Research Article
Commutative and Bounded BE-algebras

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We introduce the notions of the commutative and bounded BE-Algebras. We give some related properties of them.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras [1, 2]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In [3, 4], Hu and Li introduced a wide class of abstract algebras called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of BCH-algebras. Neggers and Kim [5] introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras [6, 7]. Jun et al. [8] introduced a new notion called BH-algebra which is another generalization of BCH/BCI/BCK-algebras. Walendziak obtained some equivalent axioms for B-algebras [9]. C. B. Kim and H. S. Kim [10] introduced the notion of BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [11], H. S. Kim and Y. H. Kim introduced the notion of BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. In [12, 13], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras and discussed some properties of the characterizations of generalized upper sets related to the structure of ideals in transitive and self-distributive BE-algebras. In [14], Ahn et al. introduced the notion of terminal section of BE-algebras and provided the characterization of the commutative BE-algebras.

In this paper we introduce the notion of bounded BE-algebras and investigate some properties of them.

2. Preliminaries

Definition 1 (see [11]). An algebra \((X; \ast, 1)\) of type \((2, 0)\) is called a BE-algebra if, for all \(x, y, \) and \(z\) in \(X\),

\[
\begin{align*}
(BE1) & \ x \ast x = 1, \\
(BE2) & \ x \ast 1 = 1, \\
(BE3) & \ 1 \ast x = x, \\
(BE4) & \ x \ast (y \ast z) = y \ast (x \ast z).
\end{align*}
\]

In \(X\), a binary relation “\(\leq\)” is defined by \(x \leq y\) if and only if \(x \ast y = 1\).

Example 2 (see [11]). Let \(X = \{1, a, b, c, d, 0\}\) be a set with the following table:

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<th>a</th>
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Then \((X; \ast, 1)\) is a BE-algebra.

Definition 3. A BE-algebra \((X; \ast, 1)\) is said to be self-distributive if \(x \ast (y \ast z) = (x \ast y) \ast (x \ast z)\) for all \(x, y, \) and \(z \in X\).
Example 4 (see [11]). Let $X = \{1, a, b, c, d\}$ be a set with the following table:

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(2)

Then $(X; *, 1)$ is a self-distributive BE-algebra.

**Proposition 5** (see [14]). Let $X$ be a self-distributive BE-algebra. If $x \leq y$, then, for all $x, y, z$ in $X$, the following inequalities hold:

(i) $z \ast x \leq z \ast y$,

(ii) $y \ast z \leq x \ast z$.

**Definition 6** (see [15]). A dual BCK-algebra is an algebra $(X; \ast, 1)$ of type (2,0) satisfying (BE1) and (BE2) and the following axioms:

(dBCK1) $x \ast y = y \ast x = 1$ implies $x = y$,

(dBCK2) $(x \ast y) \ast ((y \ast z) \ast (x \ast z)) = 1$,

(dBCK3) $x \ast ((x \ast y) \ast y) = 1$.

**Proposition 7** (see [16]). Any dual BCK-algebra is a BE-algebra.

The converse of Proposition 7 does not hold in general [16].

**Definition 8** (see [16]). Let $X$ be a BE-algebra or dual BCK-algebra. $X$ is said to be commutative if the following identity holds:

(C) $x \lor_B y = y \lor_B x$, where $x \lor_B y = (y \ast x) \ast (y \ast x)$ (3)

for all $x, y \in X$.

**Theorem 9** (see [16]). If $X$ is a commutative BE-algebra, then $(X; *, 1)$ is a dual BCK-algebra.

**Corollary 10** ([16]). $X$ is a commutative BE-algebra if and only if it is a commutative dual BCK-algebra.

If $X$ is a commutative BE-algebra, then the relation “$\leq$” is a partial order on $X$ [16].

In the following, we abbreviate $\lor_B$ as $\lor$.

3. Bounded BE-Algebras

The following definition introduces the notion of boundedness for BE-algebras.

