Hindawi Publishing Corporation
Algebra
Volume 2013, Article ID 473714, 5 pages
http://dx.doi.org/10.1155/2013/473714

Research Article

Commutative and Bounded BE-algebras

Zekiye Çiloğlu and Yılmaz Çeven

Department of Mathematics, Faculty of Arts and Sciences, Süleyman Demirel University, 32260 Isparta, Turkey

Correspondence should be addressed to Yılmaz Çeven; yilmazceven@sdu.edu.tr

Received 2 August 2013; Accepted 28 October 2013

Academic Editor: Yao-Zhong Zhang

We introduce the notions of the commutative and bounded BE-Algebras. We give some related properties of them.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras [1, 2]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In [3, 4], Hu and Li introduced a wide class of abstract algebras called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of BCH-algebras. Neggers and Kim [5] introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras [6, 7]. Jun et al. [8] introduced a new notion called BH-algebra which is another generalization of BCH/BCI/BCK-algebras. Walendziak obtained some equivalent axioms for B-algebras [9]. C. B. Kim and H. S. Kim [10] introduced the notion of BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [11], H. S. Kim and Y. H. Kim introduced the notion of BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. In [12, 13], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras and discussed some properties of the characterizations of generalized upper sets related to the structure of ideals in transitive and self-distributive BE-algebras. In [14], Ahn et al. introduced the notion of terminal section of BE-algebras and provided the characterization of the commutative BE-algebras.

In this paper we introduce the notion of bounded BE-algebras and investigate some properties of them.

2. Preliminaries

Definition 1 (see [11]). An algebra $(X; *, 1)$ of type (2, 0) is called a BE-algebra if, for all x, y, and z in X,

(BE1) $x * x = 1$,

(BE2) $x * 1 = 1$,

(BE3) $1 * x = x$,

(BE4) $x * (y * z) = y * (x * z)$.

In X, a binary relation "$\leq$" is defined by $x \leq y$ if and only if $x * y = 1$.

Example 2 (see [11]). Let $X = \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

(1)

Then $(X; *, 1)$ is a BE-algebra.

Definition 3. A BE-algebra $(X; *, 1)$ is said to be self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all x, y, and $z \in X$.

Example 4 (see [11]). Let $X = \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

(2)

Then $(X; *, 1)$ is a self-distributive BE-algebra.

Proposition 5 (see [14]). *Let X be a self-distributive BE-algebra. If $x \leq y$, then, for all x, y, and z in X, the following inequalities hold:*

(i) $z * x \leq z * y$,

(ii) $y * z \leq x * z$.

Definition 6 (see [15]). A dual BCK-algebra is an algebra $(X; *, 1)$ of type (2,0) satisfying (BE1) and (BE2) and the following axioms:

(dBCK1) $x * y = y * x = 1$ implies $x = y$,

(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$,

(dBCK3) $x * ((x * y) * y) = 1$.

Proposition 7 (see [16]). *Any dual BCK-algebra is a BE-algebra.*

The converse of Proposition 7 does not hold in general [16].

Definition 8 (see [16]). Let X be a BE-algebra or dual BCK-algebra. X is said to be commutative if the following identity holds:

$$\text{(C)} \quad x \vee_B y = y \vee_B x, \quad \text{where } x \vee_B y = (y * x) * x \tag{3}$$

for all $x, y \in X$.

Theorem 9 (see [16]). *If X is a commutative BE-algebra, then $(X; *, 1)$ is a dual BCK-algebra.*

Corollary 10 ([16]). *X is a commutative BE-algebra if and only if it is a commutative dual BCK-algebra.*

If X is a commutative BE-algebra, then the relation "$\leq$" is a partial order on X [16].

In the following, we abbreviate $\vee_B$ as $\vee$.

3. Bounded BE-Algebras

The following definition introduces the notion of boundedness for BE-algebras.

Definition 11. Let X be a BE-algebra. If there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$, then the element "0" is called unit of X. A BE-algebra with unit is called a bounded BE-algebra.

In a bounded BE-algebra, $x * 0$ denoted by xN.

Example 12. The BE-algebra in Example 2 is a bounded BE-algebra and its unit element is 0.

Example 13. The BE-algebra in Example 4 is a bounded BE-algebra and its unit element is d.

Example 14. Let $X = \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	1	1	1	d
d	1	1	b	c	1

(4)

It is clear that X is a BE-algebra, but it is not a bounded BE-algebra.

