Measures of Growth of Entire Solutions of Generalized Axially Symmetric Helmholtz Equation

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1. Introduction

The solutions of the partial differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{y} \frac{\partial u}{\partial y} + k^2 u = 0, \quad \mu > 0,
\]

are called the generalized axially symmetric Helmholtz equation functions (GASHEs). The GASHE function \( u \), regular about the origin, has the following Bessel-Gegenbauer series expansion:

\[
u (r, \theta) = \Gamma (2\mu)(kr) \sum_{n=0}^{\infty} \frac{a_n n!}{\Gamma (2\mu + n)} J_{\mu + n} (kr) C_n^\mu (\cos \theta),
\]

where \( x = r \cos \theta, y = r \sin \theta, J_{\mu + n} \) are Bessel functions of first kind, and \( C_n^\mu \) are Gegenbauer polynomials. A GASHE function \( u \) is said to be entire if the series (2) converges absolutely and uniformly on the compact subsets of the whole complex plane. For \( u \) being entire, it is known [1, page 214] that:

\[
\limsup_{n \to \infty} \left( \frac{|a_n|}{\Gamma (n + \mu + 1)} \right)^{1/n} = 0.
\]

Now we define

\[
M (r, u) = \max_{0 \leq \theta \leq 2\pi} |u (r, \theta)|.
\]

Following the usual definitions of order and type of an entire function of a complex variable \( z \), the order \( \rho \) and type \( \tau \) of \( u \) are defined as

\[
\rho (u) = \limsup_{r \to \infty} \frac{\log \log M (r, u)}{\log r}, \quad 0 < \rho (u) < \infty.
\]

Gilbert and Howard [2] have studied the order \( \rho (u) \) of an entire GASHE function \( u \) in terms of the coefficients \( a_n \)'s occurring in the series expansion (2) (see also [1, Theorem 4.5.9]). It has been noticed that the coefficients characterizations for lower order and lower type of \( u \) have not been studied so far. In this paper, we have made an attempt to bridge this gap. McCoy [3] studied the order and type of an entire function solutions of certain elliptic partial differential equation in terms of series expansion coefficients and approximation errors. Recently, Kumar [4, 5] obtained some bounds on growth parameters of entire function solutions of Helmholtz equation in \( R^2 \) in terms of Chebyshev polynomial approximation errors in sup-norm. In the present paper, we have considered the different partial differential equation from those of McCoy [3] and Kumar [4, 5] and obtained the growth parameters such as lower order and lower type of entire GASHE function in terms of the coefficients in its Bessel-Gegenbauer series expansion.
Alternative characterizations for order and type are also obtained in terms of the ratios of these successive coefficients. Our approach and method are different from all these of the above authors.

2. Auxiliary Results

In this section, we shall prove some preliminary results which will be used in the sequel.

We prove the following lemma.

Lemma 1. If \( u \) is an entire GASHE function, then for all \( r > 0 \), \( r^* > 1 \), \( K, K_* < \infty \) and for all \( n \),

\[
(K_*)^{-1} |a_n| \left( \frac{kr}{2r_*} \right)^n \left( \Gamma \left( \mu + (1/2) \right) \right)^2 \\
\leq M (r, u) \leq K \sum_{n=0}^{\infty} \frac{|a_n|}{n!} \left( \frac{kr}{2} \right)^n.
\]

Proof. First we prove right hand inequality. Using the relations

\[
|J_{\mu}(r)| \leq \left( \frac{r/2}{\Gamma(\mu+1)} \right)^\mu,
\]

\[
\max_{\mu \leq \theta \leq \mu+1} |C_{2\mu}^{\mu\mu}(\cos \theta)| \leq \frac{\Gamma(n+2\mu)}{(\Gamma(n+1))} \frac{(2\mu)}{2\mu},
\]
in (2), we get

\[
M (r, u) \leq \sum_{n=0}^{\infty} \frac{|a_n|}{n!} \left( \frac{kr}{2} \right)^n.
\]

Now to prove left hand inequality, we use the orthogonality property of Gegenbauer polynomials [1, page 173] and the uniform convergence of the series (2) as

\[
a_{n+2} \mu(n+1/2)^2(kr)^{-\mu} J_{n+1}(kr) = \int_0^\pi \sin^{2\mu} \theta C_{2\mu}^{\mu\mu} (\cos \theta) u(r, \theta) d\theta.
\]

