Research Article

Some New Explicit Values of Parameters λ_n and μ_n of Quotients of Eta-Function

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We find some new explicit values of the parameters λ_n and μ_n of quotients of eta-function by using Ramanujan's class invariants.

1. Introduction

For $q = e^{2\pi iz}$ and $\text{Im}(z) > 0$, the Dedekind eta-function $\eta(z)$ and Ramanujan's function $f(-q)$ are defined by

$$f(-q) := (q; q)_{\infty} =: q^{-1/24} \eta(z), \quad (1)$$

where $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$.

In page 212 of his lost notebook [1], Ramanujan defined

$$\lambda_n := \frac{f^6(q)}{2\sqrt{2}\sqrt{q}f^6(q^3)}, \quad q := e^{-\pi\sqrt{n/3}} \quad (2)$$

and provided a list of eleven recorded and ten unrecorded values of λ_n for positive integers n . All 21 values of λ_n and many more were established by Berndt et al. [2] by using the modular j -invariant, modular equations, Kronecker's limit formula, and the explicit Shimura reciprocity law. An account of this can also be found in Chapter 9 of [3]. Closely related to λ_n is the parameter μ_n introduced by Ramanathan [4] and is defined as

$$\mu_n := \frac{1}{3\sqrt{3}} \frac{f^6(-q)}{\sqrt{q}f^6(-q^3)}, \quad q = e^{-2\pi\sqrt{n/3}}. \quad (3)$$

Yi [5, 6] also found several values of parameters λ_n and μ_n by finding the explicit values of her parameters $r_{k,n}$ and $r'_{k,n}$, defined by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}},$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \quad (4)$$

where n and k are positive real numbers. In fact, $r_{3,n}^{1/6} = \lambda_n$, and $r_{3,n}^{1/6} = \mu_n$. Baruah and Saikia [7, 8] also evaluated several new values of λ_n and μ_n . A generalization of λ_n has been studied by Naika et al. [9]. Here we give a list of the values of n for which λ_n and μ_n were evaluated in [1, 2, 5–8] without referring them specifically to avoid repetitions. The values of n for which λ_n were evaluated in the literature are $n = 1, 3, 5, 7, 9, 11, 13, 15, 5/3, 17, 25, 33, 41, 49, 57, 65, 81, 89, 73, 97, 121, 169, 193, 217, 241, 265, 289, 361, 11/3$, and $19/3$, and the values of values of n for which μ_n were evaluated in literature are $n = 1, 2, 3, 4, 5, 6, 3/2, 7, 8, 9, 10, 5/2, 11, 13, 14, 7/2, 15, 5/3, 17, 19, 20, 5/4, 22, 11/2, 25, 26, 13/2, 44, 11/4$, and 49 . We also note from [2, page 281] and [7, page 43, Theorem 4.2] that

$$\lambda_{1/n} = \frac{1}{\lambda_n}, \quad \mu_{1/n} = \frac{1}{\mu_n}, \quad (5)$$

respectively.

In this paper we find further new values of λ_n and μ_n by using Ramanujan's class invariants which are defined as

$$\begin{aligned} G_n &= 2^{-1/4} q^{-1/24} \chi(q), \\ g_n &= 2^{-1/4} q^{-1/24} \chi(-q); \quad q := e^{-\pi\sqrt{n}}, \end{aligned} \quad (6)$$

where $\chi(q) = (-q; q^2)_\infty$. In particular, we evaluate values of λ_n for $n = 23, 31, 47, 59, 71, 2, 4, 6, 10, 5/2, 14$, and $7/2$ and new values of μ_n for $n = 23, 31, 47, 59$, and 71 . It worth to mention here that for the first time in this paper explicit values of λ_n for some even values of n are evaluated. Previously, Berndt et al. [2] calculated explicit values of λ_1, λ_3 , and μ_1 by using Ramanujan's class invariants. An account of Ramanujan's class invariants can be found in Berndt's book [10]. For further references on Ramanujan's class invariants refer see [11–14].

The explicit values of the functions λ_n and μ_n can be applied to find explicit values of Ramanujan's cubic continued fraction $G(q)$ defined by

$$G(q) := \frac{q^{1/3}}{1 + \frac{q+q^2}{1 + \frac{q^2+q^4}{1 + \frac{q^3+q^6}{1 + \dots}}}}, \quad |q| < 1. \quad (7)$$

We refer to [7] for details. To this end we define a general parameter $I_{k,n}$ [14, page 2, equation (10)] for all positive real numbers n and k and connected with parameters λ_n, μ_n , and Ramanujan's class invariants as

$$I_{k,n} := \frac{\chi(q)}{q^{(-k+1)/24} \chi(q^k)}; \quad q = e^{-\pi\sqrt{n/k}}. \quad (8)$$

Saikia [14] studied several properties of the parameter $I_{k,n}$ and evaluated many explicit values of $I_{k,n}$ by using Ramanujan's modular equations. From [14, page 4, Theorem 16(iii)], we also note here that $I_{k,n} = I_{n,k}$.

In Section 2, we record some preliminary results, and in Section 3 we evaluate new values of the parameters λ_n and μ_n .

