Research Article

Norm of a Volterra Integral Operator on Some Analytic Function Spaces

Hao Li \(^1\) and Songxiao Li \(^2\)

\(^1\) College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
\(^2\) Department of Mathematics, Jiaying University, Meizhou, Guangdong 514015, China

Correspondence should be addressed to Songxiao Li; jyulsx@163.com

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Let \(f\) be an analytic function in the unit disc \(D\). The Volterra integral operator \(I_f\) is defined as follows:

\[
I_f(h)(z) = \int_0^z f(w)h'(w)dw, h \in H(D), z \in D.
\]

In this paper, we compute the norm of \(I_f\) on some analytic function spaces.

1. Introduction

Let \(\mathbb{D} = \{z : |z| < 1\}\) be the unit disk of complex plane \(\mathbb{C}\) and \(H(\mathbb{D})\) the class of functions analytic in \(\mathbb{D}\). Denote by \(\sigma\) the normalized Lebesgue area measure in \(\mathbb{D}\) and \(g(a, z)\) the Green function with logarithmic singularity at \(a\); that is, \(g(a, z) = -\log |\varphi_a(z)|\), where \(\varphi_a(z) = (a-\bar{z})/(1-\bar{a}z)\) is the M"obius transformation of \(\mathbb{D}\).

Let \(0 < p < \infty\). The \(Q_p\) is the space of all functions \(f \in H(\mathbb{D})\) such that

\[
\|f\|_{Q_p}^2 = |f(0)|^2 + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\varphi_a(z)|^2\right)^p d\sigma(z) < \infty.
\]

From [1, 2], we see that \(Q_1 = BMO\), the space of all analytic functions of bounded mean oscillation. When \(p > 1\), the space \(Q_p\) is the same and equal to the Bloch space \(\mathfrak{B}\), which consists of all \(f \in H(\mathbb{D})\) for which

\[
\|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| \left(1 - |z|^2\right) < \infty.
\]

For \(\alpha > 0\), the \(\alpha\)-Bloch space, denoted by \(\mathfrak{B}^\alpha\), is the space of all \(f \in H(\mathbb{D})\) such that

\[
\|f\|_{\mathfrak{B}^\alpha} = \|f(0)\| + \sup_{z \in \mathbb{D}} |f'(z)| \left(1 - |z|^2\right)^\alpha < \infty.
\]

It is clear that \(\mathfrak{B}^0 \subseteq \mathfrak{B}^\alpha \subseteq \mathfrak{B}^{\alpha_2}\) for \(0 < \alpha_1 < \alpha_2 < \infty\).

Let \(1 \leq q \leq \infty\) and let \(0 \leq \alpha \leq 1\). The mean Lipschitz space \(\Lambda(q, \alpha)\) consists of those functions \(f \in H(\mathbb{D})\) for which

\[
\|f\|_{\Lambda(q, \alpha)} = |f(0)| + \sup_{0 < r < 1} \left(1 - r^2\right)^{1-\alpha} \times \left(\frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\varphi})|^q d\varphi\right)^{1/q} < \infty.
\]

It is obvious that \(\Lambda(\infty, 0)\) is just the Bloch space \(\mathfrak{B}\), which is contained in \(\Lambda(q, 0)\) for every \(1 < q < \infty\). Note that \(\Lambda(q, 1/q)\) increases with \(q \in (1, \infty)\). We refer to [5] for more information of mean Lipschitz spaces.

For \(0 \leq s < \infty\), we say that an \(f \in H(\mathbb{D})\) belongs to the growth space \(H^s_0\) if

\[
\|f\|_{H^s_0} = \sup_{0 < r < 1} |f(re^{i\varphi})| \left(1 - r^2\right)^s < \infty.
\]

It is easy to see that \(H^0_0 = H^\infty\).
For $-1 < \alpha < \infty$, an $f \in H(D)$ is said to belong to the $\mathcal{B}^\alpha$ space if
\[
\|f\|_{\mathcal{B}^\alpha}^p = |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^\alpha \, d\sigma(z) < \infty. \tag{6}
\]

For $1 < p < \infty$, the Besov space $\mathcal{B}_p$ is defined to be the space of all analytic functions $f$ in $D$ such that
\[
\|f\|_{\mathcal{B}_p}^p = |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^{p-2} \, d\sigma(z) < \infty. \tag{7}
\]

