Research Article

On $\alpha$-Cogenerated Commutative Unital $C^*$-Algebras

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Gelfand-Naimark’s theorem states that every commutative $C^*$-algebra is isomorphic to a complex valued algebra of continuous functions over a suitable compact space. We observe that for a completely regular space $X$, $\beta X$ is dense-$\alpha$-separable if and only if $C(X)$ is $\alpha$-cogenerated if and only if every family of maximal ideals of $C(X)$ with zero intersection has a subfamily with cardinal number less than $\alpha$ and zero intersection. This gives a simple characterization of $\alpha$-cogenerated commutative unital $C^*$-algebras via their maximal ideals.

1. Introduction

In this paper, by $R$ we always mean a commutative ring with identity. Let $F$ denote the reals or the complexes. For a completely regular (topological space) $X$, let $C(X)$ stand for the $F$-algebra of continuous maps $X \to F$. The reader is referred to [1] for undefined terms and notations. By $\beta X$, we mean the Stone-Čech compactification of $X$. We denote the ring of all bounded continuous functions by $C^*(X)$. It is well known that for every completely regular space $X$, we have $C^*(X) \cong C(\beta X)$ (see [1, 71]). This note is a continuation of [2], in which we showed that for a compact space $X$, the following are equivalent: $X$ is dense-separable if and only if $C(X)$ is $\aleph_0$-cogenerated if and only if $C(X)$ is separable. Here, we will drop the compactness condition of the space $X$ and improve our main result in [2]. Furthermore we generalize our results to any regular cardinal $\alpha$.

Let $\alpha$ be a regular cardinal. A set $A$ is said to be an $\alpha$-set if $|A| < \alpha$. Following Motamedi in [3], we call a ring $R$ $\alpha$-cogenerated if for any set $\{ A_i \mid i \in I \}$ of ideals of $R$ with $\bigcap_{i \in I} A_i = (0)$ there exists an $\alpha$-subset $I_0$ of $I$ such that $\bigcap_{i \in I_0} A_i = (0)$ and $\alpha$ is the least regular cardinal with this property. Any left or right Artinian ring is $\aleph_0$-cogenerated. Any ring with countably many distinct ideals is $\alpha$-cogenerated, where $\alpha$ is one of $\aleph_0$ or $\aleph_1$. In [2], it has been observed that $C[0,1]$, $C(\mathbb{K})$, where $\mathbb{K}$ is the Cantor perfect set and $C(\beta \mathbb{K})$ are $\aleph_1$-cogenerated. We call a ring $R$ $\alpha$-separable if it has the following property: if $\{ M_i \}_{i \in I}$ is a family of maximal ideals with $\bigcap_{i \in I} M_i = (0)$, then there exists an $\alpha$-subset $I_0$ of $I$ such that $\bigcap_{i \in I_0} M_i = (0)$. In this note $\aleph_1$-separable rings are also called separable. Every $\alpha$-cogenerated ring is $\alpha$-separable. However, the converse is not true. In [2], we give an example of a separable ring which is not $\aleph_1$-cogenerated.

The density of a space $X$ is defined as the smallest cardinal number of the form $|A|$, where $A$ is a dense subset of $X$; this cardinal number is denoted by $d(X)$ (see [4]). A space $X$ is called dense-$\alpha$-separable if every dense subset $A$ of $X$ has a dense-$\alpha$-subset $B$, which implies that $d(A)$ and hence $d(X)$ are less than $\alpha$. Dense-separable (or in our terminologies dense-$\aleph_1$-separable) spaces are of great interest. Dense-separable spaces were introduced and studied by Levy and McDowell in [5]. It is evident that every dense-separable space is separable and every second countable space is dense-separable. It is well known that $\mathbb{R}$, the Sorgenfrey line satisfies all the countability axioms but the second (see [6], page 195, example 3). Since every dense subset of $\mathbb{R}$ is also dense in $\mathbb{R}$, the Sorgenfrey line is dense-separable. In [5], it has been shown that $\beta \mathbb{Q}$ and $\beta \mathbb{Q} \setminus \mathbb{Q}$ are dense-separable.

2. Dense-$\alpha$-Separable Spaces

Since it is easy to observe that $C(X)$ is $\aleph_0$-cogenerated if and only if $X$ is a finite space, and in this case $C(X)$ is a finite direct product of $F$, we may suppose that in our discussion
Let $X$ be a completely regular space; then the following hold.

1. Suppose $Y$ is a subset of $X$; then $Y$ is a dense subset of $X$ if and only if $f \in C(X)$; $f_Y = 0$ implies that $f \equiv 0$.
2. Suppose $Y$ is a subset of $X$; then $Y$ is a dense subset of $X$ if and only if $f \in C(X)$; $Y \subseteq \text{int}_X(Z(f))$ implies that $f \equiv 0$.