Theorem 15. *In a bounded BE-algebra with unit 0, the following properties hold for all x, $y \in X$:*

(i) $1N = 0$, $0N = 1$,

(ii) $x \leq xNN$,

(iii) $x * yN = y * xN$,

(iv) $0 \vee x = xNN$, $x \vee 0 = x$.

Proof. (i) Using (BE3) and (BE1), we have $1N = 1 * 0 = 0$ and $0N = 0 * 0 = 1$.

(ii) Since

$$\begin{aligned} x * xNN &= x * ((x * 0) * 0) \\ &= (x * 0) * (x * 0) \\ &= 1, \end{aligned} \tag{5}$$

by (BE4) and (BE1), we get $x \leq xNN$.

(iii) Using (BE4), we have

$$\begin{aligned} x * yN &= x * (y * 0) \\ &= y * (x * 0) \\ &= y * xN. \end{aligned} \tag{6}$$

(iv) By routine operations, we have $0 \vee x = (x * 0) * 0 = xNN$ and $x \vee 0 = (0 * x) * x = 1 * x = x$. □

Theorem 16. *In a bounded and self-distributive BE-algebra with unit 0, the following properties hold for all x, $y \in X$:*

(i) $x * y \leq yN * xN$,

(ii) $x \leq y$ *implies* $yN \leq xN$.

Proof. (i) Since

$$\begin{aligned}
&(x*y)*(yN*xN)\\
&\quad=(x*y)*((y*0)*(x*0))\\
&\quad=(y*0)*((x*y)*(x*0)) \quad \text{(by BE4)}\\
&\quad=(y*0)*(x*(y*0)) \quad \text{(by distributivity)}\\
&\quad=x*((y*0)*(y*0)) \quad \text{(by BE4)}\\
&\quad=x*1 \quad \text{(by BE1)}\\
&\quad=1 \quad \text{(by BE2)},
\end{aligned} \tag{7}$$

we have $x*y\le yN*xN$.

(ii) It is trivial by Proposition 5. □

Proposition 17. *Let X be a BE-algebra. Then $x*y\le(y\vee x)*y$.*

Proof. Since

$$\begin{aligned}
(x*y)*((y\vee x)*y)&=(y\vee x)*((x*y)*y)\\
&=(y\vee x)*(y\vee x)\\
&=1,
\end{aligned} \tag{8}$$

we have $x*y\le(y\vee x)*y$. □

Proposition 18. *Let X be a self-distributive BE-algebra. Then, the next properties are valid for all $x,y\in X$:*

(i) $(y\vee x)*y\le x*y$,

(ii) $x*(x*y)=x*y$.

Proof. (i) Since

$$\begin{aligned}
x*(y\vee x)&=x*((x*y)*y)\\
&=(x*y)*(x*y)\\
&=1,
\end{aligned} \tag{9}$$

we have $x\le y\vee x$. By Proposition 5(i), we have $(y\vee x)*y\le x*y$.

(ii) Using Definition 3, (BE1) and (BE3), we have

$$\begin{aligned}
x*(x*y)&=(x*x)*(x*y)\\
&=1*(x*y)\\
&=x*y.
\end{aligned} \tag{10}$$

□

Corollary 19. *If X is a self-distributive and commutative BE-algebra, then $(y\vee x)*y=x*y$.*

Proof. It is clear by Propositions 17 and 18 and the property (dBCK1). □

Corollary 20. *If X is a self-distributive, commutative, and bounded BE-algebra with unit 0, then $xNNN=xN$.*

Proof. In Corollary 19, taking $y=0$, we have $(0\vee x)*0=x*0$. Then we get $((x*0)*0)*0=x*0$; that is, $xNNN=xN$. □

Definition 21. In a bounded BE-algebra, the element x such that $xNN=x$ is called an involution.

Let $S(X)=\{x\in X: xNN=x\}$ where X is a bounded BE-algebra. $S(X)$ is the set of all involutions in X. Moreover, since $1NN=(1*0)*0=0*0=1$ and $0NN=(0*0)*0=1*0=0$, we have $0,1\in S(X)$ and so $S(X)\neq\varnothing$.

Example 22. For the BE-algebra in Example 2, it is clear that $S(X)=X$.

Example 23. For the BE-algebra in Example 4, it is clear that $S(X)=\{1,b,c,d\}$.

Proposition 24. *Let X be a bounded BE-algebra with unit 0 and let $S(X)$ be the set of all involutions in X. Then, for all $x,y\in S(X)$, the following conditions hold:*

(i) $xNNN=xN$,

(ii) $xN*y=yN*x$.