From the series expansion of \( J_{n+1}(kr) \), we get

\[
J_{n+1}(kr) = \left( \frac{kr}{r} \right)^{\mu+1} \sum_{m=0}^{\infty} \frac{(-1)^m (kr)^{2m}}{2^{2m} m! \Gamma(n+\mu+m+1)}
\]

and for \( n \geq [(kr)^2] \), where \([x]\) denotes the integral part of \( x \), we have

\[
J_{\mu+n}(kr) \geq \frac{1}{2\Gamma(n+\mu+1)} \left( \frac{kr}{2} \right)^{\mu+n}.
\]

From (7), (9), and (11), for \( n \geq [(kr)^2] \), we now get

\[
\frac{|a_n|}{n!} \left( \frac{kr}{2} \right)^n \leq \frac{\pi^{2\mu} \Gamma(n+2\mu)}{(\Gamma(n+1))} \frac{(2\mu)}{2\mu} \frac{r_*}{K_2} \leq M (r, u).
\]

Hence the proof of Lemma 1 is complete. \( \square \)

We now define

\[
\begin{align*}
&f(z) = \sum_{n=0}^{\infty} \frac{|a_n|}{n!} \left( \frac{k}{2} \right)^n z^n, \\
g(z) = \sum_{n=0}^{\infty} \frac{|a_n|}{n!} \left( \frac{k}{2r_*} \right)^n z^n.
\end{align*}
\]

Lemma 2. If \( u \) is an entire GASHE function, then \( f \) and \( g \) are also entire functions of the complex variable \( z \). Further

\[
\frac{(\Gamma(\mu+1/2))^2}{K_2 \pi^{2\mu}} m(r, g) \leq M (r, u) \leq K M (r, f),
\]

where \( m(r, g) = \max_{\mu \leq \theta \leq \mu+1} (|a_n|/n!) (kr/2r_*)^n \) and \( M(r, f) = \max_{|z| \leq r} |f(z)| \).

Proof. Let \( u \) be an entire. In view of (3), we have

\[
\lim_{n \to \infty} \left( \frac{|a_n|}{n!} \left( \frac{k}{2} \right)^n \right)^{1/n} = \lim_{n \to \infty} \left( \frac{|a_n|}{n!} \left( \frac{k}{2r_*} \right)^n \right)^{1/n} = 0.
\]

Hence both \( f \) and \( g \) are entire. Inequalities in (15) follow from (6). \( \square \)

Lemma 3. Let \( f \) and \( g \) be entire functions of particular form defined by (14). Then orders and types of \( f \) and \( g \), respectively, are identical.
Proof. It is well known [6, pages 9–11] that if \( \phi(z) = \sum_{n=0}^{\infty} a_n z^n \) is an entire function, then the order \( \rho(\phi) \) and type \( T(\phi) \) are given as

\[
\rho(\phi) = \limsup_{n \to \infty} \frac{n \log n}{\log |a_n|^{-1}}, \quad T(\phi) = \frac{1}{\rho(\phi)} \limsup_{n \to \infty} n |a_n|^{|\rho(\phi)/n|}.
\]

Hence for the function 
\[ f(z) = \sum_{n=0}^{\infty} (a_n/|a_n|) (k/2r)^n z^n, \]
we have

\[
1 = \liminf_{n \to \infty} \frac{\log ((a_n/|a_n|) (k/2r)^n)}{n \log n} \leq \limsup_{n \to \infty} \frac{\log ((a_n/|a_n|) (k/2r)^n)}{n \log n} = \liminf_{n \to \infty} \frac{\log |a_n|^{-1}}{n \log n} + \log |a_n|^{-1}/n \log n - \frac{n \log (k/2r)}{n \log n}.
\]

Similarly, for \( g(z) = \sum_{n=0}^{\infty} (a_n/|a_n|) (k/2r)^n z^n, \) we have

\[
1 = \liminf_{n \to \infty} \frac{\log [|a_n|/n! (k/2r)^n]}{n \log n} \leq \limsup_{n \to \infty} \frac{\log |a_n|^{-1}}{n \log n} + \frac{\log n! - \log (k/2r)}{n \log n} = \liminf_{n \to \infty} \frac{\log |a_n|^{-1}}{n \log n}.
\]

It follows that \( \rho(f) = \rho(g) \). Since \( f \) and \( g \) have the same order, using (18), we can easily see that \( T(f) = T(g) \). Hence the proof is complete.

