Algorithm for Solving a New System of Generalized Variational Inclusions in Hilbert Spaces

Shamshad Husain and Sanjeev Gupta

Department of Applied Mathematics, Faculty of Engineering & Technology, Aligarh Muslim University, Aligarh 202002, India

Correspondence should be addressed to Sanjeev Gupta; guptasanmp@gmail.com

Received 23 March 2013; Accepted 29 April 2013

1. Introduction

Variational inclusions have been widely studied in recent years. The theory of variational inclusions includes variational, quasi-variational, variational-like inequalities as special cases. Various kinds of iterative methods have been studied to solve the variational inclusions. Among these methods, the resolvent operator technique to study the variational inclusions has been widely used by many authors. For details, we refer to [1–15]. For applications of variational inclusions, see [16].

2. Preliminaries

Throughout this paper, we suppose that $X$ is a real Hilbert space endowed with a norm $\| \cdot \|$ and an inner product $\langle \cdot, \cdot \rangle$, respectively. $2^X$ is the family of all the nonempty subsets of $X$.

In the sequel, let us recall some concepts.

Definition 1 (see [18, 19]). A mapping $g : X \to X$ is said to be

(i) $\lambda_g$-Lipschitz continuous if there exists a constant $\lambda_g > 0$ such that

$$\| g(x) - g(y) \| \leq \lambda_g \| x - y \| , \quad \forall x, y \in X. \quad (1)$$

(ii) monotone if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \quad \forall x, y \in X. \quad (2)$$

of relaxed cocoercive variational inclusions associated with $H(\cdot, \cdot)$-cocoercive operators in Hilbert space. For illustration of Definitions 2, 5 and main result Theorem 19 Examples 4, 6, and 20 are given, respectively. Our results can be viewed as a refinement and improvement of Bai and Yang [2], Huang and Noor [17], and Noor et al. [11].
(iii) $\xi$-strongly monotone if there exists a constant $\xi > 0$ such that
\[
\langle g(x) - g(y), x - y \rangle \geq \xi \|x - y\|^2, \quad \forall x, y \in X.
\]  
(3)

(iv) $\alpha$-expansive if there exists a constant $\alpha > 0$ such that
\[
\|g(x) - g(y)\| \geq \alpha \|x - y\|, \quad \forall x, y \in X.
\]  
(4) if $\alpha = 1$, then it is expansive.

Definition 2 (see [1]). Let $H : X \times X \to X$ and $A, B : X \to X$ be the mappings.

(i) $H(A, \cdot)$ is said to be $\mu$-cocoercive with respect to $A$ if there exists a constant $\mu > 0$ such that
\[
\langle H(Ax, u) - H(Ay, u), x - y \rangle 
\geq \mu \|Ax - Ay\|^2, \quad \forall x, y \in X.
\]  
(5)

(ii) $H(\cdot, B)$ is said to be $\gamma$-relaxed cocoercive with respect to $B$ if there exists a constant $\gamma > 0$ such that
\[
\langle H(u, Bx) - H(u, By), x - y \rangle 
\geq (-\gamma) \|Bx - By\|^2, \quad \forall x, y \in X.
\]  
(6)

(iii) $H(A, \cdot)$ is said to be $\delta_1$-Lipschitz continuous with respect to $A$ if there exists a constant $\delta_1 > 0$ such that
\[
\|H(Ax, \cdot) - H(Ay, \cdot)\| \leq \delta_1 \|x - y\|, \quad \forall x, y \in X.
\]  
(7)

(iv) $H(\cdot, B)$ is said to be $\delta_2$-Lipschitz continuous with respect to $B$ if there exists a constant $\delta_2 > 0$ such that
\[
\|H(\cdot, Bx) - H(\cdot, By)\| \leq \delta_2 \|x - y\|, \quad \forall x, y \in X.
\]  
(8)

Definition 3. A multivalued mapping $M : X \to 2^X$ is said to be $\mu'$-cocoercive if there exists a constant $\mu' > 0$ such that
\[
\langle u - v, x - y \rangle 
\geq \mu' \|u - v\|^2, \quad \forall x, y \in H \text{ for some } u \in M(x), v \in M(y).
\]  
(9)

Example 4 (see [1]). Let $X = \mathbb{R}^2$ with usual inner product. Let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
\[
Ax = (2x_1 - 2x_2, -2x_1 + 4x_2), \quad By = (y_1 - y_2, -y_2),
\]  
\[
\forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2, \quad (10)
\]

such that $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. Suppose that $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by
\[
H(Ax, By) = Ax + By.
\]  
(11)

Then $H(A, B)$ is $1/6$-cocoercive with respect to $A$ and $1/2$-relaxed cocoercive with respect to $B$.

