Research Article

The $2-(v, 11, 1)$ Designs with Unsolvable Block Transitive Automorphism

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This paper is a contribution to the study of the automorphism groups of $2-(v, k, 1)$ designs. Let $\mathcal{D}$ be $2-(v, 11, 1)$ design and $G \leq \text{Aut}(\mathcal{D})$ a block transitive and a point primitive. If G is unsolvable, then $\text{Soc}(G)$, the socle of G , is not ${}^2F_4(q^2)$.

1. Introduction

A $2-(v, k, 1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of k -subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ are contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq \text{Aut}(\mathcal{D})$ be a group of automorphisms of a $2-(v, k, 1)$ design $\mathcal{D}$. Then G is said to be block transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point transitive (point primitive) on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag transitive on $\mathcal{D}$ if G is transitive on the set of flags.

The classification of block transitive $2-(v, 3, 1)$ designs was completed about thirty years ago (see [1]). In [2], Camina and Siemons classified $2-(v, 4, 1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v, 4, 1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [3]). Tong and Li [4] classified $2-(v, 5, 1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [5] classified $2-(v, 5, 1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [6] classified $2-(v, k, 1)$ (where $k = 6, 7, 8, 9, 10$) designs with a block transitive, solvable group of automorphisms. In [7], Han and Ma classified $2-(v, 11, 1)$ designs with a block transitive classical simple group of automorphisms.

This paper is a contribution to the study of the automorphism groups of $2-(v, k, 1)$ designs. Let $\mathcal{D}$ be $2-(v, 11, 1)$

design and $G \leq \text{Aut}(\mathcal{D})$ a block transitive and a point primitive. We prove the following theorem.

Main Theorem. Let $\mathcal{D}$ be $2-(v, 11, 1)$ design and $G \leq \text{Aut}(\mathcal{D})$ a block transitive and a point primitive. If G is unsolvable, then $\text{Soc}(G) \neq {}^2F_4(q^2)$.

2. Preliminary Results

Let $\mathcal{D}$ be a $2-(v, k, 1)$ design defined on the point set $\mathcal{P}$ and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v, k, 1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G . Also, G^B denotes the permutation group induced by the action of G_B on the points of B , and so $G^B \cong G_B/G_{(B)}$.

Lemma 1 (see [8]). Let $G = {}^2F_4(q^2)$, $q^2 = 2^{2n+1}$, $n \geq 1$, then every maximal subgroup of G is conjugate to one of the following:

- (1) $P_1 = [q^{22}] : (\text{PSL}(2, q^2) \times (q^2 - 1))$;
- (2) $P_2 = [q^{20}] : ({}^2B_2(q^2) \times (q^2 - 1))$;
- (3) $SU(3, q^2) : Z_2$;
- (4) $(Z_{q^2+1} \times Z_{q^2+1}) : GL(2, 3)$;
- (5) $(Z_{q^2-\sqrt{2}q+1} \times Z_{q^2-\sqrt{2}q+1}) : [96]$, if $q^2 > 8$;

- (6) $(Z_{q^2+\sqrt{2}q+1} \times Z_{q^2+\sqrt{2}q+1}) : [96]$;
- (7) $(Z_{q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1}) : Z_{12}$;
- (8) $(Z_{q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1}) : Z_{12}$;
- (9) $PGU(3, q^2) : Z_2$;
- (10) ${}^2B_2(q^2) : Z_2$;
- (11) $B_2(q^2) : Z_2$;
- (12) ${}^2F_4(q_0^2)$, where $q_0^2 = 2^{2m+1}$ and $(2n+1)/(2m+1)$ be prime.

Lemma 2 (see [9]). Let $T = {}^2F_4(q^2)$ be an exceptional simple group of Lie type over $GF(q)$, and let G be a group with $T \trianglelefteq G \leq \text{Aut}(T)$. Suppose that M is a maximal subgroup of G not containing T , then one of the following holds:

- (1) $|M| < q^{24}|G : T|$;
- (2) $T \cap M = L(3, 3) : Z_2$, or $L(2, 25)$, if $q^2 = 2$;
- (3) $T \cap M$ is a parabolic subgroup of T .

Lemma 3 (see [7]). Let G and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq \text{Aut}(\mathcal{D})$ be block transitive and point primitive but not flag transitive. Let $\text{Soc}(G) = T$. Then

$$|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G : T|, \quad (1)$$

where $\alpha \in \mathcal{P}$ and λ is the length of the longest suborbit of G on $\mathcal{P}$.

3. Proof of the Main Theorem

Proposition 4. Let $\mathcal{D}$ be $2-(v, 11, 1)$ design and let G be block transitive and point primitive but not flag transitive, then $v = 110b_2 + 1$.

Proof. Let $b_1 = (b, v)$, $b_2 = (b, v - 1)$, $k_1 = (k, v)$, $k_2 = (k, v - 1)$. Obviously,

$$\begin{aligned} k &= k_1 k_2, & b &= b_1 b_2, \\ r &= b_2 k_2, & v &= b_1 k_1. \end{aligned} \quad (2)$$

Since $k = 11$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [9], G is flag transitive, a contradiction. Thus, $v = k(k - 1)b_2 + 1 = 110b_2 + 1$. $\square$

Proposition 5. Let $\mathcal{D}$ be $2-(v, 11, 1)$ design and let G be block transitive and point primitive but not flag transitive and $|T|$ be even. If G be unsolvable, then $|T| \leq 56|T_\alpha|^2|G : T|$.

