Research Article

A Morphism Double Category and Monoidal Structure

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We provide a recipe for “fattening” a category that leads to the construction of a double category. Motivated by an example where the underlying category has vector spaces as objects, we show how a monoidal category leads to a law of composition, satisfying certain coherence properties, on the object set of the fattened category.

1. Introduction and Geometric Background

The interaction of point particles through a gauge field can be encoded by means of Feynman diagrams, with nodes representing particles and directed edges carrying an element of the gauge group representing parallel transport along that edge. If the point particles are replaced by extended one-dimensional string-like objects, then the interaction between such objects can be encoded through diagrams of the form

\[
\begin{array}{ccc}
 x_1 & \xrightarrow{f_1} & y_1 \\
 \downarrow g_1 & & \downarrow h \\
 x_2 & \xrightarrow{f_2} & y_2 \\
\end{array}
\]

where the labels \( f_i \) and \( g_i \) describe classical parallel transport and \( h \), which may take values in a different gauge group, describes parallel transport over a space of paths.

We will now give a rapid account of some of the geometric background. We refer to our previous work [1] for further details. This material is not logically necessary for reading the rest of this paper but is presented to indicate the context and motivation for some of the ideas of this paper.

Consider a principal \( G \)-bundle \( \pi : P \to M \), where \( M \) is a smooth finite dimensional manifold and \( G \) a Lie group, and a connection \( \overline{A} \) on this bundle. In the physical context, \( M \) may be spacetime, and \( \overline{A} \) describes a gauge field. Now consider the set \( \mathcal{P}M \) of piecwise smooth paths on \( M \), equipped with a suitable smooth structure. Then, the space \( \mathcal{P} \overline{A} \) of horizontal paths in \( P \) forms a principal \( G \)-bundle over \( \mathcal{P}M \). We also use a second gauge group \( H \) (that governs parallel transport over path space), which is a Lie group along with a fixed smooth homomorphism \( \tau : H \to G \) and a smooth map

\[
\gamma: G \times H \to H: (g, h) \mapsto \alpha(g) h
\]

such that each \( \alpha(g) \) is an automorphism of \( H \), such that

\[
\tau(\alpha(g)h) = g\tau(h)g^{-1},
\]

\[
\alpha(\tau(h))h' = hh'h^{-1}
\]

for all \( g \in G \) and \( h, h' \in H \). We denote the derivative \( \tau'(e) \) by \( \tau \), viewed as a map \( LH \to LG \), and denote \( \alpha'(e) \) by \( \alpha \), to avoid notational complexity. Given also a second connection form \( A \) on \( P \) and a smooth \( \alpha \)-equivariant vertical \( LH \)-valued 2-form \( B \) on \( P \), it is possible to construct a connection form \( \omega_{(A,B)} \) on the bundle \( \mathcal{P} \overline{A} \)

\[
\omega_{(A,B)} = ev_1^*A + \tau(Z),
\]
where $Z$ is the LH-valued 1-form on $\mathcal{P}_A P$ specified by
\begin{equation}
Z = \int_0^1 B,
\end{equation}
which is a Chen integral.

Consider a path of paths in $P$ specified through a smooth map
\begin{equation}
\Gamma : [0, 1]^2 \rightarrow P : (t, s) \mapsto \Gamma(t, s) = \Gamma_1(t) \Gamma^0(s),
\end{equation}
where each $\Gamma_1$ is $A$-horizontal and the path $s \mapsto \Gamma_0(s)$ is $A$-horizontal. Let $\Gamma = \pi \circ \Gamma$. The bi-holonomy $g(t, s) \in G$ is specified as follows: parallel translate $\Gamma(0, 0)$ along $\Gamma^0_t$ by $\mathcal{A}$, then up the path $\Gamma^1_t$ by $A$, back along $\Gamma^0_t$, reversed by $\mathcal{A}$ and then down $\Gamma^0_t$. The resulting point is
\begin{equation}
\Gamma(0, 0) g(t, s).
\end{equation}

The following result is proved in [1].