Proof. (i) The proof is obvious by the definition of $S(X)$.

(ii) By Theorem 15(iii), we have

$$\begin{aligned}
xN*y&=xN*yNN\\
&=yN*xNN\\
&=yN*x.
\end{aligned} \tag{11}$$

□

In a bounded BE-algebra X, we denote $x\wedge y=(xN\vee yN)N$ where $x\vee y=(y*x)*x$ for all $x,y\in X$.

Theorem 25. *In a bounded and commutative BE-algebra X the following identities hold:*

(i) $xNN=x$,

(ii) $xN\wedge yN=(x\vee y)N$,

(iii) $xN\vee yN=(x\wedge y)N$,

(iv) $xN*yN=y*x$.

Proof. (i) Using (BE3), it is obtained that

$$\begin{aligned}
xNN&=(x*0)*0\\
&=(0*x)*x, \quad \text{by commutativity}\\
&=1*x\\
&=x.
\end{aligned} \tag{12}$$

(ii) By the definition of "$\wedge$" and (i), we have that $xN\wedge yN=(xNN\vee yNN)N=(x\vee y)N$.

(iii) By the definition of "$\wedge$" and (i), we have that $(x \wedge y)N = (xN \vee yN)NN = xN \vee yN$.

(iv) We have

$$\begin{aligned} xN * yN &= (x * 0) * (y * 0) \\ &= y * ((x * 0) * 0) \\ &= y * (xNN) \\ &= y * x. \end{aligned} \tag{13}$$

□

Corollary 26. *If X is a bounded commutative BE-algebra, then $S(X) = X$.*

Definition 27. Each of the elements a and b in a bounded BE-algebra is called the complement of the other if $a \vee b = 1$ and $a \wedge b = 0$.

Theorem 28. *Let $(X; *, 1)$ be a bounded and commutative BE-algebra. If there exists a complement of any element of X, then it is unique.*

Proof. Let $x \in X$ and a, b be two complements of x. Then we know that $x \wedge a = x \wedge b = 0$ and $x \vee a = x \vee b = 1$. Also since $x \vee a = (x * a) * a = 1$ and $a * (x * a) = x * (a * a) = x * 1 = 1$, we have $x * a \leq a$ and $a \leq x * a$. So we get $x * a = a$. Similarly $x * b = b$. Hence

$$\begin{aligned} a * b &= (x * a) * (x * b) \\ &= (aN * xN) * (bn * xN), \quad \text{by Theorem 25 (iv)} \\ &= bN * ((aN * xN) * xN), \quad \text{by BE-4} \\ &= bN * (xN \vee aN) \\ &= bN * (x \wedge a) N, \quad \text{by Theorem 25 (iii)} \\ &= (x \wedge a) * b, \quad \text{by Theorem 25 (iii)} \\ &= 0 * b \\ &= 1. \end{aligned} \tag{14}$$

With similar operations, we have $b * a = 1$. Hence we obtain $a = b$ which gives that the complement of x is unique. □

Note that for a bounded and commutative BE-algebra, an element does not need to have any complement.

Example 29. In Example 2, the complement of b is c, but a has no complement in X.

Theorem 30. *Let $(X; *, 1)$ be a commutative and bounded BE-algebra. Then the following conditions are equivalent, for all x, $y \in X$:*

(i) $x \wedge xN = 0$,

(ii) $xN \vee x = 1$,

(iii) $xN * x = x$,

(iv) $x * xN = xN$,

(v) $x * (x * y) = x * y$.

Proof. (i)⇒(ii) Let $x \wedge xN = 0$. Then it follows that

$$\begin{aligned} xN \vee x &= (xN \vee x) NN, \quad \text{by Theorem 25 (i)} \\ &= (xNN \wedge xN) N, \quad \text{by Theorem 25 (ii)} \\ &= (x \wedge xN) N, \quad \text{by Theorem 25 (i)} \\ &= 0N \\ &= 1. \end{aligned} \tag{15}$$

(ii)⇒(iii) Let $xN \vee x = 1$. Then, since $(xN * x) * x = x \vee xN = 1$ and $x * (xN * x) = xN * (x * x) = xN * 1 = 1$, we get $xN * x = x$ by (dBCK1).

(iii)⇒(iv) Let $xN * x = x$. Substituting xN for x and using Theorem 25 (i), we obtain the result.