Proof. Part 1. ($\Rightarrow$): Let $x \in X$ and $U_x$ an open set containing $x$. We must show that $U_x \cap Y \neq \emptyset$, and suppose on the contrary that $U_x \cap Y = \emptyset$; then $Y \subseteq X \setminus U_x$; by complete regularity of $X$, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $f(Y) = 0$, and this is a contradiction to our hypothesis.

($\Leftarrow$): Since $R$ is a $T_1$-space and $\emptyset$ is closed in $R$, hence $f^{-1}[0] = \emptyset$ is closed in $X$. Since $Y \subseteq f^{-1}[0]$, we conclude that $X = \overline{Y} \subseteq f^{-1}[0]$. Hence $f \equiv 0$.

Part 2. It is enough to show the necessary part: suppose that $\overline{Y} \neq X$; hence there exists $z \in X \setminus \overline{Y}$ and a function $f : X \to [0,1]$ such that $f(z) = 1$ and $f(Y) = 0$. In such as $Y$ and $z$ are contained in disjoint zero sets, there exists a function $g$ such that $\overline{Y} \subseteq \text{int} Z(g)$ and $g(z) \neq 0$ (see [1, 1.15]). This is a contradiction.

Lemma 2. For $A \subseteq \beta X$, $O^A = (0)$ if and only if $M^A = (0)$.

Proof. Suppose that $O^A = (0)$; then by the previous lemma $\overline{A} = \beta X$. Now suppose that $f \in M^A$. There is a positive unit $u$ in $C(X)$ such that $f = uf$ and $-1 \leq f \leq 1$ (see [1, 1.16]). Since $A \subseteq \text{cl}_X Z(f) \subseteq \text{Z}(\overline{f})$ and $A = \beta X$, we have $\overline{f} = 0$. Hence $f = 0$ and this implies that $f \equiv 0$.

In the next theorem, which is the main result of this note, we have generalized [2, Theorem 3] by removing the compactness hypothesis and also replacing $X_0$ with an arbitrary (regular) cardinal. Since $\beta X$, by its very definition, is compact and $\beta X = X$ whenever $X$ is compact, the earlier form of our result is just a special case of the new one. On the other hand since $\beta X$ always exists, we can (always, i.e., for an arbitrarily completely regular space $X$) judge when the ring $C(X)$ is $\alpha$-cogenerated and also $\alpha$-separating by looking at $\beta X$.

Remark 3. Before stating our main theorem, we need some useful facts. Let $X$ be a completely space and $A \subseteq \beta X$. Suppose that $O^A = \{f \in C(X) \mid A \subseteq \text{int}_X \text{cl}_X Z(f)\}$ and $M^A = \{f \in C(X) \mid A \subseteq \text{cl}_X Z(f)\}$. Lemma 1 (Lemma 2, resp.) shows that $M^A = (0)$ ($O^A = (0)$, resp.), then $A$ is dense in $X$ and vice versa. Dietrich Jr. in [7] has shown that for every ideal $I$ of $C(X)$, there exists $A \subseteq \beta X$ such that $O^A \subseteq I \subseteq M^A$. By McKnight Theorem [7, Theorem 1.3], the set $A$ is $\bigcap_{f \in I} \text{cl}_X Z(f)$. Dietrich Jr. has also proved that

Theorem 4. Let $X$ be an infinite completely regular space. The following are equivalent:

1. $\beta X$ is dense-$\alpha$-separable;
2. $C(X)$ is $\alpha$-cogenerated;
3. $C(X)$ is $\alpha$-separable.

Proof. (1) $\Rightarrow$ (2): let $\bigcap_{j \in J} I_j = (0)$. For each $j \in J$, there exists $A_j \subseteq \beta X$ such that $O^{A_j} \subseteq I_j \subseteq M^{A_j}$. By the previous observations from [7] we have

$$O^{\bigcup_{j \in J} A_j} = \bigcap_{j \in J} O^{A_j} \subseteq \bigcap_{j \in J} I_j \subseteq \bigcap_{j \in J} M^{A_j} = M^{\bigcup_{j \in J} A_j},$$

but $\bigcap_{j \in J} I_j = (0)$; therefore $O^{\bigcup_{j \in J} A_j} = \bigcap_{j \in J} O^{A_j} = (0)$, and hence by Lemma 1, $\bigcup_{j \in J} A_j$ is dense in $\beta X$. Since $\beta X$ is a dense-$\alpha$-separable space, there exists an $\alpha$-subset $B$ of $\bigcup_{j \in J} A_j$, such that $\overline{B} = X$. Hence there exists an $\alpha$-set $B_0 \subseteq J$ such that $\bigcap_{j \in B_0} A_j$ is dense in $\beta X$; that is, $O^{\bigcup_{j \in B_0} A_j} = (0)$. Now by Lemma 2, $\bigcap_{j \in B_0} M^{A_j} = (0)$, and this latter observation in its turn shows that $\bigcap_{j \in B_0} I_j = (0)$.