2. Preliminaries

Lemma 1 (see [3, page 203, Theorem 9.3.1]). *One has*

$$\frac{\lambda_n}{\mu_n} = \left(\frac{G_{n/3}}{G_{3n}} \right)^6. \quad (9)$$

Lemma 2 (see [14, page 4, Theorem 22(i)]). *One has*

$$I_{k,n} = \frac{G_{n/k}}{G_{kn}}. \quad (10)$$

Lemma 3. *One has*

$$\frac{\lambda_n}{\mu_n} = \left(\frac{G_{n/3}}{G_{3n}} \right)^6 = I_{3,n}^6. \quad (11)$$

Proof. We set $k = 3$ in Lemma 2 and apply Lemma 1 to complete the proof. $\square$

Lemma 4. *One has*

$$\begin{aligned} &9 \left\{ (\lambda_n \mu_n)^{2/3} + (\lambda_n \mu_n)^{-2/3} \right\} - 16 \\ &= \left(\frac{G_{n/3}}{G_{3n}} \right)^{12} + \left(\frac{G_{3n}}{G_{n/3}} \right)^{12}. \end{aligned} \quad (12)$$

This readily follows from [2, page 9, equation (3.4)] or [3, page 203, equation (9.3.4)].

Lemma 5. *One has*

$$\begin{aligned} &\left(\frac{G_{n/3}}{G_{3n}} \right)^{12} + \left(\frac{G_{3n}}{G_{n/3}} \right)^{12} \\ &= 8 \left\{ (G_{n/3} G_{3n})^6 + (G_{n/3} G_{3n})^{-6} \right\} - 18. \end{aligned} \quad (13)$$

Proof. From [11, page 123, equation (4.14)], we note that

$$\left(\frac{G_n}{G_{9n}} \right)^6 + \left(\frac{G_{9n}}{G_n} \right)^6 = 2\sqrt{2} \left\{ (G_n G_{9n})^3 - (G_n G_{9n})^{-3} \right\}. \quad (14)$$

Replacing n by $n/3$ in (14) and squaring, we arrive at desired result. $\square$

Lemma 6. *One has*

$$G_{31/3} = \left(\frac{39 + 7\sqrt{31}}{\sqrt{2}} \right)^{1/6} \left(\frac{\sqrt{31} - 3\sqrt{3}}{2} \right)^{1/4}, \quad (15)$$

$$G_{93} = \left(\frac{39 + 7\sqrt{31}}{\sqrt{2}} \right)^{1/6} \left(\frac{\sqrt{31} + 3\sqrt{3}}{2} \right)^{1/4}, \quad (16)$$

$$G_{117} = \left(\frac{3\sqrt{3} + 23}{\sqrt{2}} \right)^{1/6} \left(\frac{\sqrt{3} + 1}{\sqrt{2}} \right)^{1/4}, \quad (17)$$

$$G_{59/3} = \left(\frac{3\sqrt{3} - 23}{\sqrt{2}} \right)^{1/6} \left(\frac{\sqrt{3} + 1}{\sqrt{2}} \right)^{1/4}, \quad (18)$$

$$\left(\frac{G_{69}}{G_{23/3}} \right)^6 = \left(\frac{5 + \sqrt{23}}{\sqrt{2}} \right) \left(\frac{3\sqrt{3} + \sqrt{23}}{2} \right)^{3/2}, \quad (19)$$

$$\left(\frac{G_{141}}{G_{47/3}} \right)^6 = \left(\frac{7 + \sqrt{47}}{\sqrt{2}} \right) (4\sqrt{3} + \sqrt{47})^{3/2}, \quad (20)$$

$$\left(\frac{G_{213}}{G_{71/3}} \right)^6 = \left(\frac{5\sqrt{3} + \sqrt{71}}{2} \right)^{3/2} \left(\frac{59 + 7\sqrt{71}}{\sqrt{2}} \right), \quad (21)$$

$$I_{3,2} = \left(\frac{G_{2/3}}{G_6} \right)^6 = \left(-44 + 27\sqrt{3} - 3\sqrt{458 - 264\sqrt{3}} \right)^{1/12}, \quad (22)$$

$$I_{3,4} = \left(\frac{G_{4/3}}{G_{12}} \right)^6 = \left(\frac{\left(\sqrt{-2 + 3\sqrt{6}} - \sqrt{-6 + 3\sqrt{6}} \right)}{2} \right)^{1/2}. \quad (23)$$

For (15), (16), (17) and (18) see [10, page 342]. For (19), (20), and (21) see [10, page 227, equation (5.8)], [10, page 229, equation (5.16)], and [10, page 257, equation (6.61)]. For (22) and (23) see [14, page 5, Corollary 24(i & ii)].

Lemma 7. *One has*

$$g_{10/3}g_{30} = (19 + 6\sqrt{10})^{1/6}, \quad (24)$$

$$g_{2/15}g_{6/5} = (19 - 6\sqrt{10})^{1/6}, \quad (25)$$

$$g_{14/3}g_{42} = \left(\frac{5 + \sqrt{21}}{2}\right)^{1/2}, \quad (26)$$

$$g_{2/21}g_{6/7} = \left(\frac{5 - \sqrt{21}}{2}\right)^{1/2}, \quad (27)$$

$$g_{6/3}g_{18} = (\sqrt{2} + \sqrt{3})^{1/3}. \quad (28)$$

For (24) to (27) see [12]. Equation (28) follows from the values of g_2 and g_{18} from [10, page 200].