In analogy with the definitions of order and type, we define lower order \( \lambda \) and lower type \( t \) as

\[
\lambda(u) = \liminf_{r \to \infty} \frac{\log \log M(r, u)}{\log r}, \quad t(u) = \liminf_{r \to \infty} \frac{\log M(r, u)}{r^{p(u)}}, \quad 0 < p(u) < \infty.
\]

**Theorem 4.** Let \( u \) be an entire GASHE function of order \( \rho(u) \), lower order \( \lambda(u) \), type \( T(u) \), and lower type \( t(u) \). If \( f \) and \( g \) are entire functions as defined above, then

\[
\rho(f) = \rho(u) = \rho(g), \quad T(f) = T(u) = T(g), \quad \lambda(g) \leq \lambda(u) \leq \lambda(f), \quad t(g) \leq t(u) \leq t(f).
\]

**Proof.** Using (15) we get

\[
\limsup_{r \to \infty} \frac{\log \log m(r, g)}{\log r} \leq \limsup_{r \to \infty} \frac{\log \log M(r, u)}{\log r} \leq \limsup_{r \to \infty} \frac{\log \log M(r, u)}{\log r}.
\]

In view of [6, page 13] for an entire function \( \phi \) of finite order we have,

\[
\log M(r, \phi) = \log m(r, f) \quad \text{as} \quad r \to \infty.
\]

Now from above relation (26), we obtain

\[
\rho(g) \leq \rho(u) \leq \rho(f), \quad \lambda(g) \leq \lambda(u) \leq \lambda(f).
\]

Since \( \rho(g) = \rho(f) \), it proves (22) and (24). Denoting by \( \rho \) the common value of order of \( f, g, \) and \( u \), we have from (15),

\[
\limsup_{r \to \infty} \frac{\log m(r, g)}{r^{\rho}} \leq \limsup_{r \to \infty} \frac{\log M(r, u)}{r^{\rho}} \leq \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho}}.
\]

Hence by Lemma 3 we obtain (23). Similarly, we can prove (25).

**Lemma 5.** If \( (\beta_n/\beta_{n+1}) \) forms a nondecreasing function of \( n \), then \( (\gamma_n/\gamma_{n+1}) \) and \( (\delta_n/\delta_{n+1}) \) also form a nondecreasing function of \( n \), where \( \beta_n = |a_n|/\Gamma(n + \mu + 1) \), \( \gamma_n = (|a_n|/n!) (k/2r)^n \), \( \delta_n = (|a_n|/n!) (k/2r)^n \).

**Proof.** We have

\[
\gamma_n = |a_n| (n+1)! (k-1) \quad \text{for} \quad \gamma_{n+1} = |a_{n+1}| n! \quad (28)
\]

\[
\beta_n = \Gamma(n + \mu + 1) (k-2) \quad \text{for} \quad \beta_{n+1} = \Gamma(n + \mu + 2) (k-2) \quad (29)
\]

Let

\[
p(x) = \frac{(x+1) \Gamma(x+\mu+1)}{\Gamma(x+\mu+2)} = (x+1) (x+\mu+1)^{-1} \quad (30)
\]

By logarithmic differentiation, we get

\[
\frac{\log p(x)}{p(x)} = \frac{1}{x+1} - \frac{1}{x+\mu+1}.
\]

Let \( w(x) = 1/(x+1) \), \( w(x) - w(x+\mu) > 0 \) for any \( x > 0 \). Hence \( w(x) \) is a decreasing function and subsequently \( p'(x) > 0 \) for \( x > 0 \). Hence \( (\gamma_n/\gamma_{n+1}) \) is nondecreasing if \( (\beta_n/\beta_{n+1}) \) is nondecreasing. Similarly we can prove the result for \( (\delta_n/\delta_{n+1}) \).
3. Main Results

Now we prove the following theorem.