Definition 5 (see [1]). Let $A, B : X \to X, H : X \times X \to X$ be three single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping. $M$ is said to be $H(\cdot, \cdot)$-cocoercive with respect to mappings $A$ and $B$ (or simply $H(\cdot, \cdot)$-cocoercive in the sequel) if $M$ is cocoercive and $(H(A, B) + \lambda M)(X) = X$, for every $\lambda > 0$.

Example 6 (see [1]). Let $X, A, B, H$ and $M$ be the same as in Example 4, and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $M(x_1, x_2) = (0, x_2)$, for all $(x_1, x_2) \in \mathbb{R}^2$. Then $M$ is cocoercive and $(H(A, B) + \lambda M)(\mathbb{R}^2) = \mathbb{R}^2$, for all $\lambda > 0$; that is, $M$ is $H(\cdot, \cdot)$-cocoercive with respect to $A$ and $B$.

Proposition 7 (see [1]). Let $H(A, B)$ be $\mu$-cocoercive with respect to $A$ and $\gamma$-relaxed cocoercive with respect to $B$. $A$ is $\alpha$-expansive, $B$ is $\beta$-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M : X \to 2^X$ be $H(\cdot, \cdot)$-cocoercive operator. If the following inequality
\[
\langle x - y, u - v \rangle \geq 0
\]  
(12)

holds for all $(v, y) \in \text{Graph}(M)$, then $x \in Mu$, where
\[
\text{Graph}(M) = \{(x, u) \in X \times X : u \in M(x)\}.
\]  
(13)

Theorem 8 (see [1]). Let $H(A, B)$ be a $\mu$-cocoercive with respect to $A$ and $\gamma$-relaxed cocoercive with respect to $B$. $A$ is $\alpha$-expansive, $B$ is $\beta$-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M$ be an $H(\cdot, \cdot)$-cocoercive operator with respect to $A$ and $B$. Then the operator $H(A, B) + \lambda M$ is single-valued.

Definition 9 (see [1]). Let $H(A, B)$ be a $\mu$-cocoercive with respect to $A$ and $\gamma$-relaxed cocoercive with respect to $B$. $A$ is $\alpha$-expansive, $B$ is $\beta$-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M$ be an $H(\cdot, \cdot)$-cocoercive operator with respect to $A$ and $B$. The resolvent operator $R^H_{\lambda,M}(u) : X \to X$ is defined by
\[
R^H_{\lambda,M}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X.
\]  
(14)

Theorem 10 (see [1]). Let $H(A, B)$ be a $\mu$-cocoercive with respect to $A$ and $\gamma$-relaxed cocoercive with respect to $B$. $A$ is $\alpha$-expansive, $B$ is $\beta$-Lipschitz continuous, and $\mu > \gamma$, $\alpha > \beta$. Let $M$ be an $H(\cdot, \cdot)$-cocoercive operator with respect to $A$ and $B$. Then resolvent operator $R^H_{\lambda,M} : X \to X$ is $1/(\mu \alpha^2 - \gamma \beta^2)$-Lipschitz continuous; that is,
\[
\|R^H_{\lambda,M}(u) - R^H_{\lambda,M}(v)\| \leq \frac{1}{\mu \alpha^2 - \gamma \beta^2} \|u - v\|, \quad \forall u, v \in X.
\]  
(15)

3. A New System of Generalized Variational Inclusions

In this section, we will introduce a new system of generalized variational inclusions involving $H(\cdot, \cdot)$-cocoercive operators.

Let $X$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$, $\| \cdot \|$, respectively. Let $C$ be a closed and convex set in $X$. Let $H, T_1, T_2 : X \times X \to X$, and $A, B, g, h : X \to X$ be single-valued mappings. Let...
Let $M : X \to 2^X$ be a set-valued mapping such that $M$ is $H(\cdot,\cdot)$-cocoercive operator with respect to $A$ and $B$ if $\varphi : X \to R \cup \{+\infty\}$ be a continuous function. We consider the system of generalized variational inclusions of finding $(x^*, y^*) \in X$ such that
\begin{align*}
0 & \in \rho T_1 (y^*, x^*) + \rho M (g(x^*)) - g(y^*) + g(x^*) , \quad \rho > 0 , \\
0 & \in \eta T_2 (x^*, y^*) + \eta M (h(x^*)) - h(y^*) + h(x^*) , \quad \eta > 0 ,
\end{align*}
which appears to be a new one.