Lemma 8. *One has*

$$(G_n G_{9n})^2 = \frac{(g_n g_{9n})^3 + \sqrt{(g_n g_{9n})^6 + 2}}{2g_n g_{9n}}. \quad (29)$$

Proof. From [15, page 124], we note that

$$\chi(q) = 2^{1/6} \left\{ \frac{\alpha(1-\alpha)}{q} \right\}^{-1/24}, \quad (30)$$

$$\chi(-q) = 2^{1/6} (1-\alpha)^{1/12} \left(\frac{\alpha}{q} \right)^{-1/24}.$$

Employing (30) in (6) and simplifying, we deduce that

$$1 - \alpha = \left(\frac{g_n}{G_n} \right)^8. \quad (31)$$

From [15, page 230, Entry 5(ii)], we note that

$$(\alpha\beta)^{1/4} + ((1-\alpha)(1-\beta))^{1/4} = 1, \quad (32)$$

where β has degree 3 over α .

Combining (31) and (32) and simplifying, we deduce that

$$\{(G_n^8 - g_n^8)(G_{9n}^8 - g_{9n}^8)\}^{1/4} + (g_n g_{9n})^2 = (G_n G_{9n})^2. \quad (33)$$

From [10, page 187, Entry 2.2], we note that

$$(g_n G_n)^8 (G_n^8 - g_n^8) = \frac{1}{4}. \quad (34)$$

Employing (34) in (33), and simplifying, we find that

$$2(G_n G_{9n})^4 (g_n g_{9n})^2 - 2(G_n G_{9n})^2 (g_n g_{9n})^4 - 1 = 0. \quad (35)$$

Solving (35) for $(G_n G_{9n})^2$, we arrive at the desired result. $\square$

Lemma 9. *One has*

$$(G_{10/3} G_{30})^2 = \frac{(3 + \sqrt{10} + \sqrt{3}(\sqrt{5} + \sqrt{2}))(\sqrt{10} - 3)^{1/3}}{2}, \quad (36)$$

$$(G_{2/15} G_{6/5})^2 = \frac{(-3 + \sqrt{10} + \sqrt{3}(\sqrt{5} - \sqrt{2}))(\sqrt{10} + 3)^{1/3}}{2}, \quad (37)$$

$$(G_{14/3} G_{42})^2 = \frac{(6 + 3\sqrt{3} + 2\sqrt{7} + \sqrt{21})(\sqrt{7} - \sqrt{3})}{4}, \quad (38)$$

$$(G_{6/7} G_{2/21})^2 = \frac{(6 - \sqrt{21} - 3\sqrt{3} + 2\sqrt{7})(\sqrt{7} + \sqrt{3})}{4}, \quad (39)$$

$$(G_{6/3} G_{18})^2 = \frac{(1 + \sqrt{2} + \sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{2})^{1/3}}{2}. \quad (40)$$

Proof. We set $n = 10/3, 2/15, 14/3, 6/7$, and $6/3$ in Lemma 8 and employ the corresponding values of $g_n g_{9n}$ from Lemma 7 to complete the proof. $\square$

3. New Values of μ_n and λ_n

The section is devoted to evaluate new values of μ_n and λ_n .

Theorem 10. *One has*

$$\begin{aligned} \lambda_{23} &= 2^{-3/8} (24 - 5\sqrt{3})^{1/4} (25 - 3\sqrt{69})^{3/8} \\ &\quad \times \left(167 + 96\sqrt{3} + 4\sqrt{347 + 2004\sqrt{3}} \right)^{3/4}, \\ \mu_{23} &= 2^{-3/8} (24 + 5\sqrt{3})^{1/4} (25 + 3\sqrt{69})^{3/8} \\ &\quad \times \left(167 + 96\sqrt{3} + 4\sqrt{347 + 2004\sqrt{3}} \right)^{3/4}. \end{aligned} \quad (41)$$

Proof. Employing (19) in Lemma 3 with $n = 23$, we have

$$\frac{\mu_{23}}{\lambda_{23}} = \left(\frac{G_{69}}{G_{23/3}} \right)^6 = \left(\frac{5 + \sqrt{23}}{\sqrt{2}} \right) \left(\frac{3\sqrt{3} + \sqrt{23}}{2} \right)^{3/2}. \quad (42)$$

From (42), we deduce that

$$\left(\frac{G_{69}}{G_{23/3}} \right)^{12} + \left(\frac{G_{23/3}}{G_{69}} \right)^{12} = 2990 + 1728\sqrt{3}. \quad (43)$$

Setting $n = 23$ in Lemma 4, employing (43), and solving the resulting equation, we find that

$$\mu_{23} \lambda_{23} = \left(167 + 96\sqrt{3} + 4\sqrt{347 + 2004\sqrt{3}} \right)^{3/2}. \quad (44)$$

Multiplying (42) and (44) and simplifying, we easily calculate the value of μ_{23} , and dividing (44) by (42) and simplifying we evaluate the value of λ_{23} . $\square$

Remark 11. The values of $\lambda_{1/23}$ and $\mu_{1/23}$ also follow from (5) and the values of λ_{23} and μ_{23} , respectively.