(2) $\Rightarrow$ (3): it is evident.

(3) $\Rightarrow$ (1): let $D$ be a dense subset of $\beta X$. Then by Lemma 1, $\bigcap_{j \in J} O^{A_j} = O^D = (0)$. Now by Lemma 2, $M^D = (0)$. Since $C(X)$ is $\alpha$-separable, there is an $\alpha$-subset $A$ of $D$ such that $M^A = (0) = O^A$, and again by Lemma 1, this shows that $A$ is dense in $\beta X$ and hence dense in $D$.

Observe that when $X$ is finite, $C(X)$ is artinian and hence $\mathcal{N}_0$-$\alpha$-cogenerated. If $C(X)$ is separable, then $X$ is dense-separable. A ring $R$ is called von-Neumann regular if for every $a \in R$, there exists $b \in R$ such that $aba = a$. Kaplansky has shown that every ideal in a commutative von-Neumann regular ring can be written as the intersection of some family of maximal ideals. Hence, a commutative von-Neumann regular ring is $\alpha$-separable if and only if it is $\alpha$-cogenerated. This implies that for any $p$-space $X$, $C(X)$ is $\alpha$-separable if and only if $C(X)$ is $\alpha$-cogenerated. A ring is called right $V$-ring (after Villamayor) if every right simple $R$-module is injective. It is well known that over right $V$-rings, every submodule of a right $R$-module $M$ can be written as the intersection of a family of maximal submodules of $M$. Hence over a right $V$-ring, a right $R$-module is $\alpha$-cogenerated if and only if it is $\alpha$-separable. Let $\{I_j\}_{j \in J}$ be a family of ideals of $R$; we have $\bigcap_{j \in J} M_n(I_j) = M_n(\bigcap_{j \in J} I_j)$. Hence as far as one is concerned with two sided ideals of $M_n(R)$, one obtains that $M_n(R)$ is $\alpha$-separable (cogenerated, resp.) when $R$ is $\alpha$-separable (cogenerated, resp.).

Corollary 5. The following are equivalent:

1. $\beta X$ is dense-$\alpha$-separable;
2. $C(\beta X)$ is $\alpha$-cogenerated;
3. $C(\beta X)$ is $\alpha$-separable;
(4) \( C(X) \) is \( \alpha \)-cogenerated;
(5) \( C(X) \) is \( \alpha \)-separable;
(6) \( C^*(X) \) is \( \alpha \)-cogenerated;
(7) \( C^*(X) \) is \( \alpha \)-separable.

**Proof.** It is well known that \( C(\beta X) \cong C^*(X) \), and by Theorem 4 the verification is immediate. \( \square \)

**Corollary 6.** If \( X \) is either (1) separable metric or (2) separable and ordered, then \( C(X) \) is \( \aleph_1 \)-cogenerated.

**Proof.** By [5, Corollary 3.2.], for these two cases \( \beta X \) is dense-separable. Now by Theorem 3 the proof is thorough. \( \square \)

When \( C(X) \) is separable, then \( X \) is also separable. However, the converse is not true. In [5, example 5.3], a separable compact space \( Y \) has been introduced which is not dense-separable; for the space \( Y, C(Y) \) is not separable. Otherwise \( Y = \beta Y \) should be dense-separable which is not the case. Based on these observations we have the following.

**Example 7.** There exists a separable space \( Y \), such that \( C(Y) \) is not separable.

However, the converse is true when we have a much stronger property as we observe in the next proposition.

**Proposition 8.** Let \( R \) be a commutative ring. Then \( R \) is \( \alpha \)-separable if and only if \( X = \text{Max}(R) \) is dense-\( \alpha \)-separable.

**Proof.** Let \( X \) be dense-\( \alpha \)-separable and \( \mathcal{A} = \{M_i\}_{i \in I} \) a family of maximal ideals with zero intersection. This family then will be dense in \( X \). By dense-\( \alpha \)-separability of \( X \), \( \mathcal{A} \) has an \( \alpha \)-subset with zero intersection. Let \( R \) be an \( \alpha \)-separable ring and \( \mathcal{A} \) a dense subspace of \( X \). By definition \( \mathcal{A} \) has a zero intersection and by \( \alpha \)-separability of \( R \), it has an \( \alpha \)-subspace which is dense in \( X \). \( \square \)

According to Gelfand-Naimark's theorem every commutative \( C^* \)-algebra with identity is isomorphic to \( C(X, \mathbb{C}) \), where \( X \) is a suitable compact Hausdorff space. Based on Theorem 3 and Gelfand-Naimark's theorem, we have the following.

**Corollary 9.** Let \( A \) be a commutative \( C^* \)-algebra with identity and \( \alpha \) an arbitrary regular cardinal. Then the following are equivalent:

1. \( A \) is \( \alpha \)-separable;
2. \( A \) is \( \alpha \)-cogenerated.

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**References**


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