(II) If $T_1 = T_2 = T$, problem (16) is equivalent to finding $(x^*, y^*) \in X$, such that
\begin{align*}
0 & \in \rho T (y^*, x^*) + \rho M (g(x^*)) - g(y^*) + g(x^*) , \quad \rho > 0 , \\
0 & \in \eta T (x^*, y^*) + \eta M (h(x^*)) - h(y^*) + h(x^*) , \quad \eta > 0 ,
\end{align*}
which appears to be a new one.

(III) If $T_1 = T_2 = T$, $\rho = \eta$, $g = h$, and $x^* = y^* = x$, problem (17) is equivalent to finding $x \in X$, such that
\begin{equation}
0 \in T (x) + M (g(x)) ,
\end{equation}
which is known as the variational inclusion problem, or finding the zero of the sum of two (more) cocoercive operators. It is well known that a wide class of linear and nonlinear problems can be studied via variational inclusion problems.

(IV) We note that if $M(\cdot) = \partial \varphi(\cdot)$, the subdifferential of a proper, convex and lower semicontinuous function, then the system of variational inclusions (16) is equivalent to finding $(x^*, y^*) \in X$ such that
\begin{align*}
0 & \in \rho T_1 (y^*, x^*) + \rho \partial \varphi (g(x^*)) - g(y^*) + g(x^*) , \quad \rho > 0 , \\
0 & \in \eta T_2 (x^*, y^*) + \eta \partial \varphi (h(x^*)) - h(y^*) + h(x^*) , \quad \eta > 0 ,
\end{align*}
or equivalently the problem of finding $(x^*, y^*) \in X$ such that
\begin{align*}
\langle \rho T_1 (y^*, x^*) + g(x^*) - g(y^*) , x - g(x^*) \rangle \\
\geq \rho \varphi (g(x^*)) - \rho \varphi (x) , \quad x \in X , \quad \rho > 0 , \\
\langle \eta T_2 (x^*, y^*) + h(y^*) - h(x^*) , x - h(y^*) \rangle \\
\geq \eta \varphi (h(x^*)) - \eta \varphi (x) , \quad x \in X , \quad \eta > 0 ,
\end{align*}
which is called the system of mixed general variational inequalities involving four different nonlinear operators. The problem of type (21) is studied in [7].

(V) If $T_1 = T_2 = T$ is univariate operator and $g = h, \rho = \eta$, and $x^* = y^* = x$, problem (21) is equivalent to finding $x \in X$, such that
\begin{equation}
\langle Tx , y - g(x) \rangle \geq \varphi (g(x)) - \varphi (y) , \quad \forall y \in X ,
\end{equation}
which is known as the mixed general variational inequality or variational inequality of the second type. For the applications and numerical methods for solving the mixed variational inequalities, see [12].

(VI) If $\varphi(\cdot)$ is an indicator function of a closed convex set $C$ in $X$, then problem (21) is equivalent to finding $(x^*, y^*) \in X : g(x^*) , h(y^*) \in C$ such that
\begin{align*}
\langle \rho T_1 (y^*, x^*) + g(x^*) - g(y^*) , x - g(x^*) \rangle \\
& \geq 0 , \quad \forall x \in C , \quad \rho > 0 , \\
\langle \eta T_2 (x^*, y^*) + h(y^*) - h(x^*) , x - h(y^*) \rangle \\
& \geq 0 , \quad \forall x \in C , \quad \eta > 0 ,
\end{align*}
which is called the system of general variational inequalities. Such type of problem is studied in [20].

(VII) If $T_1 = T_2 = T$, then problem (21) is equivalent to finding $(x^*, y^*) \in X : g(x^*) , h(y^*) \in C$ such that
\begin{align*}
\langle \rho T (y^*, x^*) + g(x^*) - g(y^*) , x - g(x^*) \rangle \\
& \geq 0 , \quad \forall x \in C , \quad \rho > 0 , \\
\langle \eta T (x^*, y^*) + h(y^*) - h(x^*) , x - h(y^*) \rangle \\
& \geq 0 , \quad \forall x \in C , \quad \eta > 0 ,
\end{align*}
which can be viewed as a generalization of the system considered and studied in [17, 21].

(VIII) If $\varphi(\cdot)$ is the indicator function of a closed convex set $C$, then problem (22) is equivalent to finding $x^* \in X : g(x^*) \in C$ such that
\begin{equation}
\langle Tx^* , x - g(x^*) \rangle \geq 0 , \quad \forall x \in K ,
\end{equation}
which is known as the general variational inequality introduced and studied by Noor [22, 23] in 1988. This shows that the system of generalized variational inclusions (16) is more general and includes several classes of variational inclusions/inequalities and related optimization problems as special cases. For the recent applications, numerical methods, and formulations of variational inequalities and variational inclusions, see [1–24] and the references therein.