**Theorem 1.** Suppose that
\begin{equation}
\Gamma : [0, 1]^2 \rightarrow P : (t, s) \mapsto \Gamma(t, s) = \Gamma_1(t) \Gamma^0(s)
\end{equation}
is smooth, with each $\Gamma_1$ being $\mathcal{A}$-horizontal and the path $s \mapsto \Gamma_0(s)$ being $A$-horizontal. Then, the parallel translate of $\Gamma_0$ by the connection $\alpha_{(A,B)}$ along the path $[0, s] \rightarrow \mathcal{P}M : u \mapsto \Gamma_u$, where $\Gamma = \pi \circ \Gamma$, results in
\begin{equation}
\Gamma_1 g(1, s) \tau(h_0(s)),
\end{equation}
with $g(1, s)$ being the “bi-holonomy” specified as in (7), and $s \mapsto h_0(s) \in H$ solving the differential equation
\begin{equation}
\frac{dh_0(s)}{ds} h_0(s)^{-1} = - \alpha(g(1, s)^{-1})
\end{equation}
\begin{equation}
\times \int_0^1 B(\partial_1 \Gamma(t, s), \partial_2 \Gamma(t, s)) dt
\end{equation}
with initial condition $h_0(0)$ being the identity in $H$.

Consider the category $C_0$ whose objects are fibers of a given vector bundle $E$ over $M$ and whose arrows are piecewise smooth paths in $M$ (up to “backtrack equivalence”; for more on this notion see [2]) along with parallel transport operators, by a connection $\mathcal{A}$, along such paths. Note that all arrows are invertible. In Figure 1, $E_{p_1}$ is the vector space which is the fiber over the corresponding point $p_1$. For the path $c_1$, there is a parallel transport operator $f_1 : E_{p_1} \rightarrow E_{q_1}$. Next, if $c_2$ is a path from the base of the fiber $E_{p_1}$ to the base of $E_{q_1}$, then there is a corresponding parallel transport operator $f_2 : E_{p_1} \rightarrow E_{q_2}$.

A “higher” morphism $c_1 \rightarrow c_2$ is obtained from any suitably smooth path of paths, starting with the initial path $c_1$ and ending with $c_2$ (again backtracks need to be erased). Using the connection $\mathcal{A}$, this produces parallel transport operators and paths $E_{p_1} \rightarrow E_{p_2}$ and $E_{q_1} \rightarrow E_{q_2}$. Moreover, another connection $\mathcal{A}$ and 2-form $B$, along with a path of paths lead to a linear map $\text{Mor}(E_{p_1}, E_{q_1}) \rightarrow \text{Mor}(E_{p_2}, E_{q_2})$, where $\text{Mor}(E, F)$ is the vector space of all linear maps $E \rightarrow F$. We view this, in a “first approximation,” as a morphism from the object $\text{Mor}(E_{p_1}, E_{q_1})$ to the object $\text{Mor}(E_{p_2}, E_{q_2})$ (say, mapping all paths from $p_1$ to $q_1$ to the path $c_1$). In this paper, we will not develop this framework in full detail (that would build on the theory from our earlier work [1]) but focus on more algebraic aspects and other purely algebraic issues (such as monoidal structures).

Instead of vector bundles, one could also work with the principal bundle $P$ itself, taking as objects of a category $C_0$ all the fibers of the bundle $P$ and as morphisms $f : P_p \rightarrow P_q$ the $G$-equivariant bijections $P_p \rightarrow P_q$, where $P_p$ and $P_q$ are fibers of $P$, over points $p$ and $q$, and paths running from $p$ to $q$.

The interface between gauge theory and category theory, in various forms and cases, has been studied in many works, for instance [1, 3–7]. In the present paper, we extract the abstract essence of some of these structures in a category theory setting, leaving the differential geometry behind as the concrete context. We abstract the process of passing from the point-particle picture to a string-like picture to a theory setting, leaving the differential geometry behind as the concrete context. We abstract the process of passing from the point-particle picture to a string-like picture to a funtor which generates a category $\mathcal{F}(C)$ from a category $C$. Proposition 5 describes properties of a natural product operation on the objects of $\mathcal{F}(C)$ when $C$ is a monoidal category. An excellent review of monoidal categories in relation to topological quantum field theory can be found in [8]. Symmetric monoidal bicategories are discussed in [9] in a context different from ours.