Let $f \in H(D)$. The Volterra integral operators $I_f$ and $J_f$ are defined as follows:
\[
I_f(h)(z) = \int_0^z h'(w) f(w) \, dw, \tag{8}
\]
\[
J_f(h)(z) = \int_0^z h(w) f'(w) \, dw, \tag{9}
\]
where $M_f$ denotes the multiplication operator; that is, $M_f(h) = fh$. If $f$ is a constant, then all results about $I_f$, $J_f$, or $M_f$ are trivial. In this paper, we assume that $f$ is a nonconstant. Both operators have been studied by many authors. See [6–23] and the references therein.

Norms of some special operators, such as composition operator, weighted composition operator, and some integral operators, have been studied by many authors. The interested readers can refer [13, 24–32], for example. Recently, Liu and Xiong studied the norm of integral operators in some function spaces in the unit disk. In this paper, we study the norm of integral operator $I_f$ on some function spaces in the unit disk.

2. Main Results

In this section, we state and prove our main results. In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1 (see [5, page 14]). If $f \in H^p(0 < p \leq \infty)$, then $|f(z)| \leq (1 - |z|^2)^{-1/p} \|f\|_p, |z| < 1$, and the inequality is sharp for each fixed $z$.

Lemma 2. Let $-1 < \alpha < \infty$ and $0 < p < \infty$. For any $f \in H(D)$, the following one has:
\[
|f(\alpha)|^p \leq (\alpha + 1) \int_D |f(z)|^p \left(1 - |z|^2\right)^{2\alpha} \left(1 - |z|^2\right)^\alpha \, d\sigma(z), \tag{10}
\]
where $a$ is any point in $D$.

Proof. For any $f \in H(D)$, taking $z = re^{i\theta}$ and the subharmonicity of $|f(z)|^p$, we get
\[
|f(0)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, \tag{11}
\]
and so
\[
|f(0)|^p \leq (\alpha + 1) \frac{1}{\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{1}{1 - r^2} \, d\theta \, dr \tag{12}
\]
For any $a \in D$, let $f_a(z) = (a - z)/(1 - \overline{a}z)$. Replacing $f$ by $f \circ f_a(z)$ and applying the change of variable formula give the following:
\[
|f(a)|^p \leq (\alpha + 1) \int_D |f \circ f_a(z)|^p \left(1 - |z|^2\right)^\alpha \, d\sigma(z),
\]
\[
= (\alpha + 1) \int_D |f(z)|^p \left(1 - |z|^2\right)^\alpha \left(1 - |a|^2\right)^\alpha \left(1 - |\overline{a}z|^2\right)^\alpha \, d\sigma(z).
\]
The proof is complete. \hfill \Box

Theorem 3. Let $f \in H(D)$. The integral operator $I_f$ is bounded on $A(1, 1)$ if and only if $f \in H^\infty$. Moreover, one has
\[
\|I_f\| = \|f\|_{H^\infty}. \tag{14}
\]

Proof. If $f \in H^\infty$, by (4), we have
\[
\|f \circ h\|_{A(1, 1)} \leq \sup_{0 < \nu < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})^\nu h'(re^{i\theta})| \, d\theta\right) \|h\|_{A(1, 1)} = \|f\|_{H^\infty} \sup_{0 < \nu < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})| \, d\theta\right).
\]
Thus $\|I_f\| \leq \|f\|_{H^\infty}$.

On the other hand, denote $c = \sup_{z \in D} |f(z)|$. Given any $\epsilon > 0$, there exists $z_1 \in D$ such that $|f(z_1)| > c - \epsilon$. Let
\[
h(z) = \frac{z_1 - z}{1 - \overline{z}_1z} - z_1. \tag{16}
\]
Then we have \( \|h\|_{\Lambda(1,1)} = 1 \). In fact, taking \( z_1 = r_1 e^{i\psi_1} \) and \( z = r e^{i\psi} \) and using Poisson integral, we get

\[
\|h\|_{\Lambda(1,1)} = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |h'(r e^{i\psi})| \, d\psi \right) = 1.
\]

Taking \( z_1 = r_1 e^{i\psi_1} \), we obtain

\[
\sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f'(r e^{i\psi})| \, d\psi \right) = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(r e^{i\psi})| \right) = 1.
\]

