mu_n, & \text{for } (x, y) \in \Omega_n \setminus \Omega_{n-1}, \quad n \geq n_0 + 1, \end{cases} \tag{73}$$

where

$$\begin{aligned}
\mu_n &= \left[2^{2n} \cdot \prod_{s=1}^n h_s \right]^{-1}, \\
h_s &= \|f_s\|_C + \max_{N_{s-1} \leq \bar{n}, \bar{m} < N_s} \left\| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right\|_C \\
& + \max_{\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s} \left\| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right\|_C + 1.
\end{aligned} \tag{74}$$

From (68) and (70)–(74) we obtain the following:

(A) $0 < \mu(x, y) \leq 1$, $\mu(x, y)$ is a measurable function and

$$|\{(x, y) \in T : \mu(x, y) \neq 1\}| < \varepsilon. \tag{75}$$

(B) Consider $\sum_{k, \nu=1}^{\infty} |c_{k, \nu}|^q < \infty$ for all $q > 2$.

Hence, obviously we have (see (68) and (70))

$$\lim_{\min\{k, \nu\} \rightarrow \infty} c_{k, \nu} = 0. \tag{76}$$

It follows from (72)–(74) that for all $s \geq n_0$ and $N_{s-1} \leq \bar{n}, \bar{m} < N_s$

$$\begin{aligned}
& \iint_{T \setminus \Omega_s} \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu(x, y) dx dy \\
& = \sum_{n=s+1}^{\infty} \left[\iint_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \mu_n dx dy \right] \\
& \leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[\iint_T \left| \sum_{k, \nu=N_{s-1}}^{\bar{n}, \bar{m}} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| h_s^{-1} dx dy \right] \\
& < \frac{1}{3} 2^{-2s}.
\end{aligned} \tag{77}$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have

$$\begin{aligned}
& \iint_{T \setminus \Omega_s} \left| \sum_{2N_{s-1}^2 \leq k^2 + \nu^2 \leq R^2} c_{k, \nu}^{(s)} \psi_k(x) \cdot \psi_\nu(y) \right| \\
& \quad \times \mu(x, y) dx dy < \frac{1}{3} 2^{-2s}.
\end{aligned} \tag{78}$$

By (65) and (72)–(74) for all $s \geq n_0$ we have

$$\begin{aligned}
& \iint_T |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\
& = \iint_{\Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\
& \quad + \iint_{T \setminus \Omega_s} |P_s(x, y) - f_s(x, y)| \mu(x, y) dx dy \\
& = \sum_{n=s+1}^{\infty} \left[\iint_{\Omega_n \setminus \Omega_{n-1}} |P_s(x, y) - f_s(x, y)| \mu_n dx dy \right] \\
& \leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[\iint_T \left(|f_s(x, y)| + \sum_{k, \nu=N_{s-1}}^{N_s-1} c_{k, \nu}^{(s)} \psi_k(x) \right. \right. \\
& \quad \left. \left. \cdot \psi_\nu(y) \right) h_s^{-1} dx dy \right] \\
& < \frac{1}{3} 2^{-2s} < 2^{-2s}.
\end{aligned} \tag{79}$$

By (69) and (72)–(77) for all $N_{s-1} \leq \bar{n}$, $\bar{m} < N_s$ and $s \geq n_0 + 1$ we obtain