We now show that the system of generalized variational inclusions (16) is equivalent to the fixed-point problem, and this is the motivation of our next result.

Lemma 11. Let $M$ be $H(\cdot,\cdot)$-cocoercive operator. Then $(x^*, y^*) \in X$ is a solution of problem (16) if and only if $(x^*, y^*) \in X$ satisfies the following:
\begin{align*}
g (x^*) & = R_{\rho T_1}^{H(\cdot,\cdot)} [ H (A (g(y^*)) , B (g(y^*)) ) - \rho T_1 (y^*, x^*) ] , \\
h (y^*) & = R_{\eta T_2}^{H(\cdot,\cdot)} [ H (A (h(x^*)) , B (h(x^*)) ) - \eta T_2 (x^*, y^*) ] .
\end{align*}
where \( p_{\rho M}^{H} (u) = (H(A, B) + \rho M)^{-1}(u) \) and \( p_{\eta M}^{H} (u) = (H(A, B) + \eta M)^{-1}(u) \).

**Proof.** The conclusion can be drawn directly from the definition of resolvent operators \( R_{\rho M}^{H} \) and \( R_{\eta M}^{H} \).

This equivalent formulation is used to suggest and analyze a number of iterative methods for solving the system of generalized variational inclusions (16). To do so, one rewrites the equations in the following form:

\[
\begin{align*}
x^* &= x^* - g(x^*) \\
&\quad + R_{\rho M}^{H} \left[ H(A(g(y^*)), B(g(y^*))) - \rho T_1(y^*, x^*) \right], \\
y^* &= y^* - h(y^*) \\
&\quad + R_{\eta M}^{H} \left[ H(A(h(x^*)), B(h(x^*))) - \eta T_2(x^*, y^*) \right].
\end{align*}
\]

(27)

Based on Lemma 11, we construct the following iterative algorithm for solving (16).

**Algorithm 12.** For a given \((x_0, y_0) \in X\), compute the sequences \(\{x_n\}\) and \(\{y_n\}\) from the iterative schemes:

\[
\begin{align*}
x_{n+1} &= (1 - \omega_n) x_n + \omega_n (x_n - g(x_n)) \\
&\quad + \omega_n R_{\rho M}^{H} \left[ H(A(g(y_n)), B(g(y_n))) - \rho T_1(y_n, x_n) \right], \quad n \geq 0, \\
y_n &= y_n - h(y_n) \\
&\quad + R_{\eta M}^{H} \left[ H(A(h(x_n)), B(h(x_n))) - \eta T_2(x_n, y_n) \right], \quad n \geq 1,
\end{align*}
\]

(28)

where \(\omega_n \in [0, 1]\).

If \(\omega_n = 1\), then Algorithm 12 reduces to Algorithm 13.

**Algorithm 13.** For a given \((x_0, y_0) \in X\), compute the sequences \(\{x_n\}\) and \(\{y_n\}\) from the iterative schemes:

\[
\begin{align*}
x_{n+1} &= x_n - g(x_n) \\
&\quad + R_{\rho M}^{H} \left[ H(A(g(y_n)), B(g(y_n))) - \rho T_1(y_n, x_n) \right], \quad n \geq 0, \\
h(y_n) &= \left[ R_{\eta M}^{H} \left[ H(A(h(x_n)), B(h(x_n))) - \eta T_2(x_n, y_n) \right] \right], \quad n \geq 1.
\end{align*}
\]

(29)

For suitable and appropriate choice of the operators \(M, H, T_1, T_2, A, B, g, h\), and spaces, one can obtain a wide class of iterative methods for solving different classes of variational inclusions and related optimization problems. This shows that Algorithm 12 is quite flexible and general and includes various known and new algorithms for solving variational inequalities and related optimization problems as special cases.