2. The Fat Category

Let $C$ be a category. We define a new category $\mathcal{F}(C)$ as follows. The objects of $\mathcal{F}(C)$ are the morphisms of $C$. A morphism in $\mathcal{F}(C)$ from the object $x_1 \rightarrow y_1$ to the object $x_2 \rightarrow y_2$ consists of morphisms $x_1 \rightarrow y_1 \rightarrow x_2$ and $y_1 \rightarrow y_2$ in $C$, along with a set-mapping
\begin{equation}
h : \text{Mor}(x_1, y_1) \rightarrow \text{Mor}(x_2, y_2),
\end{equation}
which maps $f_1$ to $f_2$ as follows:
\begin{equation}
h(f_1) = f_2.
\end{equation}
(In a later section we require that the hom-sets \( \text{Mor}(x, y) \) themselves also have algebraic structure that should be preserved by such \( h \).) Here is a diagram displaying a morphism \( u \) of \( \mathbb{F}(C) \):

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g_1} \downarrow h \downarrow g_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

It is clear that this does specify a category, which we call the fat category for \( C \) (composition is “vertical,” with successive \( h \)s composed). Sometimes it will be easier on the eye to write

\[(x, y, f)\]

for \( x \xymatrix{ f \ar[r] & y } \). Thus, diagram (13) can also be displayed as

\[
\begin{array}{c}
(x_1, y_1, f_1) \\
\downarrow u \\
(x_2, y_2, f_2)
\end{array}
\]

The composition \( v \circ u \) of morphisms in \( \mathbb{F}(C) \) is defined “vertically” by drawing the diagram of \( v \) below that of \( u \) and composing vertically downward.

Commutative diagrams in \( C \) lead to morphisms of \( \mathbb{F}(C) \) in a natural way and yield a subcategory of \( \mathbb{F}(C) \) that is recognizable as the “category of arrows” [10, §1.4], sometimes denoted as \( \text{Arr}(C) \).

**Lemma 2.** Any commutative diagram

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g_1} \downarrow g_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

in \( C \), in which \( g_1 \) is an isomorphism, generates a morphism

\[
(x_1, y_1, f_1) \xymatrix{ u \ar[r] & (x_2, y_2, f_2) }
\]

in \( \mathbb{F}(C) \),

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g_1} \downarrow h \downarrow g_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

where

\[
h_u : \text{Mor}(x_1, y_1) \longrightarrow \text{Mor}(x_2, y_2) : \phi \mapsto g_2 \phi g_1^{-1}.
\]

Moreover, if

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g_1} \downarrow g'_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

is a commutative diagram in \( C \), where \( g_1 \) and \( g'_1 \) are isomorphisms, then the composite of the induced morphisms,

\[
\begin{array}{c}
(x_1, y_1, f_1) \\
\downarrow u \\
(x_2, y_2, f_2)
\end{array}
\]

\[
\begin{array}{c}
(x_2, y_2, f_2) \\
\downarrow v \\
(x_3, y_3, f_3)
\end{array}
\]

is the morphism in \( \mathbb{F}(C) \) induced by the commutative diagram

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g'_1} \downarrow g_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

3. A Double Category of Isomorphisms

Let \( \mathbb{F}(C)_h \) be the category whose objects are the invertible arrows of \( C \) and whose arrows are the arrows

\[
\begin{array}{c}
x_1 \xymatrix{ \ar[r]^{f_1} & y_1 } \\
\ar[d]_{g_1} \downarrow h \downarrow g_2 \\
x_2 \xymatrix{ \ar[r]^{f_2} & y_2 }
\end{array}
\]

in \( \mathbb{F}(C) \) in which the verticals \( g_1 \) and \( g_2 \) are also isomorphisms in \( C \). This is, for all purposes here, as good as assuming that all arrows of \( C \) are invertible, since we will only work with such arrows. In the geometric context, the arrows represent parallel transports and so the invertibility assumption is natural. The mapping \( h \) is motivated by the “surface” parallel transport mentioned briefly in (10).
4 Algebra

Let us define horizontal composition of morphisms in \( \mathbb{F}(C)_0 \) as follows:

\[
\begin{array}{ccc}
  x_1 & \xrightarrow{f_1} & y_1 \\
  g_1 & \downarrow & h \\
  x_2 & \xrightarrow{f_2} & y_2 \\
\end{array}
\quad
\begin{array}{ccc}
  y_1 & \xrightarrow{f'_1} & z_1 \\
  g_1' & \downarrow & h' \\
  y_2 & \xrightarrow{f'_2} & z_2 \\
\end{array}
\]

where the composition is defined only when \( g'_1 = g_2 \), and \( h'' \) is given by

\[
h'' : \text{Mor}(x_1, z_1) \rightarrow \text{Mor}(x_2, z_2) : f \mapsto h' \left( ff_1^{-1} \right) h(f_1) \]