$$\begin{aligned}
 & \int \int_T \left| \sum_{k,v=N_{s-1}}^{\bar{n},\bar{m}} c_{k,v}^{(s)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \\
 & \int \int_{\Omega_s} \left| \sum_{k,v=N_{s-1}}^{\bar{n},\bar{m}} c_{k,v}^{(s)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \\
 & \int \int_{T \setminus \Omega_s} \left| \sum_{k,v=N_{s-1}}^{\bar{n},\bar{m}} c_{k,v}^{(s)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \\
 & < \sum_{n=n_0+1}^s \left[\int \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{k,v=N_{s-1}}^{\bar{n},\bar{m}} c_{k,v}^{(s)} \psi_k(x) \cdot \psi_v(y) \right| \cdot \mu_n dx dy \right] \\
 & \quad + \frac{1}{3} 2^{-2s} \\
 & < \sum_{n=n_0+1}^s \left(2^{-2(s+1)} + 2 \cdot \int \int_{\Omega_n \setminus \Omega_{n-1}} |f_s(x, y)| dx dy \right) \cdot \mu_n \\
 & \quad + \frac{1}{3} 2^{-2s} \\
 & = 2^{-2(s+1)} \cdot \sum_{n=n_0+1}^s \mu_n + \int \int_{\Omega_s} |f_s(x, y)| \mu(x, y) dx dy \\
 & \quad + \frac{1}{3} 2^{-2s} \\
 & < 2 \cdot \int \int_T |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}.
 \end{aligned} \tag{80}$$

Analogously for all $s \geq n_0$ and $\sqrt{2}N_{s-1} \leq R \leq \sqrt{2}N_s$ we have (see (78))

$$\begin{aligned}
 & \int \int_T \left| \sum_{2N_{s-1}^2 \leq k^2 + v^2 \leq R^2} c_{k,v}^{(s)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \\
 & < 2 \cdot \int \int_T |f_s(x, y)| \mu(x, y) dx dy + 2^{-2s}.
 \end{aligned} \tag{81}$$

Now we will show that the series (70) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums.

Let $f(x, y) \in L^1_\mu(T)$, that is,

$$\int \int_T |f(x, y)| \mu(x, y) dx dy < \infty. \tag{82}$$

It is easy to see that we can choose a function $f_{n_1}(x, y)$ from the sequence (64) such that

$$\int \int_T |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy < 2^{-2}, \tag{83}$$

$$n_1 > n_0 + 1.$$

Hence, we have

$$\begin{aligned}
 & \int \int_T |f_{n_1}(x, y)| \mu(x, y) dx dy \\
 & < 2^{-2} + \int \int_T |f(x, y)| \mu(x, y) dx dy.
 \end{aligned} \tag{84}$$

From (80) and (83) we get

$$\begin{aligned}
 & \int \int_T |f(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy \\
 & \leq \int \int_T |f(x, y) - f_{n_1}(x, y)| \mu(x, y) dx dy \\
 & \quad + \int \int_T |f_{n_1}(x, y) - P_{n_1}(x, y)| \mu(x, y) dx dy \\
 & < 2 \cdot 2^{-2}.
 \end{aligned} \tag{85}$$

Assume that numbers $n_1 < n_2 < \dots < n_{q-1}$ are chosen in such a way that the following condition is satisfied:

$$\begin{aligned}
 & \int \int_T \left| f(x, y) - \sum_{s=1}^j P_{n_s}(x, y) \right| \mu(x, y) dx dy < 2 \cdot 2^{-2j}, \\
 & 1 \leq j \leq q-1.
 \end{aligned} \tag{86}$$

Now we choose a function $f_{n_q}(x, y)$ from the sequence (64) such that

$$\begin{aligned}
 & \int \int_T \left| \left(f(x, y) - \sum_{s=1}^{q-1} P_{n_s}(x, y) \right) - f_{n_q}(x, y) \right| \mu(x, y) dx dy \\
 & < 2 \cdot 2^{-2q}, \quad n_q > n_{q-1}.
 \end{aligned} \tag{87}$$

This with (86) implies

$$\begin{aligned}
 & \int \int_T |f_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2q} + 2 \cdot 2^{-2(q-1)} \\
 & = 9 \cdot 2^{-2q}.
 \end{aligned} \tag{88}$$