Theorem 12. *One has*

$$\begin{aligned}\lambda_{47} &= (48 - 7\sqrt{47})^{1/4} (756\sqrt{3} - 191\sqrt{47})^{1/4} \\ &\quad \times \left(6983 + 4032\sqrt{3} + 12\sqrt{677315 + 391048\sqrt{3}} \right)^{3/4}, \\ \mu_{47} &= (48 + 7\sqrt{47})^{1/4} (756\sqrt{3} + 191\sqrt{47})^{1/4} \\ &\quad \times \left(6983 + 4032\sqrt{3} + 12\sqrt{677315 + 391048\sqrt{3}} \right)^{3/4}.\end{aligned}\quad (45)$$

Proof. Employing (20) in Lemma 3 with $n = 47$, we have

$$\frac{\mu_{47}}{\lambda_{47}} = \left(\frac{G_{141}}{G_{47/3}} \right)^6 = \left(\frac{7 + \sqrt{47}}{\sqrt{2}} \right) (4\sqrt{3} + \sqrt{47})^{3/2}. \quad (46)$$

From (46), we deduce that

$$\left(\frac{G_{141}}{G_{47/3}} \right)^{12} + \left(\frac{G_{47/3}}{G_{141}} \right)^{12} = 125854 + 72576\sqrt{3}. \quad (47)$$

Setting $n = 47$ in Lemma 4, employing (47), and solving the resulting equation, we obtain

$$\mu_{47}\lambda_{47} = \left(6983 + 4032\sqrt{3} + 12\sqrt{677315 + 391048\sqrt{3}} \right)^{3/2}. \quad (48)$$

Multiplying (46) and (48) and simplifying, we easily calculate the value of μ_{47} , and dividing (48) by (46) and simplifying we evaluate the value of λ_{47} . $\square$

Remark 13. The values of $\lambda_{1/47}$ and $\mu_{1/47}$ also follow from (5) and the values of λ_{47} and μ_{47} , respectively.

Theorem 14. *One has*

$$\begin{aligned}\lambda_{71} &= 2^{-1/4} (59 - 7\sqrt{71})^{1/2} (180\sqrt{3} - 37\sqrt{71})^{1/4} \\ &\quad \times \left(120551 + 69600\sqrt{3} + 20\sqrt{72662559 + 41951748\sqrt{3}} \right)^{3/4}, \\ \mu_{71} &= 2^{-1/4} (59 + 7\sqrt{71})^{1/2} (180\sqrt{3} + 37\sqrt{71})^{1/4} \\ &\quad \times \left(120551 + 69600\sqrt{3} + 20\sqrt{72662559 + 41951748\sqrt{3}} \right)^{3/4}.\end{aligned}\quad (49)$$

Proof. Employing (21) in Lemma 3 with $n = 71$, we have

$$\frac{\mu_{71}}{\lambda_{71}} = \left(\frac{G_{213}}{G_{71/3}} \right)^6 = \left(\frac{5\sqrt{3} + \sqrt{71}}{2} \right)^{3/2} \left(\frac{59 + 7\sqrt{71}}{\sqrt{2}} \right). \quad (50)$$

From (50), we deduce that

$$\left(\frac{G_{213}}{G_{71/3}} \right)^{12} + \left(\frac{G_{71/3}}{G_{213}} \right)^{12} = 2169902 + 1252800\sqrt{3}. \quad (51)$$

Setting $n = 23$ in Lemma 4, employing (51), and solving the resulting equation, we obtain

$$\begin{aligned}\mu_{71}\lambda_{71} &= \left(120551 + 69600\sqrt{3} + 20\sqrt{72662559 + 41951748\sqrt{3}} \right)^{3/2}.\end{aligned}\quad (52)$$

Combining (50) and (52) we easily evaluate the values of μ_{71} and λ_{71} . $\square$

Remark 15. The values of $\lambda_{1/71}$ and $\mu_{1/71}$ also follow from (5) and the values of λ_{71} and μ_{71} , respectively.

Theorem 16. *One has*

$$\begin{aligned}\lambda_{31} &= 2^{-3/2} (1351 + 780\sqrt{3})^{3/4} (\sqrt{31} - 3\sqrt{3})^{1/4}, \\ \mu_{31} &= 2^{-3/2} (1351 + 780\sqrt{3})^{3/4} (\sqrt{31} + 3\sqrt{3})^{1/4}.\end{aligned}\quad (53)$$

Proof. Combining (15), (16) and Lemma 3, we deduce that

$$\frac{\mu_{31}}{\lambda_{31}} = \left(\frac{G_{93}}{G_{31/3}} \right)^6 = \frac{(\sqrt{31} + 3\sqrt{3})^3}{8}. \quad (54)$$

From (54), we find that

$$\left(\frac{G_{93}}{G_{31/3}} \right)^{12} + \left(\frac{G_{31/3}}{G_{93}} \right)^{12} = 24302. \quad (55)$$

Setting $n = 31$ in Lemma 4, employing (55), and solving the resulting equation, we obtain

$$\lambda_{31}\mu_{31} = (1351 + 780\sqrt{3})^{3/2}. \quad (56)$$

Combining (54) and (56) we easily evaluate the values of λ_{31} and μ_{31} . $\square$

Remark 17. The values of $\lambda_{1/31}$ and $\mu_{1/31}$ can easily be calculated by employing the values of λ_{31} and μ_{31} , respectively, in (5).