Proof. Let $B = \{1, 2, \dots, 11\} \in \mathcal{B}$. Since G is unsolvable, then the structure of G^B and the rank and subdegree of G do not occur:

$$\begin{array}{ccc} G^B & \text{Rank} & \text{Subdegree} \\ \langle 1 \rangle & 111 & \overbrace{1, b_2, \dots, b_2}^{110} \end{array} \quad (3)$$

Otherwise, $|G^B| = 1$ is odd and $|B| = 11$. We have $|G_B|$ and b_2 are odd. Since $v = 110b_2 + 1 = b_1$ and $b = b_1 b_2$, then b is odd and $|G| = b|G_B|$ is also odd, a contradiction with $|T|$ be even. Thus, $\lambda \geq 2b_2$. By Lemma 3,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda} \cdot |G : T| \leq \frac{v}{2b_2} \cdot |G : T|. \quad (4)$$

By Proposition 4,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{110b_2 + 1}{2b_2} \cdot |G : T| \leq 56|G : T|. \quad (5)$$

$\square$

Now we may prove our main theorem.

Suppose that $\text{Soc}(G) = {}^2F_4(q^2) = T$, then $T \trianglelefteq G \leq \text{Aut}(T)$. We have $G = T : \langle x \rangle$, where $x \in \text{Out}(T)$, the outer automorphisms group of T which may be generated by an automorphism of field. By [8], we may assume that x is an automorphism of field. Set ${}^\circ(x) = f$, then $f \mid (2n + 1)$. Obviously, $|{}^2F_4(q^2)| = q^{24}(q^2 - 1)(q^6 + 1)(q^8 - 1)(q^{12} + 1)$. By [10] and $k = 11$, G is not flag transitive. Since G is point primitive, G_α ($\alpha \in \mathcal{P}$) is the maximal subgroup of G and T is block transitive in $\mathcal{D}$. Hence, $M = G_\alpha$ satisfies one of the three cases in Lemma 2. We will rule out these cases one by one.

Case 1 ($|M| < q^{24}|G : T|$). By Proposition 5, we have an upper bound of $|T|$,

$$|T| < 56|T_\alpha|^2|G : T| < 56q^{48}f. \quad (6)$$

We get

$$\begin{aligned} q^{24}(q^2 - 1)(q^6 + 1)(q^8 - 1) \\ \times (q^{12} + 1) < 56q^{48}f < 56q^{48}q^2, \end{aligned} \quad (7)$$

that is,

$$(q - 1)^2 < 56. \quad (8)$$

Thus, $q^2 = 2^5$ or 2^3 and $T = {}^2F_4(2^5)$ or ${}^2F_4(2^3)$. Since $v = 110b_2 + 1$ is odd, then $|T_\alpha|$ contains a Sylow 2-subgroup of T . Clearly T_α is contained in some maximal subgroups of T . By Lemma 1, $T_\alpha \cong T_i$, where $T_i \leq P_i$ ($i = 1, 2$). Then $v \mid (2^5 - 1)(2^{15} + 1)(2^{20} - 1)(2^{30} + 1)$ or $v \mid (2^3 - 1)(2^9 + 1)(2^{12} - 1)(2^{18} + 1)$. By Proposition 4 and rather long and repetitive numerical calculations, we get a contradiction.

Case 2 ($T \cap M = L(3, 3) : Z_2$, or $L(2, 25)$, if $q^2 = 2$). Obviously, $|T| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 13$ and $v = |T : T_\alpha| = 2^6 \cdot 5^2$ or $2^5 \cdot 3$. By Proposition 4, we have $110 \mid 2^6 \cdot 5^2$ or $110 \mid 2^5 \cdot 3$, a contradiction.

Case 3 ($T \cap M$ is a parabolic subgroup of T). By Lemma 1, the parabolic subgroup of ${}^2F_4(q^2)$ is conjugate to P_1 or P_2 . Then the order of parabolic subgroup is $q^{24}(q^2 - 1)^2(q^2 + 1)$ or $q^{24}(q^2 - 1)^2(q^4 + 1)$. We get $v = (q^4 + 1)(q^6 + 1)(q^{12} + 1)$ or

$(q^2 + 1)(q^6 + 1)(q^{12} + 1)$. By Proposition 4, we have $55 \mid (q^{18} + q^{14} + q^{12} + q^8 + q^6 + q^2 + 1)$ or $55 \mid (q^{18} + q^{16} + q^{12} + q^{10} + q^6 + q^4 + 1)$. But

$$\begin{aligned} & q^{18} + q^{14} + q^{12} + q^8 + q^6 + q^2 + 1 \\ & \equiv \begin{cases} 2 \pmod{5} & \text{if } q \equiv 1 \pmod{5} \\ 1 \pmod{5} & \text{if } q \equiv 2 \pmod{5} \\ 4 \pmod{5} & \text{if } q \equiv 3 \pmod{5} \\ 2 \pmod{5} & \text{if } q \equiv 4 \pmod{5}, \end{cases} \\ & q^{18} + q^{16} + q^{12} + q^{10} + q^6 + q^4 + 1 \\ & \equiv \begin{cases} 2 \pmod{5} & \text{if } q \equiv 1 \pmod{5} \\ 1 \pmod{5} & \text{if } q \equiv 2 \pmod{5} \\ 1 \pmod{5} & \text{if } q \equiv 3 \pmod{5} \\ 2 \pmod{5} & \text{if } q \equiv 4 \pmod{5}. \end{cases} \end{aligned} \quad (9)$$

Obviously, $5 \nmid (q^{18} + q^{14} + q^{12} + q^8 + q^6 + q^2 + 1)$ or $5 \nmid (q^{18} + q^{16} + q^{12} + q^{10} + q^6 + q^4 + 1)$, a contradiction.

This completes the proof the Main Theorem.

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