**Definition 11.** Let $X$ be a BE-algebra. If there exists an element 0 satisfying $0 \leq x$ (or $0 \ast x = 1$) for all $x \in X$, then the element “0” is called unit of $X$. A BE-algebra with unit is called a bounded BE-algebra.

In a bounded BE-algebra, $x \ast 0$ denoted by $xN$.

Example 12. The BE-algebra in Example 2 is a bounded BE-algebra and its unit element is 0.

Example 13. The BE-algebra in Example 4 is a bounded BE-algebra and its unit element is $d$.

Example 14. Let $X = \{1, a, b, c, d\}$ be a set with the following table:

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(4)

It is clear that $X$ is a BE-algebra, but it is not a bounded BE-algebra.

**Theorem 15.** In a bounded BE-algebra with unit 0, the following properties hold for all $x, y \in X$:

(i) $1N = 0$, $0N = 1$,

(ii) $x \leq yN$ implies $yN \leq xN$.

(iii) $x \ast yN = y \ast xN$.

(iv) $0 \lor x = xN$, $x \lor 0 = x$.

**Proof.** (i) Using (BE3) and (BE1), we have $1N = 1 \ast 0 = 0$ and $0N = 0 \ast 0 = 1$.

(ii) Since

$x \ast xN = x \ast ((x \ast 0) \ast 0)$

by (BE4) and (BE1), we get $x \leq xN$.

(iii) Using (BE4), we have

$x \ast yN = x \ast (y \ast 0)$

by (BE4) and (BE1).

(iv) By routine operations, we have $0 \lor x = (x \ast 0) \ast 0 = xN$ and $x \lor 0 = (0 \ast x) \ast x = 1 \ast x = x$. 

**Theorem 16.** In a bounded and self-distributive BE-algebra with unit 0, the following properties hold for all $x, y \in X$:

(i) $x \ast y \leq yN \ast xN$.

(ii) $x \leq y$ implies $yN \leq xN$. 


Proof. (i) Since
\[
(x \ast y) \ast (y_N \ast x_N)
= (x \ast y) \ast ((y \ast 0) \ast (x \ast 0))
= (y \ast 0) \ast ((x \ast y) \ast (x \ast 0)) \quad \text{(by BE4)}
= (y \ast 0) \ast (x \ast (y \ast 0)) \quad \text{(by distributivity)}
= x \ast ((y \ast 0) \ast (y \ast 0)) \quad \text{(by BE4)}
= x \ast 1 \quad \text{(by BE1)}
= 1 \quad \text{(by BE2)}
\]
we have \(x \ast y \leq y_N \ast x_N\).

(ii) It is trivial by Proposition 5.

Proposition 17. Let \(X\) be a BE-algebra. Then \(x \ast y \leq (y \lor x) \ast y\).

Proof. Since
\[
(x \ast y) \ast ((y \lor x) \ast y) = (y \lor x) \ast ((x \ast y) \ast y)
= (y \lor x) \ast (y \lor x)
= 1,
\]
we have \(x \ast y \leq (y \lor x) \ast y\).

Proposition 18. Let \(X\) be a self-distributive BE-algebra. Then, the next properties are valid for all \(x, y \in X\):

(i) \((y \lor x) \ast y \leq x \ast y\),
(ii) \(x \ast (x \ast y) = x \ast y\).

Proof. (i) Since
\[
x \ast (y \lor x) = x \ast ((x \ast y) \ast y)
= (x \ast y) \ast (x \ast y)
= 1,
\]
we have \(x \leq y \lor x\). By Proposition 5(i), we have \((y \lor x) \ast y \leq x \ast y\).

(ii) Using Definition 3, (BE1) and (BE3), we have
\[
x \ast (x \ast y) = (x \ast x) \ast (x \ast y)
= 1 \ast (x \ast y)
= x \ast y.
\]

Corollary 19. If \(X\) is a self-distributive and commutative BE-algebra, then \((y \lor x) \ast y = x \ast y\).