**Theorem 6.** Let \( u \) be an entire GASHE function of order \( \rho(u) \) (0 < \( \rho(u) < \infty \)), type \( T(u) \), and lower type \( t(u) \). Then

\[
\liminf_{n \to \infty} \left( \frac{k}{2r} \right) \rho(u) \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho(u)} \leq t(u) \leq T(u) \leq \limsup_{n \to \infty} \left( \frac{k}{2r} \right) \rho(u) \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho(u)} .
\]  

(34)

Further, if \( \beta_{n+1}/\beta_n \) forms a nondecreasing function of \( n \) for all \( n > n_0 \), then

\[
\limsup_{n \to \infty} \frac{n}{\rho(u)} \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho(u)} \leq eT(u) \left( \frac{2}{k} \right)^{\rho(u)} ,
\]

(35)

\[
\limsup_{n \to \infty} \frac{\log n}{\log (\beta_n/\beta_{n+1})} = \rho(u).
\]

Proof. If \( \phi(z) = \sum_{n=0}^{\infty} a_n z^n \) is an entire function of order \( \rho(\phi) \), type \( T(\phi) \), and lower type \( t(\phi) \), then we have [7, Theorem 1]

\[
\liminf_{n \to \infty} \frac{n}{\rho(\phi)} \left( \frac{a_{n+1}}{a_n} \right)^{\rho(\phi)} \leq t(\phi) \leq T(\phi) \leq \limsup_{n \to \infty} \frac{n}{\rho(\phi)} \left( \frac{a_{n+1}}{a_n} \right)^{\rho(\phi)} .
\]

(36)

Applying right hand inequality to \( f(z) = \sum y_n z^n \), we get

\[
T(f) \leq \limsup_{n \to \infty} \frac{n}{\rho(f)} \left[ \frac{a_{n+1}}{a_n} \right] \frac{n!}{(n+1)!} \left( \frac{k}{2r} \right)^{\rho(f)} \]

(37)

\[
= \limsup_{n \to \infty} \frac{n}{\rho} \left[ \frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu+2)} \right] \left( \frac{\beta_{n+1}}{\beta_n} \right) \frac{k}{2r} \]

Since

\[
\frac{\Gamma(n+\mu+2)}{\Gamma(n+\mu+1)} \frac{1}{(n+1)/(n+1)} \to 1 \quad \text{as} \quad n \to \infty,
\]

\[
T(u) = T(f) \leq \limsup_{n \to \infty} \left( \frac{k^{\rho(u)}}{2r} \right) \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho(u)} .
\]

(38)

To prove left hand inequality in (34), we consider the entire function \( g(z) = \sum_{n=0}^{\infty} \delta_n z^n \). Then

\[
\liminf_{n \to \infty} \frac{n}{\rho} \left[ \frac{|a_{n+1}|}{|a_n|} \right] \left( \frac{k}{2r} \right)^n \leq t(g)
\]

(39)

\[
= \liminf_{n \to \infty} \frac{n}{\rho} \left[ \frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu+2)} \right] \left( \frac{\beta_{n+1}}{\beta_n} \right) \frac{k}{2r} \]

Thus the proof of (34) is complete. To prove (35), consider an entire function \( \phi(z) = \sum_{n=0}^{\infty} a_n z^n \) of order \( \rho(\phi) \) and type \( T(\phi) \). If \( a_n/\alpha_{n+1} \) forms a nondecreasing function of \( n \) for \( n > n_0 \), then we know ([8], [9, Theorem 2]) that

\[
\rho(\phi) = \limsup_{n \to \infty} \frac{\log n}{\log |a_n/\alpha_{n+1}|}.
\]

(40)

Further, we have [8, Theorem 3]

\[
\limsup_{n \to \infty} \frac{n}{\rho(\phi)} \left[ \frac{\alpha_{n+1}}{\alpha_n} \right] \frac{\rho(\phi)}{\alpha_n} \leq eT(\phi).
\]

(41)

Now let us suppose that \( \beta_n/\beta_{n+1} \) forms a nondecreasing function of \( n \) for \( n > n_0 \). From Lemma 5, \( (y_n/y_{n+1}) \) also forms a nondecreasing function of \( n \) for \( n > n_0 \). Using (40) to \( f(z) = \sum_{n=0}^{\infty} y_n z^n \), we get

\[
\rho(f) = \limsup_{n \to \infty} \frac{\log n}{\log (y_n/y_{n+1})} = \limsup_{n \to \infty} \log \frac{n}{\log n} \times \log \frac{\beta_n}{\beta_{n+1}} \left( \frac{k}{2r} \right)^n \]

(42)

\[
= \log \left( \frac{\beta_n}{\beta_{n+1}} \right) - \log (n+1) - \log (n+\mu + 1) \quad \text{as} \quad n \to \infty.
\]
\[ \rho(f) = \limsup_{n \to \infty} \frac{\log n}{\log (y_n/y_{n+1})} \]

\[ = \limsup_{n \to \infty} \frac{\log (\beta_n/\beta_{n+1})}{2} \left[ \log \left( \frac{(\beta_n/\beta_{n+1})^k}{2} \right) \right] \]

\[ \sim \log \left( \frac{\beta_n}{\beta_{n+1}} \right) \text{ as } n \to \infty. \]

\[ (43) \]