(iv)⇒(v) Let $x * xN = xN$. Then we get $yN * (x * xN) = yN * xN$. Hence we have $x * (yN * xN) = yN * xN$. Using Theorem 25(iv), we obtain $x * (x * y) = x * y$.

(v)⇒(ii) Let $x * (x * y) = x * y$. Then we have

$$\begin{aligned} xN \vee x &= (x * (xN)) * xN \\ &= (x * (x * 0)) * xN \\ &= (x * 0) * (x * 0) \\ &= 1. \end{aligned} \tag{16}$$

(ii)⇒(i) Let $xN \vee x = 1$. Then we obtain

$$\begin{aligned} xN \wedge x &= xN \wedge xNN \\ &= (x \vee xN) N, \quad \text{by Theorem 25 (ii)} \\ &= 1N \\ &= 0. \end{aligned} \tag{17}$$

□

Note that, if X is a self-distributive BE-algebra, then $x * (x * y) = x * y$ by Proposition 18. In this case, Theorem 30(v) is true. If X is also commutative and bounded, then xN is the complement of x by Theorem 30(i) and (ii).

Now we obtain a bounded BE-algebra from a non-bounded BE-algebra as the following theorem.

Theorem 31. *Let $(X; *, 1)$ be a BE-algebra and $0 \notin X$. Define the operation $\circledast$ on $\overline{X} = X \cup \{0\}$ as follows:*

$$x \circledast y = \begin{cases} x * y & \text{if } x, y \in X, \\ 1 & \text{if } x = 0,\ y \in X, \\ 0 & \text{if } x \in X,\ y = 0, \\ 1 & \text{if } x = y = 0. \end{cases} \tag{18}$$

Then $(\overline{X}; \circledast, 1)$ is a bounded BE-algebra with unit 0.

Proof. It is clear that BE1, BE2, and BE3 are satisfied. It suffices to verify BE4. Note that

if $x = 0$ and $y, z \in X$, then $0 \circledast (y * z) = 1$ and $y \circledast (0 \circledast z) = y \circledast 1 = 1$;

if $y = 0$ and $x, z \in X$, then $x \circledast (0 \circledast z) = x \circledast 1 = 1$ and $0 \circledast (x \circledast z) = 1$;

if $z = 0$ and $x, y \in X$, then $x \circledast (y \circledast 0) = x \circledast 0 = 0$ and $y \circledast (x \circledast 0) = y \circledast 0 = 0$.

For the remain situations, it is clearly seen that BE4 is satisfied with similar arguments. □

We call the BE-algebra $(\overline{X}; \circledast, 1)$ in the previous theorem the extension of $(X; 1, *)$. Note that for the BE-algebra $(\overline{X}; \circledast, 1)$, if for $1 \neq x \in X$, we have $x \notin S(X)$ since $xNN = (x \circledast 0) \circledast 0 = 0 * 0 = 1 \neq x$. Hence $S(X) = \{0, 1\}$.

Theorem 32. *The extension of a self-distributive BE-algebra $(X; *, 1)$ is also self-distributive.*

Proof. Let $0 \notin X$ and denote $\overline{X} = X \cup \{0\}$. Let $\circledast$ be defined as in Theorem 31. We want to see that $x \circledast (y \circledast z) = (x \circledast y) \circledast (x \circledast z)$ for all x, y, and $z \in \overline{X}$. For all $x, y \in X$, we know $x \circledast y = x * y$, $0 \circledast y = 1$, $x \circledast 0 = 0$, and $0 \circledast 0 = 1$. So we obtain the following, for all x, y, and $z \in X$:

$$\begin{aligned}
0 \circledast (0 \circledast 0) &= (0 \circledast 0) \circledast (0 \circledast 0), \\
0 \circledast (y \circledast 0) &= (0 \circledast y) \circledast (0 \circledast 0), \\
x \circledast (0 \circledast 0) &= (x \circledast 0) \circledast (x \circledast 0), \\
0 \circledast (0 \circledast z) &= (0 \circledast 0) \circledast (0 \circledast z), \\
0 \circledast (y \circledast z) &= (0 \circledast y) \circledast (0 \circledast z), \\
x \circledast (0 \circledast z) &= (x \circledast 0) \circledast (x \circledast z), \\
x \circledast (y \circledast 0) &= (x \circledast y) \circledast (x \circledast 0), \\
x \circledast (y \circledast z) &= (x \circledast y) \circledast (x \circledast z).
\end{aligned} \tag{19}$$