Proof. It is clear by Propositions 17 and 18 and the property (dBCK1).

Corollary 20. If \(X\) is a self-distributive, commutative, and bounded BE-algebra with unit 0, then \(xNNN = xN\).

Proof. In Corollary 19, taking \(y = 0\), we have \((0 \lor x) \ast 0 = x \ast 0\). Then we get \(((x \ast 0) \ast 0) \ast 0 = x \ast 0\); that is, \(xNNN = xN\).

Definition 21. In a bounded BE-algebra, the element \(x\) such that \(xNN = x\) is called an involution.

Let \(S(X) = \{x \in X : xNN = x\}\) where \(X\) is a bounded BE-algebra. \(S(X)\) is the set of all involutions in \(X\). Moreover, since \(1NN = (1 \ast 0) \ast 0 = 0 \ast 0 = 1\) and \(0NN = (0 \ast 0) \ast 0 = 1 \ast 0 = 0\), we have \(0, 1 \in S(X)\) and so \(S(X) \neq \emptyset\).

Example 22. For the BE-algebra in Example 2, it is clear that \(S(X) = X\).

Example 23. For the BE-algebra in Example 4, it is clear that \(S(X) = \{1, b, c, d\}\).

Proposition 24. Let \(X\) be a bounded BE-algebra with unit 0 and let \(S(X)\) be the set of all involutions in \(X\). Then, for all \(x, y \in S(X)\), the following conditions hold:

(i) \(xNNN = xN\),
(ii) \(xN \ast y = yN \ast x\).

Proof. (i) The proof is obvious by the definition of \(S(X)\).

(ii) By Theorem 15(iii), we have
\[
xN \ast y = xN \ast yNN
= yN \ast xN
= yN \ast x.
\]

In a bounded BE-algebra \(X\), we denote \(x \land y = (xN \lor yN)N\) where \(x \lor y = (x \ast y) \ast x\) for all \(x, y \in X\).

Theorem 25. In a bounded and commutative BE-algebra \(X\) the following identities hold:

(i) \(xNNN = xN\),
(ii) \(xN \land yN = (x \lor y)N\),
(iii) \(xN \lor yN = (x \land y)N\),
(iv) \(xN \ast yN = y \ast x\).

Proof. (i) Using (BE3), it is obtained that
\[
xNNN = (x \ast 0) \ast 0
= (0 \ast x) \ast x \quad \text{by commutativity}
= 1 \ast x
= x.
\]
(iii) By the definition of "∧" and (i), we have that \((x \wedge y)N = (xN \vee yN)N = xN \vee yN\).

(iv) We have
\[
xN \ast yN = (x \ast 0) \ast (y \ast 0) \\
= y \ast ((x \ast 0) \ast 0) \\
= y \ast (xN) \\
= y \ast x.
\] (13)

### Corollary 26
If \(X\) is a bounded commutative BE-algebra, then \(S(X) = X\).

### Definition 27
Each of the elements \(a\) and \(b\) in a bounded BE-algebra is called the complement of the other if \(a \vee b = 1\) and \(a \wedge b = 0\).

### Theorem 28
Let \((X; \ast, 1)\) be a bounded and commutative BE-algebra. If there exists a complement of any element of \(X\), then it is unique.

**Proof.** Let \(x \in X\) and \(a, b\) be two complements of \(x\). Then we know that \(x \wedge a = x \wedge b = 0\) and \(x \vee a = x \vee b = 1\). Also since \(x \vee a = (x \ast a) \ast a = 1\) and \(a \ast (x \ast a) = x \ast (a \ast a) = x \ast 1 = 1\), we have \(x \ast a \leq a\) and \(a \leq x \ast a\). So we get \(x \ast a = a\). Similarly \(x \ast b = b\). Hence
\[
a \ast b = (x \ast a) \ast (x \ast b) \\
= (aN \ast xN) \ast (bN \ast xN), \quad \text{by Theorem 25(iv)} \\
= bN \ast ((aN \ast xN) \ast xN), \quad \text{by BE-4} \\
= bN \ast (xN \vee aN) \\
= bN \ast (x \vee a)N, \quad \text{by Theorem 25(iii)} \\
= (x \vee a) \ast b, \quad \text{by Theorem 25(iii)} \\
= 0 \ast b \\
= 1.
\] (14)

With similar operations, we have \(b \ast a = 1\). Hence we obtain \(a = b\) which gives that the complement of \(x\) is unique. \(\square\)

Note that for a bounded and commutative BE-algebra, an element does not need to have any complement.