Hence and from (65) and (79)–(81) we obtain

$$\int \int_T |f_{n_q}(x, y) - P_{n_q}(x, y)| \mu(x, y) dx dy < 2^{-2n_q}, \tag{89}$$

where

$$P_{n_q}(x, y) = \sum_{k,v=N_{n_q-1}}^{N_{n_q}-1} c_{k,v}^{(n_q)} \psi_k(x) \psi_v(y), \tag{90}$$

$$\begin{aligned}
 & \max_{N_{n_q-1} \leq \bar{n}, \bar{m} < N_{n_q}} \left[\int \int_T \left| \sum_{k,v=N_{n_q-1}}^{\bar{n},\bar{m}} c_{k,v}^{(n_q)} \psi_k(x) \right. \right. \\
 & \quad \left. \left. \cdot \psi_v(y) \right| \mu(x, y) dx dy \right] \\
 & < 19 \cdot 2^{-2q}.
 \end{aligned} \tag{91}$$

Analogously we have

$$\max_{\sqrt{2}N_{n_q-1} \leq R \leq \sqrt{2}N_{n_q}} \left[\int \int_T \left| \sum_{2N_{n_q-1}^2 \leq k^2 + y^2 \leq R^2} c_{k,v}^{(n_q)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \right] < 19 \cdot 2^{-2q}. \quad (92)$$

In quality subseries of the theorem we will take

$$\sum_{q=1}^{\infty} P_{n_q}(x, y) = \sum_{q=1}^{\infty} \left[\sum_{k,v=N_{n_q-1}}^{N_{n_q}-1} c_{k,v}^{(n_q)} \psi_k(x) \psi_v(y) \right]. \quad (93)$$

From (87) and (88) we have

$$\begin{aligned} & \int \int_T \left| f(x, y) - \sum_{s=1}^q P_{n_s}(x, y) \right| \mu(x, y) dx dy \\ & \leq \int \int_T \left| \left(f(x, y) - \sum_{s=1}^{q-1} P_{n_s}(x, y) \right) - f_{n_q}(x, y) \right| \\ & \quad \times \mu(x, y) dx dy \\ & + \int \int_T \left| f_{n_q}(x, y) - P_{n_q}(x, y) \right| \mu(x, y) dx dy < 2 \cdot 2^{-2q}. \end{aligned} \quad (94)$$

Let $\bar{n}$ and $\bar{m}$ be arbitrary natural numbers. Then for some natural number q we have

$$N_{n_q-1} \leq \min \{\bar{n}, \bar{m}\} < N_{n_q}. \quad (95)$$

Taking into account (89) and (93) for rectangular partial sums $S_{\bar{n}, \bar{m}}(x, y)$ of (91) we get

$$\begin{aligned} & \int \int_T |S_{\bar{n}, \bar{m}}(x, y) - f(x, y)| \mu(x, y) dx dy \\ & \leq \int \int_T \left| f(x, y) - \sum_{s=1}^q P_{n_s}(x, y) \right| \mu(x, y) dx dy \\ & + \max_{N_{n_q-1} \leq \bar{n}, \bar{m} < N_{n_q}} \left[\int \int_T \left| \sum_{k,v=N_{n_q-1}}^{\bar{n}, \bar{m}} c_{k,v}^{(n_q)} \psi_k(x) \cdot \psi_v(y) \right| \mu(x, y) dx dy \right] \\ & < 21 \cdot 2^{-2q}. \end{aligned} \quad (96)$$

Analogously for $\sqrt{2}N_{n_q-1} \leq R \leq \sqrt{2}N_{n_q}$ we have

$$\int \int_T |S_R(x, y) - f(x, y)| \mu(x, y) dx dy < 21 \cdot 2^{-2q}, \quad (97)$$

where $S_R(x, y)$ is the spherical partial sums of (91).

From (96) and (97) we conclude that the series (70) is universal in $L^1_\mu(T)$ concerning subseries with respect to convergence by both spherical and rectangular partial sums (see Definition 2).

Theorem 3 is proved. $\square$

Remark 8. We can prove Theorem 4 by the same method used in the proof of Theorem 3.

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