(24)

Note that \( h'' \) satisfies

\[
h'' \left( f'_1 f_1 \right) = h' \left( f'_1 \right) h(f_1) = f'_2 f_2.
\]

(25)

Consider now the following diagram:

\[
\begin{array}{ccc}
  x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{f'_1} & z_1 \\
  g_1 & \downarrow & h & \downarrow & g_1' \\
  x_2 & \xrightarrow{f_2} & y_2 & \xrightarrow{f'_2} & z_2 \\
\end{array}
\]

(27)

The morphisms of \( \mathbb{F}(C)_0 \) thus have two laws of composition: \( \circ_V \) and \( \circ_H \). As we see below, these compositions obey a consistency condition (28), which thereby specifies a double category [10, 11, §1.5].

**Proposition 3.** The morphisms of \( \mathbb{F}(C)_0 \) form a double category under the laws of composition \( \circ_V \) and \( \circ_H \) in the sense that for diagram (27), with notation as explained above,

\[
\left( u_{j'} \circ_H u_j \right) \circ_V \left( u_{k'} \circ_H u_k \right) = \left( u_{j'} \circ_V u_j \right) \circ_H \left( u_{k'} \circ_V u_k \right),
\]

(28)

for all morphisms \( u_j, u_{j'}, u_k, u_{k'} \) in \( \text{Mor}(\mathbb{F}(C)_0) \) for which the compositions on both sides of (28) are meaningful.

**Proof.** Denote by \( u_h \) the morphism of \( \mathbb{F}(C)_0 \) specified by the upper left square in (27), by \( u_{ij} \) the morphism specified by the upper right square, by \( u_i \) the morphism specified by the lower left square, and, lastly, by \( u_{ij'} \) the morphism specified by the lower right square.

Let \( f \in \text{Mor}(x_1, z_1) \). Then,

\[
\left( \left( u_j \circ_H u_{j'} \right) \circ_V \left( u_k \circ_H u_{k'} \right) \right) (f) = \left( u_j \circ_H u_{j'} \right) \circ_V \left( u_k \circ_H u_{k'} \right)(f),
\]

(29)

Comparing (29) and (30), we have the claimed equality (28). \( \square \)

Then, \( \mathbb{F}(C)_0 \) equipped with both laws of composition \( \circ_V \) and \( \circ_H \) is a double category [11]. In the geometric context, this is expressed as a flatness condition for the connection \( \omega_{\mathbb{F}A,B} \) described in the Introduction; for more, see, for instance, [1, 3].

**4. Enrichment for Morphisms**

We continue with the notation and structures as before; \( C \) is a category and \( \mathbb{F}(C) \) is the “fat” category described in Section 2. Now let \( \mathbb{F}(C)_1 \) be a subcategory of \( \mathbb{F}(C)_0 \), having the same objects but possibly fewer morphisms. The idea is that the hom-sets in \( \mathbb{F}(C) \) could have additional structure; for example, if \( C \) has only one object \( E_\rho \), a fiber of a vector bundle, then \( \text{Mor}(E_\rho, E_\rho) \) is a group under composition. The morphisms of \( \mathbb{F}(C)_1 \) could be required to be group automorphisms. We require that for any objects \( x, y, z \) of \( C \) and isomorphism \( g : y \rightarrow x \), the map

\[
r_g : \text{Mor}(x, z) \rightarrow \text{Mor}(y, z) : f \mapsto fg
\]

(31)

is a morphism of \( \mathbb{F}(C)_1 \).

**Proposition 4.** Let \( \mathbb{F}(C)_1 \) be any subcategory of \( \mathbb{F}(C)_0 \) having the same objects as \( \mathbb{F}(C)_0 \), and satisfying the condition (31) as explained above. Both horizontal and vertical composites of morphisms in \( \mathbb{F}(C)_1 \) are in \( \mathbb{F}(C)_1 \). Thus, \( \mathbb{F}(C)_1 \) is a double category.