**Definition 14.** A mapping \(T : X \times X \rightarrow X\) is called \(p\)-strongly monotone in the first variable if there exists a constant \(p > 0\) such that, for all \(x, y \in X\),

\[
\langle T(x, x') - T(y, y'), x - y \rangle \geq p \|x - y\|^2, \quad \forall x', y' \in X.
\]

(30)

**Definition 15.** A mapping \(T : X \times X \rightarrow X\) is called relaxed \((p, q)\)-cocoercive if there exists a constant \(q > 0\) such that, for all \(x, y \in X\),

\[
\langle T(x, x') - T(y, y'), x - y \rangle \geq -q \|T(x, x') - T(y, y')\|^2, \quad \forall x', y' \in X.
\]

(31)

**Definition 16.** A mapping \(T : X \times X \rightarrow X\) is called relaxed \((p, q)\)-cocoercive in the first variable if there exist constants \(p > 0, q > 0\) such that, for all \(x, y \in X\),

\[
\langle T(x, x') - T(y, y'), x - y \rangle \geq -p \|T(x, x') - T(y, y')\|^2 + q \|x - y\|^2, \quad \forall x', y' \in X.
\]

(32)

The class of relaxed \((p, q)\)-cocoercive mappings is more general than the class of strongly monotone mappings.

**Definition 17.** A mapping \(T : X \times X \rightarrow X\) is called \(r\)-Lipschitz continuous in the first variable if there exists a constant \(r > 0\) such that, for all \(x, y \in X\),

\[
\|T(x, x') - T(y, y')\| \leq r \|x - y\|, \quad \forall x', y' \in X.
\]

(33)

**Lemma 18** (see [24]). Assume that \(\{a_n\}\) is a sequence of nonnegative real numbers such that

\[
a_{n+1} \leq (1 - \lambda_n) a_n + b_n, \quad \forall n \geq 0,
\]

(34)

where \(\lambda_n\) is a sequence in \([0, 1]\) with \(\sum_{n=0}^{\infty} \lambda_n = \infty, b_n = o(\lambda_n)\), and then \(\lim_{n \rightarrow \infty} a_n = 0\).

**Theorem 19.** Let \(X\) be a real Hilbert space. Suppose that \(H, T_1, T_2 : X \times X \rightarrow X, A, B, g, h : X \rightarrow X\) are single-valued mappings and \(M : X \rightarrow 2^X\) is a set-valued mapping such that \(M = H(\cdot, \cdot)\)-cocoercive operator with respect to \(A\) and \(B\). Assume that

(i) \(H(A, B)\) is \(\mu\)-cocoercive with respect to \(A, \gamma\)-relaxed cocoercive with respect to \(B\), and \(\mu > \gamma\);

(ii) \(A\) is \(\alpha\)-expansive, \(B\) is \(\beta\)-Lipschitz continuous, and \(\alpha > \beta\);

(iii) \(H(A, B)\) is \(\delta_1\)-Lipschitz continuous with respect to \(A\) and \(\delta_2\)-Lipschitz continuous with respect to \(B\);

(iv) \(T_1 : X \times X \rightarrow X\) is relaxed \((p_1, q_1)\)-cocoercive and \(r_1\)-Lipschitz continuous in the first variable;
(v) \( T_2 : X \times X \rightarrow X \) is relaxed \((p_2, q_2)\)-cocoercive and \( r_2 \)-Lipschitz continuous in the first variable;

(vi) \( g : X \rightarrow X \) is relaxed \((s_1, t_1)\)-cocoercive and \( l_1 \)-Lipschitz continuous;

(vii) \( h : X \rightarrow X \) is relaxed \((s_2, t_2)\)-cocoercive and \( l_2 \)-Lipschitz continuous;

(viii) \( \omega_n \in [0, 1] \) and \( \sum_{n=0}^{\infty} \omega_n = \infty \);

(ix) \( \theta_4 < 1 \) and \((1 - \theta_3)(1 - \theta_4) > \frac{L_2^2}{\theta_5}(\theta_3 + \theta_2 + \theta_6)\), where
\[
\begin{align*}
\theta_1 &= \sqrt{1 + 2p_1r_1^2 - 2p_1q_1 + \rho^2 r_1^2}, \\
\theta_2 &= \sqrt{1 + 2n_2r_2^2 - 2n_2q_2 + \eta^2 r_2^2}, \\
\theta_3 &= \sqrt{1 + 2s_1l_1^2 - 2t_1 + l_1^2}, \\
\theta_4 &= \sqrt{1 + 2s_2l_2^2 - 2t_2 + l_2^2}, \\
\theta_5 &= \sqrt{1 + l_2^2\delta_1^2 + l_1^2\delta_2^2 - 2l_1^2 r}, \\
\theta_6 &= \sqrt{1 + l_1^2\delta_1^2 + l_2^2\delta_2^2 - 2l_2^2 r},
\end{align*}
\]

and \( r = \mu \alpha^2 - y^2 \delta, L_n = (1/(\mu \alpha^2 - y^2 \delta)). \)

Then the iterative sequences \(\{x_n\} \) and \(\{y_n\} \) generated by Algorithm 12 converge strongly to \(x^*\) and \(y^*\), respectively, and \((x^*, y^*)\) is a solution of problem (16).