Now using (41) for \( f(z) = \sum_{n=0}^{\infty} y_n z^n \), we get

\[ \limsup_{n \to \infty} \frac{n}{n+1} \sum_{n=0}^{\infty} \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho(f)} \leq eT(f). \]

\[ (44) \]

Since \( \rho(f) = \rho \), \( T(f) = T \), thus, we get

\[ \limsup_{n \to \infty} \frac{n}{n+1} \sum_{n=0}^{\infty} \left( \frac{\beta_{n+1}}{\beta_n} \right)^{\rho} \leq e\left( \frac{2}{k} \right)^T. \]

\[ (45) \]

Hence the proof is complete. \( \square \)

**Theorem 7.** Let \( u \) be an entire GASHF function of order \( \rho \), \( 0 < \rho < \infty \), lower order \( \lambda \), and lower type \( t \). If \( (\beta_n/\beta_{n+1}) \) forms a nondecreasing function of \( n \) for \( n > n_0 \), then

\[ \lambda = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n)^t}, \]

\[ (46) \]

\[ t = \liminf_{n \to \infty} \frac{\log (\beta_n)^{\rho/n}}{e^\rho}. \]

\[ (47) \]

**Proof.** For entire function \( \phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \), if \( |\alpha_n/\alpha_{n+1}| \) forms a nondecreasing function of \( n \) for \( n > n_0 \), then we have \([8], [10, Theorem 2])

\[ \lambda(\phi) = \liminf_{n \to \infty} \frac{n \log n}{\log (\alpha_n)^{-1}}. \]

\[ (48) \]

Let \( (\beta_n/\beta_{n+1}) \) forms a nondecreasing function of \( n \) for \( n > n_0 \). Applying Lemma 5 and (48) to \( f(z) = \sum_{n=0}^{\infty} y_n z^n \), we get

\[ \lambda(f) = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n)^{-1}}. \]

\[ (49) \]

Similarly, using Lemma 5 and (48) for entire function \( g(z) = \sum_{n=0}^{\infty} \delta_n z^n \), we have

\[ \lambda(g) = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n)^{-1}}. \]

\[ (50) \]

The result (46) is now followed by (24) and above two relations for \( \lambda(f) \) and \( \lambda(g) \).

If \( \phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \) is an entire function of order \( \rho(\phi) \), lower type \( t(\phi) \), and \( |\alpha_n/\alpha_{n+1}| \) forms a nondecreasing function of \( n \) for \( n > n_0 \), then by a result of Shah [11], we have

\[ t(\phi) = \liminf_{n \to \infty} \frac{n}{e^\rho(\phi)} (\alpha_n)^{\rho(\phi)/n}, \]

\[ t(g) = \liminf_{n \to \infty} \frac{n}{e^\rho(g)} (\beta_n)^{\rho(g)/n}. \]

\[ (51) \]

Equation (47) now follows in view of (22) and (25). Hence the proof is complete. \( \square \)

**Theorem 8.** Let \( u \) be an entire GASHE function of lower order \( \lambda \), and let \( |\alpha_n/\alpha_{n+1}| \) forms a nondecreasing function of \( n \) for \( n > n_0 \), then

\[ \lambda = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n/\beta_{n+1})}. \]

\[ (52) \]

**Proof.** For an entire function \( \phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \), from [12, Corollary, page 312], we get

\[ \lambda(\phi) = \liminf_{n \to \infty} \frac{n \log n}{\log (\alpha_n)^{-1}}. \]

\[ (53) \]

provided \( |\alpha_n/\alpha_{n+1}| \) forms a nondecreasing function of \( n \) for \( n > n_0 \). Applying this condition on \( \{\beta_n\} \), we can easily show, as in Theorem 6, that

\[ \lambda(f) = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n/\beta_{n+1})}. \]

\[ (54) \]

Applying (53) to \( g(z) = \sum_{n=0}^{\infty} \delta_n z^n \) also, we have

\[ \lambda(g) = \liminf_{n \to \infty} \frac{n \log n}{\log (\beta_n/\beta_{n+1})}. \]

\[ (55) \]

The relation (48) now follows on using (24). Hence the proof is complete. \( \square \)

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