The above results show that $(\overline{X}; \circledast, 1)$ is self-distributive. □

Theorem 33. *If $(X; *, 1)$ is a commutative, self-distributive, and bounded BE-algebra, then it is a lattice with respect to $\vee$ and $\wedge$ where $x \vee y = y * (y * x)$ and $x \wedge y = (xN \vee yN)N$ for any $x, y \in X$.*

Proof. From Theorem 3.6 in [14], we know that $(X; *, 1)$ is a semilattice with respect to $\vee$. Then we need to show that the set $\{x, y\}$ for all $x, y \in X$ has a greatest lower bound. We know that $xN \leq xN \vee yN$ and $yN \leq xN \vee yN$. By Proposition 5, we have $(xN \vee yN)N \leq xNN$ and $(xN \vee yN)N \leq yNN$. Since $xNN = x$ and $yNN = y$ by Theorem 25(i), we have $x \wedge y \leq x$ and $x \wedge y \leq y$. So $x \wedge y$ is a lower bound of x and y. Next we must show that if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$. Let $z \leq x$ and $z \leq y$. We have $xN \leq zN$ and $yN \leq zN$ by Proposition 5. Since X is an upper semilattice with respect to $\vee$, then we obtain $xN \vee yN \leq zN$ and hence $zNN \leq (xN \vee yN)N$. So we have $z \leq x \wedge y$. Then $x \wedge y$ is the greatest lower bound of x and y. So, we can say that X is a lower semilattice. Consequently $(X, \leq)$ is a lattice with respect to $\vee$ and $\wedge$. □

References

[1] Y. Imai and K. Iséki, "On axiom systems of propositional calculi. XIV," *Proceedings of the Japan Academy*, vol. 42, pp. 19–22, 1966.

[2] K. Iséki, "On BCI-algebras," *Kobe Journal of Mathematics*, vol. 8, no. 1, pp. 125–130, 1980.

[3] Q. P. Hu and X. Li, "On BCH-algebras," *Kobe Journal of Mathematics*, vol. 11, no. 2, part 2, pp. 313–320, 1983.

[4] Q. P. Hu and X. Li, "On proper BCH-algebras," *Mathematica Japonica*, vol. 30, no. 4, pp. 659–661, 1985.

[5] J. Neggers and H. S. Kim, "On *d*-algebras," *Mathematica Slovaca*, vol. 49, no. 1, pp. 19–26, 1999.

[6] J. Neggers and H. S. Kim, "On *B*-algebras," *Matematichki Vesnik*, vol. 54, no. 1-2, pp. 21–29, 2002.

[7] J. Neggers and H. S. Kim, "A fundamental theorem of *B*-homomorphism for *B*-algebras," *International Mathematical Journal*, vol. 2, no. 3, pp. 215–219, 2002.

[8] Y. B. Jun, E. H. Roh, and H. S. Kim, "On BH-algebras," *Scientiae Mathematicae*, vol. 1, no. 3, pp. 347–354, 1998.

[9] A. Walendziak, "Some axiomatizations of *B*-algebras," *Mathematica Slovaca*, vol. 56, no. 3, pp. 301–306, 2006.

[10] C. B. Kim and H. S. Kim, "On *BM*-algebras," *Scientiae Mathematicae Japonicae*, vol. 63, no. 3, pp. 421–427, 2006.

[11] H. S. Kim and Y. H. Kim, "On BE-algebras," *Scientiae Mathematicae Japonicae*, vol. 66, no. 1, pp. 113–116, 2007.

[12] S. S. Ahn and K. S. So, "On ideals and upper sets in *BE*-algebras," *Scientiae Mathematicae Japonicae*, vol. 68, no. 2, pp. 279–285, 2008.

[13] S. S. Ahn and K. S. So, "On generalized upper sets in BE-algebras," *Bulletin of the Korean Mathematical Society*, vol. 46, no. 2, pp. 281–287, 2009.

[14] S. S. Ahn, Y. H. Kim, and J. M. Ko, "Filters in commutative *BE*-algebras," *Korean Mathematical Society*, vol. 27, no. 2, pp. 233–242, 2012.

[15] K. H. Kim and Y. H. Yon, "Dual BCK-algebra and MV-algebra," *Scientiae Mathematicae Japonicae*, vol. 66, no. 2, pp. 247–253, 2007.

[16] A. Walendziak, "On commutative BE-algebras," *Scientiae Mathematicae Japonicae*, vol. 69, no. 2, pp. 281–284, 2009.