### Example 29
In Example 2, the complement of \(b\) is \(c\), but \(a\) has no complement in \(X\).

### Theorem 30
Let \((X; \ast, 1)\) be a commutative and bounded BE-algebra. Then the following conditions are equivalent, for all \(x, y \in X\):

(i) \(x \wedge xN = 0\),
(ii) \(xN \vee x = 1\),
(iii) \(xN \ast x = x\),
(iv) \(x \ast xN = xN\),
(v) \(x \ast (x \ast y) = x \ast y\).

**Proof.** (i)\(\Rightarrow\)(ii) Let \(x \wedge xN = 0\). Then it follows that
\[
xN \vee x = (xN \vee x)N, \quad \text{by Theorem 25(i)} \\
= (xN \wedge xN)N, \quad \text{by Theorem 25(ii)} \\
= (x \wedge xN), \quad \text{by Theorem 25(i)}
\] (15)
\[= 0N = 1.\]

(ii)\(\Rightarrow\)(iii) Let \(xN \vee x = 1\). Then, since \((x \ast x) \ast x = x \vee xN = 1\) and \(x \ast (xN \ast x) = xN \ast (x \ast x) = xN \ast 1 = 1\), we get \(xN \ast x = x\) by (dBCK1).

(iii)\(\Rightarrow\)(iv) Let \(xN \ast x = x\). Substituting \(xN\) for \(x\) and using Theorem 25(iv), we obtain the result.

(iv)\(\Rightarrow\)(v) Let \(x \ast xN = xN\). Then we get \(yN \ast (x \ast xN) = yN \ast xN\). Hence we have \(x \ast (yN \ast xN) = yN \ast xN\). Using Theorem 25(iv), we obtain \(x \ast (x \ast y) = x \ast y\).

(v)\(\Rightarrow\)(ii) Let \(x \ast (x \ast y) = x \ast y\). Then we have
\[
xN \vee x = (x \ast (xN)) \ast xN \\
= (x \ast (x \ast 0)) \ast xN \\
= (x \ast 0) \ast (x \ast 0) \\
= 1.
\] (16)

(ii)\(\Rightarrow\)(i) Let \(xN \vee x = 1\). Then we obtain
\[
xN \wedge x = xN \wedge xN \\
= (x \vee xN)N, \quad \text{by Theorem 25(ii)} \\
= 1N \\
= 0.
\] (17)

Note that, if \(X\) is a self-distributive BE-algebra, then \(x \ast (x \ast y) = x \ast y\) by Proposition 18. In this case, Theorem 30(v) is true. If \(X\) is also commutative and bounded, then \(xN\) is the complement of \(x\) by Theorem 30(i) and (ii).

Now we obtain a bounded BE-algebra from a non-bounded BE-algebra as the following theorem.

### Theorem 31
Let \((X; \ast, 1)\) be a BE-algebra and \(0 \notin X\). Define the operation \(@\) on \(\overline{X} = X \cup \{0\}\) as follows:
\[
x \ast y = \begin{cases} 
  x \ast y & \text{if } x, y \in X, \\
  1 & \text{if } x = 0, y \in X, \\
  0 & \text{if } x \in X, y = 0, \\
  1 & \text{if } x = y = 0.
\end{cases}
\] (18)