Theorem 18. *One has*

$$\begin{aligned}\lambda_{59} &= 2^{-1/2} (3\sqrt{59} - 23) (62423 + 4692\sqrt{177})^{3/4}, \\ \mu_{59} &= 2^{-1/2} (3\sqrt{59} + 23) (62423 + 4692\sqrt{177})^{3/4}.\end{aligned}\quad (57)$$

Proof. Combining (17), (18) and Lemma 3, we deduce that

$$\frac{\mu_{59}}{\lambda_{59}} = \left(\frac{G_{177}}{G_{59/3}} \right)^6 = \frac{(3\sqrt{59} + 23)^2}{2}. \quad (58)$$

From (58), we find that

$$\left(\frac{G_{177}}{G_{59/3}}\right)^{12} + \left(\frac{G_{59/3}}{G_{177}}\right)^{12} = 1123598. \quad (59)$$

Setting $n = 59$ in Lemma 4, employing (59), and solving the resulting equation, we obtain

$$\lambda_{59}\mu_{59} = (62423 + 4692\sqrt{177})^{3/2}. \quad (60)$$

Combining (58) and (60) we easily calculate the values of λ_{59} and μ_{59} . $\square$

Remark 19. The values of $\lambda_{1/59}$ and $\mu_{1/59}$ can easily be calculated by employing the values of λ_{59} and μ_{59} , respectively, in (5).

Next, we evaluate some values of λ_n and μ_n for even values of n . The values of λ_n are new. The values of μ_n are classical, but the method of evaluations is new.

Theorem 20. *One has*

$$\begin{aligned} \lambda_2 &= \left(-4 + 3\sqrt{3} + \sqrt{42 - 24\sqrt{3}}\right)^{3/4} \\ &\quad \times \left(-44 + 27\sqrt{3} - 3\sqrt{458 - 264\sqrt{3}}\right)^{1/4}, \\ \mu_2 &= \left(-4 + 3\sqrt{3} + \sqrt{42 - 24\sqrt{3}}\right)^{3/4} \\ &\quad \times \left(-44 + 27\sqrt{3} + 3\sqrt{458 - 264\sqrt{3}}\right)^{1/4}. \end{aligned} \quad (61)$$

Proof. Combining (22) and Lemma 3 with $n = 2$, we note that

$$\begin{aligned} \frac{\lambda_2}{\mu_2} &= \left(\frac{G_{2/3}}{G_6}\right)^6 \\ &= \left(-44 + 27\sqrt{3} - 3\sqrt{458 - 264\sqrt{3}}\right)^{1/2}. \end{aligned} \quad (62)$$

From (62), we deduce that

$$\left(\frac{G_{2/3}}{G_6}\right)^{12} + \left(\frac{G_6}{G_{2/3}}\right)^{12} = -88 + 54\sqrt{3}. \quad (63)$$

Setting $n = 2$ in Lemma 4, employing (63), and solving the resulting equation, we obtain

$$\lambda_2\mu_2 = \left(-4 + 3\sqrt{3} + \sqrt{42 - 24\sqrt{3}}\right)^{3/2}. \quad (64)$$

Combining (62) and (64) we easily evaluate the values of λ_2 and μ_2 . $\square$

Remark 21. The values of $\lambda_{1/2}$ and $\mu_{1/2}$ can easily be calculated by employing the values of λ_2 and μ_2 , respectively, in (5).

Theorem 22. *One has*

$$\begin{aligned} \lambda_4 &= 2^{-9/4} \left(-76 + 33\sqrt{6} + \sqrt{12306 - 5016\sqrt{6}}\right)^{3/4} \\ &\quad \times \left(\sqrt{-2 + 3\sqrt{6}} - \sqrt{-6 + 3\sqrt{6}}\right)^{3/2}, \\ \mu_4 &= 2^{-9/4} \left(-76 + 33\sqrt{6} + \sqrt{12306 - 5016\sqrt{6}}\right)^{3/4} \\ &\quad \times \left(\sqrt{-2 + 3\sqrt{6}} + \sqrt{-6 + 3\sqrt{6}}\right)^{3/2}. \end{aligned} \quad (65)$$

Proof. Combining (23) and Lemma 3 with $n = 2$, we note that

$$\frac{\lambda_4}{\mu_4} = \left(\frac{G_{4/3}}{G_{12}}\right)^6 = \left(\frac{\sqrt{-2 + 3\sqrt{6}} - \sqrt{-6 + 3\sqrt{6}}}{2}\right)^3. \quad (66)$$

From (66), we deduce that

$$\left(\frac{G_{4/3}}{G_{12}}\right)^{12} + \left(\frac{G_{12}}{G_{4/3}}\right)^{12} = -700 + 297\sqrt{6}. \quad (67)$$

Setting $n = 2$ in Lemma 4, employing (67), and solving the resulting equation, we obtain

$$\lambda_4\mu_4 = \left(\frac{-76 + 33\sqrt{6} + \sqrt{12306 - 5016\sqrt{6}}}{2}\right)^{3/2}. \quad (68)$$

Combining (66) and (68) we easily evaluate the values of λ_4 and μ_4 . $\square$

Remark 23. The values of $\lambda_{1/4}$ and $\mu_{1/4}$ can easily be calculated by employing the values of λ_4 and μ_4 , respectively, in (5).