**Proof.** To prove the result, we need first to evaluate \(\|x_{n+1} - x^*\|\) for all \(n \geq 0\). From (28), and the Lipschitz continuity of the resolvent operator \(R_{PM}^{(H, \cdot, \cdot)}\), we have

\[
\begin{align*}
\|x_{n+1} - x^*\| \\
= \| & (1 - \omega_n) x_n + \omega_n \\
& \times \left[ x_n - g(x_n) \\
& + R_{PM}^{(H, \cdot, \cdot)}[AH(g(y_n)), B(g(y_n)) - \rho T_1(y_n, x_n)] - x^* \right]\| \\
= \left\| & (1 - \omega_n) x_n + \omega_n \\
& \times \left[ x_n - g(x_n) \\
& + R_{PM}^{(H, \cdot, \cdot)}[AH(g(y_n)), B(g(y_n)) - \rho T_1(y_n, x_n)] \\
& - (1 - \omega_n)x^* - \omega_n \\
& \times [x^* - g(x^*)] \\
& + R_{PM}^{(H, \cdot, \cdot)}[AH(g(y^*)), B(g(y^*)) - \rho T_1(y^*, x^*)]\right]\| \| x^* - x^* \|
\end{align*}
\]

By the assumption that \( T_1 \) is relaxed \((p_1, q_1)\)-cocoercive and \( r_1 \)-Lipschitz continuous in the first variable, we obtain that

\[
\begin{align*}
\| & y_n - y^* \| \\
= \| & y_n - y^* \| - \rho \| T_1(y_n, x_n) - T_1(y^*, x^*) \| \\
& + \rho^2 \| T_1(y_n, x_n) - T_1(y^*, x^*) \| \\
\leq \| & y_n - y^* \| \\
& - 2\rho \| - p_1 \| T_1(y_n, x_n) - T_1(y^*, x^*) \|^2 + q_1 \| y_n - y^* \|^2 \\
& \leq \| & y_n - y^* \| \\
& - 2\rho \| - p_1 \| T_1(y_n, x_n) - T_1(y^*, x^*) \|^2 + q_1 \| y_n - y^* \|^2 \\
& \leq \| & y_n - y^* \| \\
& - 2\rho \| - p_1 \| T_1(y_n, x_n) - T_1(y^*, x^*) \|^2 + q_1 \| y_n - y^* \|^2 \\
& = \theta_3^2 \| y_n - y^* \|^2,
\end{align*}
\]

where \( \theta_3 = \sqrt{1 + 2s_1l_1^2 - 2t_1 + l_1^2} \). By the assumption that \( g \) is relaxed \((s_1, t_1)\)-cocoercive and \( l_1 \)-Lipschitz continuous, we arrive at

\[
\begin{align*}
\| x_n - x^* - (g(x_n) - g(x^*)) \| \\
= \| & x_n - x^* - (g(x_n) - g(x^*)) \| \\
& - 2\| - s_1 \| g(x_n) - g(x^*) \| + t_1 \| x_n - x^* \|^2 \\
& \leq \| x_n - x^* \| \\
& + 2\| - s_1 \| g(x_n) - g(x^*) \| + t_1 \| x_n - x^* \|^2 \\
& = \theta_3^2 \| x_n - x^* \|^2,
\end{align*}
\]

where \( \theta_3 = \sqrt{1 + 2s_1l_1^2 - 2t_1 + l_1^2} \). Now, we estimate

\[
\begin{align*}
\| y_n - y^* \| \\
= \| & [H(A(g(y_n)), B(g(y_n))) \\
& - H(A(g(y^*)), B(g(y^*)))) \| \\
& \leq \| y_n - y^* \| \\
& - 2\| H(A(g(y_n)), B(g(y_n))) \\
& - H(A(g(y^*)), B(g(y^*)))) \| \\
& \leq \| y_n - y^* \| \\
& - 2\| H(A(g(y_n)), B(g(y_n))) \\
& - H(A(g(y^*)), B(g(y^*)))) \| \\
& \leq \| y_n - y^* \| \\
& - 2\| H(A(g(y_n)), B(g(y_n))) \\
& - H(A(g(y^*)), B(g(y^*)))) \| \\
& \leq \| y_n - y^* \|;
\end{align*}
\]
−H (A (g (y∗)), B (g (y∗))), y_n − y∗
+ ∥H (A (g (y_n)), B (g (y_n)))
−H (A (g (y∗)), B (g (y∗))))∥²
+ ∥H (A (g (y∗)), B (g (y∗)))
−H (A (g (y∗)), B (g (y∗))))∥²
≤ ∥y_n − y∗∥² − 2μα²∥g (y_n) − g (y∗)∥²
+ 2β²∥g (y_n) − g (y∗)∥²
+ δ₁2∥g (y_n) − g (y∗)∥² + δ₂2∥g (y_n) − g (y∗)∥²
≤ [1 + l₁²δ₁2 + l₂²δ₂2 − 2l²(μα² − γ²β²)] ∥y_n − y∗∥²
= [1 + l₁²δ₁2 + l₂²δ₂2 − 2l²] ∥y_n − y∗∥²
= θ₂³∥y_n − y∗∥²,
(39)