Now we need only to show the reverse inequality. Denote \( c = \sup_{z \in \mathbb{D}} |f(z)| \). Given any \( \epsilon > 0 \), there exists \( z_1 \in \mathbb{D} \) such that \( |f(z_1)| > c - \epsilon \). Let

\[
h_1(z) = \int_{\Gamma(z)} \frac{(1 - |z|^2)\Gamma(\alpha/2)}{(1 - z_1 z)^{3\alpha}} \, d\zeta,
\]

where \( \Gamma(z) \) is any path in \( \mathbb{D} \) from 0 to \( z \). By Theorem 3.11 in [33, page 274], we know \( h_1 \) is an analytic function in \( \mathbb{D} \) and \( h_1^2(z) = (1 - |z|^2)^{-\alpha/2} / (1 - z_1 z)^{2\alpha} \). Also it is easy to check that \( \|h_1\|_{L^\alpha} = 1/(\alpha + 1) \). Indeed, by using the method of the proof of Lemma 4.2.2 in [4], we have

\[
\|h_1\|^2_{L^\alpha} = \int_{\mathbb{D}} \frac{(1 - |z|^2)^{2\alpha}}{(1 - z_1 z)^{3\alpha}} \, d\sigma(z)
\]

and the proof is complete.
Proof. If \( f \in H^\infty \), then by (7), we have
\[
\|I_fh\|_{B^p} = \int_D |f(z)h'(z)|^p (1-|z|^2)^{-2} d\sigma(z)
\]
< \|f\|_{H^\infty} \int_D |h'(z)|^p (1-|z|^2)^{-2} d\sigma(z) \tag{28}
\]
and so \( \|I_f\| \leq \|f\|_{H^\infty} \).

Now we need only to show the reverse inequality. Denote \( c = \sup_{z \in D} |f(z)| \). Given any \( \varepsilon > 0 \), there exists \( z_1 \in D \) such that \( |f(z_1)| > c - \varepsilon \). Let
\[
h_1(z) = \frac{z_1 - z}{1 - z_1 z}, \quad z \in D. \tag{29}
\]
We see that \( \|h_1\|_{B^p} = 1/(p-1) \). Indeed,
\[
\|h_1\|_{B^p} = \int_D \frac{|z_1|^p (1-|z|^2)^{-2}}{|1 - z_1 z|^{2p}} d\sigma(z)
\]
\[
= \frac{(1-|z_1|^2)^p}{|1 - z_1 z|^{2p}} \int_D \frac{(1-|z|^2)^{-2}}{p} d\sigma(z)
\]
\[
= \frac{1}{p} \left( \frac{1-|z_1|^2}{1-|z|^2} \right)^p = \frac{1}{p-1}.
\]
Let \( h(z) = h_1(z)/\|h_1\|_{B^p} \). Then \( \|h\|_{B^p} = 1 \). Thus by Lemma 2, we have
\[
\|I_f\| \geq \|I_fh(z)\|_{B^p} = \int_D |f(z)h'(z)|^p (1-|z|^2)^{-2} d\sigma(z)
\]
\[
= (p-1) \int_D |f(z)|^p \frac{(1-|z|^2)^p}{|1 - z_1 z|^{2p}}
\]
\[
\times (1-|z|^2)^{-2} d\sigma(z)
\]
\[
\geq |f(z_1)|^p > (c - \varepsilon)^p. \tag{31}
\]
Since \( \varepsilon \) is arbitrary, we obtain the desired result. The proof is complete. \( \Box \)

Theorem 6. Let \( f \in H(D) \) and let \( 0 < \alpha \leq \beta < \infty \). The integral operator \( I_f \) is bounded from \( \mathfrak{B}^\alpha \) to \( \mathfrak{B}^\beta \) if and only if \( f \in H^{\infty}_{\beta-\alpha} \). Moreover, one has
\[
\|I_f\| = \sup_{z \in D} |f(z)| (1-|z|^2)^{\beta-\alpha}. \tag{32}
\]

Proof. If \( f \in H^{\infty}_{\beta-\alpha} \), then by (3), we have
\[
\|I_fh\|_{B^p} = \sup_{z \in D} |f(z)h'(z)| (1-|z|^2)^{\beta-\alpha}
\]
\[
\leq \|f\|_{H^{\infty}_{\beta-\alpha}} \sup_{z \in D} |h'(z)| (1-|z|^2)^{\alpha} \tag{33}
\]
\[
\leq \|f\|_{H^{\infty}_{\beta-\alpha}} \|h\|_{B^\alpha}.
\]
Hence \( \|I_f\| \leq \|f\|_{H^{\infty}_{\beta-\alpha}} \).