Theorem 24. *One has*

$$\begin{aligned} \mu_6 &= \frac{1}{3\sqrt{3}}(s+t)^{1/4}(u+v)^{3/4}, \\ \lambda_6 &= \frac{1}{3\sqrt{3}}(s+t)^{-1/4}(u+v)^{3/4}, \end{aligned} \quad (69)$$

where

$$\begin{aligned} s &= -900 + 650\sqrt{2} + 525\sqrt{3} - 364\sqrt{6}, \\ t &= 5\sqrt{131074 - 92664\sqrt{2} - 75656\sqrt{3} + 53508\sqrt{6}}, \\ u &= -892 + 650\sqrt{2} + 525\sqrt{3} - 364\sqrt{6}, \\ v &= \sqrt{3262434 - 2306200\sqrt{2} - 1883000\sqrt{3} + 1331876\sqrt{6}}. \end{aligned} \quad (70)$$

Proof. Employing (40) in Lemma 5 with $n = 6$, solving the resulting equation, and noting the result $\lambda_n/\mu_n = (G_{n/3}/G_{3n})^6$ from Lemma 3, we obtain

$$\left(\frac{\mu_6}{\lambda_6}\right)^2 = s + t, \quad (71)$$

where

$$\begin{aligned} s &= -900 + 650\sqrt{2} + 525\sqrt{3} - 364\sqrt{6}, \\ t &= 5\sqrt{131074 - 92664\sqrt{2} - 75656\sqrt{3} + 53508\sqrt{6}}. \end{aligned} \quad (72)$$

Employing (71) in Lemma 4 and solving the resulting equation, we obtain

$$(\lambda_6\mu_6)^{2/3} = u + v, \quad (73)$$

where

$$\begin{aligned} u &= -892 + 650\sqrt{2} + 525\sqrt{3} - 364\sqrt{6}, \\ v &= \sqrt{3262434 - 2306200\sqrt{2} - 1883000\sqrt{3} + 1331876\sqrt{6}}. \end{aligned} \quad (74)$$

Combining (71) and (73) we calculate the values of μ_6 and λ_6 . $\square$

Remark 25. The values of $\lambda_{1/6}$ and $\mu_{1/6}$ can easily be calculated by employing the values of λ_6 and μ_6 , respectively, in (5).

Theorem 26. *One has*

$$\begin{aligned} \mu_{10} &= (a + b)^{1/4}(c + d)^{3/4}, \\ \lambda_{10} &= (c + d)^{3/4}(a + b)^{-1/4}, \end{aligned} \quad (75)$$

where

$$\begin{aligned} a &= -11960 + 4914\sqrt{6} - 3780\sqrt{10} + 3105\sqrt{15}, \\ b &= 3\sqrt{30(2131205 - 870044\sqrt{6} + 673946\sqrt{10} - 550264\sqrt{15})}, \\ c &= -1328 + 546\sqrt{6} - 420\sqrt{10} + 345\sqrt{15}, \\ d &= \sqrt{6(1183609 - 483196\sqrt{6} + 374290\sqrt{10} - 305600\sqrt{15})}. \end{aligned} \quad (76)$$

Proof. Employing (36) in Lemma 5 with $n = 10$, solving the resulting equation, and noting the result $\lambda_n/\mu_n = (G_{n/3}/G_{3n})^6$ from Lemma 3, we obtain

$$\left(\frac{\mu_{10}}{\lambda_{10}}\right)^2 = \left(\frac{G_{30}}{G_{10/3}}\right)^{12} = a + b, \quad (77)$$

where

$$\begin{aligned} a &= -11960 + 4914\sqrt{6} - 3780\sqrt{10} + 3105\sqrt{15}, \\ b &= 3\sqrt{30(2131205 - 870044\sqrt{6} + 673946\sqrt{10} - 550264\sqrt{15})}. \end{aligned} \quad (78)$$

Employing (77) in Lemma 4 and solving the resulting equation, we obtain

$$(\lambda_{10}\mu_{10})^{2/3} = c + d, \quad (79)$$

where

$$\begin{aligned} c &= -1328 + 546\sqrt{6} - 420\sqrt{10} + 345\sqrt{15}, \\ d &= \sqrt{6(1183609 - 483196\sqrt{6} + 374290\sqrt{10} - 305600\sqrt{15})}. \end{aligned} \quad (80)$$

Combining (77) and (79) we calculate the values of μ_{10} and λ_{10} . $\square$

Remark 27. The values of $\lambda_{1/10}$ and $\mu_{1/10}$ can easily be calculated by employing the values of λ_{10} and μ_{10} , respectively, in (5).