where θ₂ = √{1 + l₁²δ₁2 + l₂²δ₂2 − 2l²r} and r = μα² − γ²β².

Substituting (37)–(39) into (36) yields
∥x_n−1 − x∗∥ ≤ [1 − ω_n (1 − θ₂)] ∥x_n − x∗∥
+ ω_nL_n (θ₁ + θ₂) ∥y_n − y∗∥,
(40)

where L_n = (1/(μα² − γ²β²)).

Next we estimate
∥y_n − y∗∥
= ∥y_n − h (y_n)
+ R_{H}^{(11)} [H (A (h (x_n)), B (h (x_n))) − ηT₂ (x_n, y_n)]
− y∗ + h (y∗)
− R_{H}^{(11)} [H (A (h (x∗)), B (h (x∗))) − ηT₂ (x∗, y∗)]
= ∥y_n − y∗ − (h (y_n) − h (y∗))
+ 1/μα² − γ²β² × ∥x_n − x∗
− [H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))]
+ ∥x_n − x∗ − η [T₂ (x_n, y_n) − T₂ (x∗, y∗)]∥∥.
(41)

By the assumption that T₂ is relaxed (p₂, q₂)-coercive and r₂-Lipschitz continuous in the first variable, we see that
∥x_n − x∗ − η [T₂ (x_n, y_n) − T₂ (x∗, y∗)]∥²
= ∥x_n − x∗∥² − 2η ⟨T₂ (x_n, y_n) − T₂ (x∗, y∗), x_n − x∗⟩
+ η² ∥T₂ (x_n, y_n) − T₂ (x∗, y∗)∥²
≤ ∥x_n − x∗∥²
− 2η [−p₂∥T₂ (x_n, y_n) − T₂ (x∗, y∗)∥² + q₂∥x_n − x∗∥²]
+ η² ∥T₂ (x_n, y_n) − T₂ (x∗, y∗)∥²
≤ ∥x_n − x∗∥²
− 2η [−2ηp₂∥x_n − x∗∥² + η²q₂∥x_n − x∗∥²]
+ η² ∥T₂ (x_n, y_n) − T₂ (x∗, y∗)∥²
≤ ∥x_n − x∗∥² + 2ηp₂r₂²∥x_n − x∗∥²
− 2ηq₂∥x_n − x∗∥² + η²q₂∥x_n − x∗∥²
= θ₃²∥x_n − x∗∥²,
(42)

where θ₃ = √{1 + 2ηp₂r₂² − 2ηq₂ + η²q₂}. From the proof of (38), we can obtain that
∥y_n − y∗ − (h (y_n) − h (y∗))∥² ≤ θ₄²∥y_n − y∗∥²,
(43)

where θ₄ = √{1 + 2s₂r₂² − 2t₂ + l₂²}.

Now, we estimate
∥x_n − x∗ − η [H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
≤ ∥x_n − x∗∥²
− 2 (H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))) × x_n − x∗
+ ∥H (A (h (x_n)), B (h (x_n)))
− H (A (h (x∗)), B (h (x∗)))∥²
= θ₅²∥x_n − x∗∥²,
(44)

where θ₅ = √{1 + l₁²δ₁2 + l₂²δ₂2 − 2l²} and r = μα² − γ²β².