For the converse, denote \( c = \sup_{z \in D} |f(z)| (1-|z|^2)^{\beta-\alpha} \). Given any \( \varepsilon > 0 \), there exists \( z_1 \in D \) such that \( |f(z_1)| > c - \varepsilon \). Let
\[
h(z) = \frac{z_1 - z}{1 - z_1 z}, \quad z \in D. \tag{38}
\]
Then by the proof of Theorem 3, we see that \( \|h\|_{\Lambda(1,1)} = 1 \).
In the meantime, we know that \( |h'(z_1)|(1 - |z_1|^2) = 1 \), which gives
\[
\|I_f\| \geq \|I_fh\|_{\Lambda(1,1)} = \sup_{z \in \mathbb{D}} \left| f(z) h'(z) \right|(1 - |z|^2)
\geq \left| f(z_1) h'(z_1) \right|(1 - |z|^2)
= \left| f(z_1) \right| > c - \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, we get the desired result. The proof is complete.

Finally, we consider the norm of \( I_f \) from \( \Lambda(\infty,1) \) to some Banach spaces.

**Theorem 8.** If \( f \in H(\mathbb{D}) \), then the following assertions hold.

1. Let \( 0 < p < \infty \). The integral operator \( I_f \) is bounded from \( \Lambda(\infty,1) \) space to \( Q_p \) space if and only if \( f \) satisfies
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1 - |\varphi_a(z)|^2)^2 d\sigma(z) < \infty. \tag{40}
\]
Moreover, one has
\[
\|I_f\| = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1 - |\varphi_a(z)|^2)^2 d\sigma(z). \tag{41}
\]

2. Let \( 0 \leq \alpha \leq 1 \) and let \( 0 \leq q \leq \infty \). The integral operator \( I_f \) is bounded from \( \Lambda(\infty,1) \) space to \( \mathcal{B}^q \) space if and only if \( f \) satisfies
\[
\sup_{0 < r < 1} (1 - r^2)^{-1 - \alpha} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^q d\varphi \right)^{1/q} < \infty. \tag{42}
\]
Moreover, one has
\[
\|I_f\| = \sup_{0 < r < 1} (1 - r^2)^{-1 - \alpha} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^q d\varphi \right)^{1/q}. \tag{43}
\]

3. Let \( 0 < \alpha < \infty \). The integral operator \( I_f \) is bounded from \( \Lambda(\infty,1) \) space to \( \Omega^p \) space if and only if \( f \) satisfies
\[
\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{p\alpha} d\sigma(z) < \infty. \tag{44}
\]
Moreover, one has
\[
\|I_f\| = \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{p\alpha} d\sigma(z). \tag{45}
\]

4. Let \( -1 < \alpha < \infty \). The integral operator \( I_f \) is bounded from \( \Lambda(\infty,1) \) space to \( \Psi^p \) space if and only if \( f \) satisfies
\[
\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{p\alpha} d\sigma(z) < \infty. \tag{46}
\]
Moreover, one has
\[
\|I_f\| = \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{p\alpha} d\sigma(z). \tag{47}
\]

**Proof.** The assertion (1) will be proved only here, and the conclusions of (2), (3), and (4) follow by using the similar arguments to that used in proving (1), and so the proofs are omitted.

If \( h \in \Lambda(\infty,1) \), then by (1), we have
\[
\|I_fh\|^2_{Q_p} = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z) h'(z)|^2 (1 - |\varphi_a(z)|^2)^2 d\sigma(z)
\leq \|h\|^2_{\Lambda(\infty,1)} \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^2 d\sigma(z), \tag{48}
\]
and so
\[
\|I_f\| \leq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^2 d\sigma(z). \tag{49}
\]
For the converse, let \( h(z) = z \). It is easy to see that \( \|h\|_{\Lambda(\infty,1)} = 1 \). Thus
\[
\|I_f\| = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\varphi_a(z)|^2)^2 d\sigma(z),
\]
The desired result follows by (47) and (48). The proof is complete.

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**References**