Theorem 28. *One has*

$$\begin{aligned} \mu_{5/2} &= (p + q)^{1/4}(r + s)^{3/4}, \\ \lambda_{5/2} &= (p + q)^{-1/4}(r + s)^{3/4}, \end{aligned} \quad (81)$$

where

$$\begin{aligned} p &= -11960 + 4914\sqrt{6} + 3780\sqrt{10} - 3105\sqrt{15}, \\ q &= 3\sqrt{30(2131205 - 870044\sqrt{6} - 673946\sqrt{10} + 550264\sqrt{15})}, \\ r &= -1328 + 546\sqrt{6} + 420\sqrt{10} - 345\sqrt{15}, \\ s &= \sqrt{6(1183609 - 483196\sqrt{6} - 374290\sqrt{10} + 305600\sqrt{15})}. \end{aligned} \quad (82)$$

Proof. Employing (37) in Lemma 5 with $n = 5/2$, solving the resulting equation, and noting the result $\lambda_n/\mu_n = (G_{n/3}/G_{3n})^6$ from Lemma 3, we obtain

$$\left(\frac{\mu_{5/2}}{\lambda_{5/2}}\right)^2 = \left(\frac{G_{15/2}}{G_{5/6}}\right)^{12} = p + q, \quad (83)$$

where

$$\begin{aligned} p &= -11960 + 4914\sqrt{6} + 3780\sqrt{10} - 3105\sqrt{15}, \\ q &= 3\sqrt{30(2131205 - 870044\sqrt{6} - 673946\sqrt{10} + 550264\sqrt{15})}. \end{aligned} \quad (84)$$

Employing (83) in Lemma 4 and solving the resulting equation, we obtain

$$(\lambda_{5/2}\mu_{5/2})^{2/3} = r + s, \quad (85)$$

where

$$\begin{aligned} r &= -1328 + 546\sqrt{6} + 420\sqrt{10} - 345\sqrt{15}, \\ s &= \sqrt{6(1183609 - 483196\sqrt{6} - 374290\sqrt{10} + 305600\sqrt{15})}. \end{aligned} \quad (86)$$

Combining (83) and (85) we calculate the values of $\mu_{5/2}$ and $\lambda_{5/2}$. $\square$

Remark 29. The values of $\lambda_{2/5}$ and $\mu_{2/5}$ can easily be calculated by employing the values of $\lambda_{5/2}$ and $\mu_{5/2}$, respectively, in (5).

Theorem 30. One has

$$\begin{aligned}\mu_{14} &= (e + f)^{1/4}(g + h)^{3/4}, \\ \lambda_{14} &= (e + f)^{-1/4}(g + h)^{3/4},\end{aligned}\quad (87)$$

where

$$\begin{aligned}e &= -98000 + 56700\sqrt{3} + 4125\sqrt{7} - 2376\sqrt{21}, \\ f &= 15\sqrt{171108062 - 98789040\sqrt{3} - 64672608\sqrt{7} + 37338840\sqrt{21}}, \\ g &= -10888 + 6300\sqrt{3} + 4125\sqrt{7} - 2376\sqrt{21}, \\ h &= \sqrt{6(79213469 - 45733800\sqrt{3} - 29939800\sqrt{7} + 17285796\sqrt{21})}.\end{aligned}\quad (88)$$

Proof. Employing (38) in Lemma 5 with $n = 14$, solving the resulting equation, and noting the result $\lambda_n/\mu_n = (G_{n/3}/G_{3n})^6$ from Lemma 3, we obtain

$$\left(\frac{\mu_{14}}{\lambda_{14}}\right)^2 = \left(\frac{G_{42}}{G_{14/3}}\right)^{12} = e + f, \quad (89)$$

where

$$\begin{aligned}e &= -98000 + 56700\sqrt{3} + 4125\sqrt{7} - 2376\sqrt{21}, \\ f &= 15\sqrt{171108062 - 98789040\sqrt{3} - 64672608\sqrt{7} + 37338840\sqrt{21}}.\end{aligned}\quad (90)$$

Employing (89) in Lemma 4 and solving the resulting equation, we obtain

$$(\lambda_{14}\mu_{14})^{2/3} = g + h, \quad (91)$$

where

$$\begin{aligned}g &= -10888 + 6300\sqrt{3} + 4125\sqrt{7} - 2376\sqrt{21}, \\ h &= \sqrt{6(79213469 - 45733800\sqrt{3} - 29939800\sqrt{7} + 17285796\sqrt{21})}.\end{aligned}\quad (92)$$

Combining (89) and (91) we calculate the values of μ_{14} and λ_{14} . $\square$

Remark 31. The values of $\lambda_{1/14}$ and $\mu_{1/14}$ can easily be calculated by employing the values of λ_{14} and μ_{14} , respectively, in (5).