**Proof.** The consistency condition between horizontal and vertical composites has already been checked in Proposition 3. Thus, we need only to check that horizontal composition, specified in (25) as

\[
h'' : \text{Mor}(x_1, z_1) \rightarrow \text{Mor}(x_2, z_2) : f \mapsto h' \left( ff_1^{-1} \right) h(f_1),
\]

(32)

is a morphism of \( \mathbb{F}(C)_1 \), for all invertible \( f_1 \in \text{Mor}(x_1, y_1) \) and all \( h : \text{Mor}(x_1, y_1) \rightarrow \text{Mor}(x'_1, y'_1), h' \in \text{Mor}(y_1, z_1) \rightarrow \text{Mor}(y'_1, z'_1) \) morphisms in \( \mathbb{F}(C)_1 \). Observe that

\[
h''(f) = h' \left( ff_1^{-1} \right) h(f_1) = r_{h(f_1)} \circ h' \circ r_{f_1^{-1}}(f),
\]

(33)

where the notation \( r_g \) is as in (31). Thus, \( h'' \) is a composite of morphisms in \( \mathbb{F}(C)_1 \). \( \square \)
5. Monoidal Structures

In this section we will explore some algebraic structural enhancements of the fattened category $F(C)_0$. The discussion is motivated by intrinsic algebraic considerations, but we discuss briefly now the relationship with the geometric context.

Consider the very special case where $C$ is the category with only one object $E_0$, the fiber over a fixed point $o$ in a vector bundle, and a morphism $f : E_0 \to E_0$ is an ordered pair as follow:

$$f = (c, T),$$  \hspace{1cm} (34)

consisting of a piecewise smooth loop $c$ based at $o$ (with backtracks erased) along with a linear map $T : E_0 \to E_0$ representing parallel transport around the loop. For $F(C)_0$ in this special case, a morphism $h : \text{Mor}(E_0, E_0) \to \text{Mor}(E_0, E_0)$ arises from paths of paths along with a linear map $\text{End}(E_0) \to \text{End}(E_0)$, where $\text{End}(E_0)$ is the vector space of all linear maps $E_0 \to E_0$. Each hom-set $\text{Mor}(E_0, E_0)$ is a monoid: composition

$$\text{Mor}(E_0, E_0) \times \text{Mor}(E_0, E_0) \to \text{Mor}(E_0, E_0) : (f, f') \mapsto f \otimes f'$$  \hspace{1cm} (35)

is given by concatenation of loops along with ordinary composition of linear maps in $\text{End}(E_0)$:

$$(c, T) \otimes (c', T') = (c \ast c', T \circ T'),$$  \hspace{1cm} (36)

where $c \ast c'$ is the loop $c'$ followed by the loop $c$. (Since this discussion is primarily for motivation, we leave out technical details of “backtrack erasure.”)

Turning to the abstract setting, we assume henceforth that $C$ is a monoidal category. This means that there is a bifunctor

$$\otimes : C \times C \to C$$  \hspace{1cm} (37)

and there is an identity object $1$ in $C$ for which certain natural coherence conditions hold as we now describe. In addition, there exists a natural isomorphism $\alpha$, the associator, which associates to any of the objects $A, B, C$ of $C$ an isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$  \hspace{1cm} (38)

such that the following diagram commutes:

There are also natural isomorphisms $l$ and $r$, the left and right unitors, associating to each object $A$ in $C$ morphisms

$$l_A : 1 \otimes A \to A, \quad r_A : A \otimes 1 \to A$$  \hspace{1cm} (40)

such that

$$l_A \otimes l_B : 1 \otimes 1 \otimes A \otimes B \to A \otimes B$$  \hspace{1cm} (41)

commutes for all objects $A$ and $B$ in $C$.

Note that naturality means there are certain other conditions as well. For example, that the left unitor is a natural transformation means that for any morphism $x \xrightarrow{f} y$ in $C$ the diagram

$$\begin{array}{ccc}
1 \otimes x & \xrightarrow{1 \otimes f} & 1 \otimes y \\
\downarrow{l_x} & & \downarrow{l_y} \\
x & \xrightarrow{f} & y
\end{array}$$  \hspace{1cm} (42)

commutes; here, in the upper horizontal arrow, $1$ is the unique morphism $i_1 : 1 \to 1$ in $C$.