Then \((\overline{X}; @, 1)\) is a bounded BE-algebra with unit 0.
Proof. It is clear that BE1, BE2, and BE3 are satisfied. It suffices to verify BE4. Note that
if \( x = 0 \) and \( y, z \in X \), then \( 0 \ast (y \ast z) = 1 \) and \( y \ast (0 \ast z) = y \ast 1 = 1 \);
if \( y = 0 \) and \( x, z \in X \), then \( x \ast (0 \ast z) = x \ast 1 = 1 \) and \( 0 \ast (x \ast z) = 1 \);
if \( z = 0 \) and \( x, y \in X \), then \( x \ast (y \ast 0) = x \ast 0 = 0 \) and \( y \ast (x \ast 0) = y \ast 0 = 0 \).

For the remain situations, it is clearly seen that BE4 is satisfied with similar arguments.

We call the BE-algebra \((\overline{X}; \ast, 1)\) in the previous theorem the extension of \((X; 1, \ast)\), if for \( 1 \neq x \in X \), we have \( x \notin S(X) \) since \( x \ast 1 = 1 \). Hence \( S(X) = \{0, 1\} \).

Theorem 32. The extension of a self-distributive BE-algebra \((X; \ast, 1)\) is also self-distributive.

Proof. Let \( 0 \notin X \) and denote \( \overline{X} = X \cup \{0\} \). Let \( \ast \) be defined as in Theorem 31. We want to see that for \( x \ast (y \ast z) = (x \ast y) \ast (x \ast z) \) and \( x \ast (0 \ast z) = (x \ast 0) \ast (x \ast z) \) for all \( x, y, z \in \overline{X} \). For all \( x, y \in X \), we know \( x \ast y = x \ast y, 0 \ast y = 1, x \ast 0 = 0 \), and \( 0 \ast 0 = 1 \). So we obtain the following, for all \( x, y, z \in X \):

\[
\begin{align*}
0 \ast (0 \ast 0) &= (0 \ast 0) \ast (0 \ast 0), \\
0 \ast (y \ast 0) &= (0 \ast y) \ast (0 \ast 0), \\
x \ast (0 \ast 0) &= (x \ast 0) \ast (x \ast 0), \\
0 \ast (0 \ast z) &= (0 \ast 0) \ast (0 \ast z), \\
0 \ast (y \ast z) &= (0 \ast y) \ast (0 \ast z), \\
x \ast (0 \ast z) &= (x \ast 0) \ast (x \ast z), \\
x \ast (y \ast 0) &= (x \ast y) \ast (x \ast 0), \\
x \ast (y \ast z) &= (x \ast y) \ast (x \ast z).
\end{align*}
\]

Equation (19)

The above results show that \((\overline{X}; \ast, 1)\) is self-distributive.

Theorem 33. If \((X; \ast, 1)\) is a commutative, self-distributive, and bounded BE-algebra, then it is a lattice with respect to \( \land \) and \( \lor \) where \( x \lor y = y \ast (y \ast x) \) and \( x \land y = (x \lor y) \land N \) for any \( x, y \in X \).

Proof. From Theorem 3.6 in [14], we know that \((X; \ast, 1)\) is a semilattice with respect to \( \lor \). Then we need to show that the set \{\( x, y \)\} for all \( x, y \in X \) has a greatest lower bound. We know that \( x \lor y \leq x \lor y \lor y \lor y \). By Proposition 5, we have \( (x \lor y) \lor y \leq x \lor (y \lor y) \). Since \( x \lor y \leq x \lor y \lor y \), we have \( x \lor y \leq x \lor y \land y \). We must show that \( x \land y \leq y \land y \). Let \( x \land y \leq x \land y \). We have \( x \lor y \leq z \lor y \lor y \). Since \( x \lor y \) is an upper semilattice with respect to \( \lor \), then we obtain \( x \lor (y \lor y) \leq z \lor y \). Hence \( z \lor y \leq z \lor y \). So we can say that \( X \) is a lower semilattice. Consequently \((X; \land)\) is a lattice with respect to \( \lor \) and \( \land \).

References