Substituting (42)–(44) into (41) yields
∥y_n − y∗∥ ≤ θ₄∥y_n − y∗∥ + L_n (θ₁ + θ₂) ∥x_n − x∗∥,
(45)
where \( L_n = \frac{1}{(\mu \alpha^2 - \gamma \beta^2)} \). Since \( \theta_4 < 1 \), we observe that
\[
\|y_n - y^*\| \leq L_n \|x_n - x^*\|.
\] (46)

Substituting (46) into (40) yields
\[
\|x_{n+1} - x^*\| \leq [1 - \omega_n (1 - \theta_3)] \|x_n - x^*\|
+ \omega_n L_n^2 \frac{(\theta_1 + \theta_3)(\theta_2 + \theta_6)}{1 - \theta_4} \|x_n - x^*\|
= \left[ 1 - \omega_n \left( 1 - \theta_3 - L_n^2 \frac{(\theta_1 + \theta_3)(\theta_2 + \theta_6)}{1 - \theta_4} \right) \right]
\times \|x_n - x^*\|.
\] (47)

Noticing condition (ix) and applying Lemma 18 to (47), we get the desired conclusion easily. This completes the proof.

Example 20. Let \( X = \mathbb{R}^2 \) with usual inner product. Let \( A, B : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by
\[
A x = \begin{pmatrix} 7x_1 \\ 7x_2 \end{pmatrix}, \quad B x = \begin{pmatrix} 1 - 2x_1 \\ 1 - 2x_2 \end{pmatrix}, \quad \forall x \in \mathbb{R}^2.
\] (48)

Suppose that \( H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by
\[
H(Ax, By) = Ax + By, \quad \forall x \in \mathbb{R}^2.
\] (49)

Then, it is easy to check the following.

(i) \( H(A, B) \) is \( 1/n \)-cocoercive with respect to \( A \), for \( n = 7, 8 \), and \( 7/n \)-relaxed cocoercive with respect to \( B \), for \( n = 1, 2 \).

(ii) \( A \) is \( n \)-expansive, for \( n = 6, 7 \), and \( B \) is \( 1/n \)-Lipschitz continuous, for \( n = 6, 7 \).

(iii) \( H(A, B) \) is \( n \)-Lipschitz continuous with respect to \( A \), for \( n = 7, 8 \), and \( 1/n \)-Lipschitz continuous with respect to \( B \), for \( n = 6, 7 \).

Let \( T_1, T_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by
\[
T_1(x, x') = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad T_2(y, y') = \begin{pmatrix} \frac{1}{2}y_1 - \frac{1}{4}y_2 \\ \frac{1}{4}y_1 + \frac{1}{2}y_2 \end{pmatrix},
\] \forall x, x', y, y' \in \mathbb{R}^2.
\] (50)

Then, it is easy to verify the following.

(iv) \( T_1 \) is relaxed \( (1, 2) \)-cocoercive and \( 1 \)-Lipschitz continuous.

(v) \( T_2 \) is relaxed \( (1, (3/n)) \)-cocoercive, for \( n = 16, 17 \), and \( \sqrt{5}/n \)-Lipschitz continuous, for \( n = 3, 4 \).

Let \( g, h : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by
\[
g(x) = \begin{pmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \end{pmatrix}, \quad h(x) = \begin{pmatrix} \frac{2}{3}x_1 \\ \frac{2}{3}x_2 \end{pmatrix}, \quad \forall x \in \mathbb{R}^2.
\] (51)

Then, it is easy to verify the following.

(vi) \( g \) is relaxed \((1, (3/n))\)-cocoercive, for \( n = 4, 5 \), and \( 1/n \)-Lipschitz continuous, for \( n = 1, 2 \).

(vii) \( h \) is relaxed \((1, (10/n))\)-cocoercive, for \( n = 9, 10 \), and \( 2/n \)-Lipschitz continuous, for \( n = 2, 3 \).

(viii) Clearly, for the constants
\[
\mu = \frac{1}{7}, \quad \alpha = 7, \quad \gamma = 7, \quad \beta = \frac{1}{7}, \quad \delta_1 = 7,
\]
\[
\delta_2 = \frac{1}{7}, \quad L_n = \frac{7}{48}, \quad p_1 = 1, \quad q_1 = 2,
\]
\[
r_1 = 1, \quad p_2 = 1, \quad q_2 = \frac{13}{16}, \quad r_2 = \frac{\sqrt{5}}{4},
\]
\[
s_1 = 1, \quad t_1 = \frac{3}{4}, \quad l_1 = \frac{1}{2}, \quad s_2 = 1,
\]
\[
t_2 = \frac{10}{9}, \quad l_2 = \frac{2}{3}, \quad \theta_1 = 0.1, \quad \theta_2 = 0.56,
\]
\[
\theta_3 = 0.5, \quad \theta_4 = 0.33, \quad \theta_5 = 3.13, \quad \theta_6 = 4.08.
\] (52)

obtained in (i) to (vii) above, the conditions of Theorem 19 are satisfied for the inclusion system (16) for \( \rho = 0.9, \eta = 1 \).

References


Submit your manuscripts at
http://www.hindawi.com