We now define a product on the objects of $F(C)$

$$\text{Obj}(F(C)) \times \text{Obj}(F(C)) \to \text{Obj}(F(C)) : (u, v) \mapsto u \otimes v$$  \hspace{1cm} (43)
as follows:

\[(x_1 \xrightarrow{f_1} y_1) \otimes (x_2 \xrightarrow{f_2} y_2) \overset{\text{def}}{=} x_1 \otimes x_2 \xrightarrow{f_1 \otimes f_2} y_1 \otimes y_2. \tag{44}\]

In the fat category \(F(C)\), we then have associators and unitors as follows. First, the unit is

\[1_F = 1 \xrightarrow{i_1} 1, \tag{45}\]

where 1 denotes the identity object in \(C\) and \(i_1\) the identity map on 1. We will often denote \(i_1\) also simply as 1, the meaning being clear from context. For any object \(x \xrightarrow{f} y\), there is the left unitor

\[l_{x,y,f} : \begin{array}{c}
\text{Mor}(1 \otimes x, 1 \otimes y) \\
\xrightarrow{x \otimes l_f}
\end{array} \xrightarrow{\phi} \begin{array}{c}
\text{Mor}(x \otimes y) \\
\xrightarrow{\phi \otimes 1}
\end{array}, \tag{46}\]

where the mapping

\[l_{x,y,f} : \text{Mor}(1 \otimes x, 1 \otimes y) \rightarrow \text{Mor}(x \otimes y) : \phi \rightarrow l_y \phi l_x^{-1}. \tag{47}\]

The two triangles at the two ends of this “trough” commute because of coherence in \(C\), the top rectangle also commutes because of the naturality of \(\alpha\). Then, it is entertaining to check that the two rectangular “slanted sides” are also commutative. In fact, the slant side on the left is

\[r_{(x,y,f)} : \begin{array}{c}
\text{Mor}(x \otimes 1, y \otimes 1) \\
\xrightarrow{r_x}
\end{array} \xrightarrow{\phi} \begin{array}{c}
\text{Mor}(x \otimes y) \\
\xrightarrow{r_y \phi r_x^{-1}}
\end{array} \tag{48}\]

Again, this is indeed a morphism in \(F(C)\) by essentially the same argument that was used above in (46) for the left unitor. The associator in \(F(C)\) is given as follows. Consider objects \(x_i \xrightarrow{f_i} y_i\) in \(F(C)\), for \(i \in \{1, 2, 3\}\). The fact that \(\alpha\) is a natural transformation means that the diagram

\[
\begin{array}{c}
\text{Mor}(x_1 \otimes x_2, x_3) \\
\xrightarrow{(f_1 \otimes f_2) \circ f_3}
\end{array} \xrightarrow{\alpha_{x_1,x_2,x_3}} \begin{array}{c}
\text{Mor}(y_1 \otimes y_2, y_3) \\
\xrightarrow{(y_1 \otimes y_2) \circ y_3}
\end{array}
\]

is commutative. Hence, by the first half of Lemma 2, this induces a morphism

\[
\begin{array}{c}
\text{Mor}(x_1 \otimes x_2, x_3) \\
\xrightarrow{(f_1 \otimes f_2) \circ f_3}
\end{array} \xrightarrow{h} \begin{array}{c}
\text{Mor}(y_1 \otimes y_2, y_3) \\
\xrightarrow{(y_1 \otimes y_2) \circ y_3}
\end{array}
\]

\[\begin{array}{c}
\text{Mor}(x_1 \otimes 1, x_2) \\
\xrightarrow{f_1 \otimes 1}
\end{array} \xrightarrow{h} \begin{array}{c}
\text{Mor}(y_1 \otimes 1, y_2) \\
\xrightarrow{1 \otimes y_2}
\end{array}, \tag{50}\]

\[\begin{array}{c}
\text{Mor}(1 \otimes x_1, 1 \otimes x_2) \\
\xrightarrow{l_y \circ l_x}
\end{array} \xrightarrow{h} \begin{array}{c}
\text{Mor}(x_1 \otimes x_2, x_3) \\
\xrightarrow{f_1 \otimes f_2}
\end{array} \xrightarrow{h} \begin{array}{c}
\text{Mor}(y_1 \otimes y_2, y_3) \\
\xrightarrow{1 \otimes y_2}
\end{array}. \tag{51}\]