Theorem 32. One has

$$\begin{aligned}\mu_{7/2} &= (l + m)^{1/4}(u + v)^{3/4}, \\ \lambda_{7/2} &= (l + m)^{-1/4}(u + v)^{3/4},\end{aligned}\quad (93)$$

where

$$\begin{aligned}l &= -98000 - 56700\sqrt{3} + 4125\sqrt{7} + 2376\sqrt{21}, \\ m &= 15\sqrt{171108062 + 98789040\sqrt{3} - 64672608\sqrt{7} - 37338840\sqrt{21}}, \\ u &= -10888 - 6300\sqrt{3} + 4125\sqrt{7} + 2376\sqrt{21}, \\ v &= \sqrt{6(79213469 + 45733800\sqrt{3} - 29939800\sqrt{7} - 17285796\sqrt{21})}.\end{aligned}\quad (94)$$

Proof. Employing (39) in Lemma 5 with $n = 7/2$, solving the resulting equation, and noting the result $\lambda_n/\mu_n = (G_{n/3}/G_{3n})^6$ from Lemma 3, we obtain

$$\left(\frac{\mu_{7/2}}{\lambda_{7/2}}\right)^2 = \left(\frac{G_{21/2}}{G_{7/6}}\right)^{12} = l + m, \quad (95)$$

where

$$\begin{aligned}l &= -98000 - 56700\sqrt{3} + 4125\sqrt{7} + 2376\sqrt{21}, \\ m &= 15\sqrt{171108062 + 98789040\sqrt{3} - 64672608\sqrt{7} - 37338840\sqrt{21}}.\end{aligned}\quad (96)$$

Employing (95) in Lemma 4 and solving the resulting equation, we obtain

$$(\lambda_{7/2}\mu_{7/2})^{2/3} = u + v, \quad (97)$$

where

$$\begin{aligned}u &= -10888 - 6300\sqrt{3} + 4125\sqrt{7} + 2376\sqrt{21}, \\ v &= \sqrt{6(79213469 + 45733800\sqrt{3} - 29939800\sqrt{7} - 17285796\sqrt{21})}.\end{aligned}\quad (98)$$

Combining (95) and (97) we calculate the values of $\mu_{7/2}$ and $\lambda_{7/2}$. $\square$

Remark 33. The values of $\lambda_{2/7}$ and $\mu_{2/7}$ can easily be calculated by employing the values of $\lambda_{7/2}$ and $\mu_{7/2}$, respectively, in (5).

Remark 34. (i) From (36) and (77) we can easily deduce the values of class invariants $G_{10/3}$ and G_{30} . (ii) Combining (37) and (83) we can calculate the values of class invariants $G_{15/2}$ and $G_{6/5}$. (iii) The values of class invariants $G_{14/3}$ and G_{42} can be calculated from (38) and (89). (iv) Combining (39) and (95) we can obtain the values of class invariants $G_{21/2}$ and $G_{7/6}$. (v) Combining (40), (71), and Lemma 3 we can calculate the values of class invariants G_2 and G_{18} .

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References

- [1] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, India, 1988.
- [2] B. C. Berndt, H. H. Chan, S.-Y. Kang, and L.-C. Zhang, "A certain quotient of eta-functions found in Ramanujan's lost notebook," *Pacific Journal of Mathematics*, vol. 202, no. 2, pp. 267–304, 2002.
- [3] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook. Part II*, Springer, New York, NY, USA, 2009.
- [4] K. G. Ramanathan, "On some theorems stated by Ramanujan," in *Number Theory and Related Topics (Bombay, 1988)*, vol. 12, pp. 151–160, Oxford University Press, Bombay, India, 1989.
- [5] J. Yi, *Construction and application of modular equation [Ph.D. thesis]*, University of Illinois, Chicago, Ill, USA, 2001.
- [6] J. Yi, "Some new modular equations and their applications," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 531–546, 2006.
- [7] N. D. Baruah and N. Saikia, "Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction," *The Journal of Computational and Applied Mathematics*, vol. 160, no. 1-2, pp. 37–51, 2003.
- [8] N. D. Baruah and N. Saikia, "Some new explicit values of Ramanujan's continued fractions," *Indian Journal of Mathematics*, vol. 46, no. 2-3, pp. 197–222, 2004.
- [9] M. S. M. Naika, M. C. Maheshkumar, and K. S. Bairy, "Certain quotient of eta-function identities," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 1, pp. 121–136, 2008.
- [10] B. C. Berndt, *Ramanujan's Notebooks. V*, Springer, New York, NY, USA, 1998.
- [11] N. D. Baruah, "On some class invariants of Ramanujan," *The Journal of the Indian Mathematical Society. New Series*, vol. 68, no. 1-4, pp. 113–131, 2001.
- [12] N. Saikia, "Ramanujan's Schläfli-type modular equations and class invariants g_n ," *Functiones et Approximatio Commentarii Mathematici*. In press.
- [13] N. Saikia, "Ramanujan's modular equations and Weber-Ramanujan class invariants G_n and g_n ," *Bulletin of Mathematical Sciences*, vol. 2, no. 1, pp. 205–223, 2012.
- [14] N. Saikia, "A parameter for Ramanujan's function $\chi(q)$: its explicit values and applications," *ISRN Computational Mathematics*, vol. 2012, Article ID 169050, 14 pages, 2012.
- [15] B. C. Berndt, *Ramanujan's Notebooks. III*, Springer, New York, NY, USA, 1991.