The two triangles at the two ends of this “trough” commute because of coherence in \(C\), the top rectangle also commutes because of the naturality of \(\alpha\). Then, it is entertaining to check that the two rectangular “slanted sides” are also commutative. In fact, the slant side on the left is

\[r^F_{(x_1,y_1,f_1)} \circ l^F_{(x_2,y_2,f_2)} : \begin{array}{c}
\text{Mor}(x_1, y_1, f_1) \\
\xrightarrow{(x_1, y_1, f_1) \circ x_2, y_2, f_2}
\end{array} \xrightarrow{\alpha_{x_1,x_2,x_3}} \begin{array}{c}
\text{Mor}(x_1, y_1, f_1) \\
\xrightarrow{(x_1, y_1, f_1) \circ x_2, y_2, f_2}
\end{array}
\]

as a morphism in \(F(C)\), and the slant side on the right is

\[l^F_{(x_1,y_1,f_1)} \circ r^F_{(x_2,y_2,f_2)} \tag{54}\]
Thus, viewed as a diagram in $\mathcal{C}$, the “trough” looks like

\[
\begin{array}{c}
((x_1, y_1, f_1) \otimes 1_{\mathbb{F}}) \otimes (x_2, y_2, f_2) \\
\xrightarrow{a_{x_1, y_1, f_1}} (x_1, y_1, f_1) \otimes (x_2, y_2, f_2) \\
\xrightarrow{\alpha_{x_1, y_1, f_1}} (1_{\mathbb{F}} \otimes (x_2, y_2, f_2)) \otimes (x_1, y_1, f_1) \\
\xrightarrow{\beta_{x_2, y_2, f_2}} (x_1, y_1, f_1) \otimes (x_2, y_2, f_2)
\end{array}
\]

(55)

Since the trough commutes in $\mathcal{C}$, so does its avatar (55) in $\mathcal{C}$, thanks to the second half of Lemma 2. This verifies the coherence property in $\mathcal{C}$ involving the unitors.

Now, we turn to coherence for the associators. In the following diagram, where we leave out the $\otimes$ products for ease of viewing, the slant arrows are all tensor products of the $f_i$ and the horizontal and vertical arrows are various associators:

(56)

Coherence in the monoidal category $\mathcal{C}$ implies that the two rectangles at the end of this box are commutative, as mentioned earlier. Naturality of the associator implies that the top, bottom, and sides are also commutative. Thus, the entire diagram is commutative. If we abbreviate the objects in $\mathcal{F}(\mathcal{C})$ as

\[
X_i = (x_i, y_i, f_i),
\]

for $i \in \{1, 2, 3, 4\}$, we can read the full diagram as a diagram in the category $\mathcal{F}(\mathcal{C})$ as follows:

\[
\mathcal{F}(\mathcal{C}) : (X_1 X_2 X_3) X_4 \to (X_1 (X_2 X_3)) X_4 \to X_1 ((X_2 X_3) X_4) \to X_1 (X_2 (X_3 X_4)) \to X_1 (X_2 X_3) X_4
\]

(58)

As a diagram in $\mathcal{F}(\mathcal{C})$, this is commutative, by Lemma 2. This establishes coherence of the associator in $\mathcal{F}(\mathcal{C})$.

We have completed the proof of Proposition 5.

**Proposition 5.** Suppose that $\mathcal{C}$ is a monoidal category and let $\mathcal{F}(\mathcal{C})$ be the category specified above in the context of (i). Then, with tensor product as defined in (44), $\mathcal{F}(\mathcal{C})$ satisfies all conditions of a monoidal category at the level of objects.

6. **Concluding Remarks**

In this paper, we have presented certain “fattened” categories $\mathcal{F}(\mathcal{C}), \mathcal{F}(\mathcal{C})_0,$ and $\mathcal{F}(\mathcal{C})_1$ constructed out of a given category $\mathcal{C}$; the morphisms of $\mathcal{F}(\mathcal{C})_0$ form a double category. It is shown how a monoidal structure on $\mathcal{C}$ induces a multiplication on the objects of $\mathcal{F}(\mathcal{C})$ that satisfies certain coherence properties.

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**References**


