Research Article

On the Continuity of Hausdorff Dimension of Julia Sets Concerning Potts Models

Gang Liu and Junyang Gao

1 Department of Mathematics and Computational Science, Hengyang Normal University, Hengyang 421002, China
2 School of Science, China University of Mining and Technology (Beijing), Beijing 100083, China

Correspondence should be addressed to Gang Liu; liugangmath@sina.cn

Received 26 February 2013; Accepted 28 March 2013

Academic Editors: G. L. Karakostas, D.-X. Zhou, and C. Zhu

Copyright © 2013 G. Liu and J. Gao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Considering the Julia sets of a family of rational maps concerning two-dimensional diamond hierarchical Potts models in statistical mechanics, we show the continuity of their Hausdorff dimension.

1. Introduction

The continuity of Hausdorff dimension of Julia sets is an important and interesting problem for rational maps with degree \(d \geq 2\). In general, this problem adheres to the continuity of Julia sets which is response to the stability of system. It is well known that both the Julia set \(J(R)\) and its Hausdorff dimension of a rational map \(R\) vary continuously in the parameter space \(R^n\) if \(R\) is hyperbolic [1,2]. However, as we know, there are no direct relationship between them when \(R\) is not hyperbolic though there are many works devoted to the two problems [1, 3, 4].

In this paper, we discuss a family of rational maps \(T_{n\lambda}: \mathbb{C} \to \mathbb{C}\) for \(n = 3\); here

\[
T_{n\lambda}(z) = \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^n
\]

with two parameters \(n \in \mathbb{N}\) and \(\lambda \in \mathbb{R}\). \(T_{n\lambda}\) is a renormalization transformation of \(\lambda\)-state Potts models on the two-dimensional diamond-like hierarchical lattice with bifurcation number \(n\) in statistical mechanics [5]. In turn, the zeros of the partition function for the model with bifurcation number \(n\) condense to the Julia sets of \(T_{n\lambda}\) [6]. It has been shown that there exists some relationship between the critical temperatures, the critical amplitudes, and the structures of the Julia sets [7]. Therefore, much interest has been devoted to these physical models, since they exhibit a connection between statistical mechanics and complex dynamics [6, 8–15].

We have known that, for any given \(n \in \mathbb{N}\), the Julia set \(J(T_{n\lambda})\) of \(T_{n\lambda}\) is continuous in the Hausdorff distance for any \(\lambda \in \mathbb{R}\) except two points [11]. Whether the Hausdorff dimension of \(J(T_{n\lambda})\) is also continuous for any \(\lambda \in \mathbb{R}\) except two points? From the proof of the main result in [10, 11], for even integer \(n\), it is easy to see that \(T_{n\lambda}\) is hyperbolic in the real axis \(\mathbb{R}\) except countable points. Except at most three points from those countable points, \(T_{n\lambda}\) is subhyperbolic but not hyperbolic; though the dynamical property of \(T_{n\lambda}\) is simple, it is difficult to compute all the iteration number of critical points which are eventually equal to the repelling fixed points in the iteration of \(T_{n\lambda}\). Therefore, we cannot give a quantitative analysis for the corresponding critical points when the parameter is close to the above points. For any odd integer \(n \geq 5\), there exist at least two real numbers \(\lambda_1, \lambda_2 \in (1, 2)\) such that \(T_{n\lambda_1}\) and \(T_{n\lambda_2}\) are Feigenbaum-like maps [15]. As we have seen, for the simplest Feigenbaum quadratic polynomials, the continuity of Hausdorff dimension of its Julia sets is unknown. Based on the above reason, we just consider the case for \(n = 3\).

We define the following constants:

\[
\alpha = 2 + \min_{0 \leq t \leq 1} \frac{t^6 - 2t^4 + 1}{t - 1},
\]

\[
\beta = 2 + \max_{-2 \leq t \leq 0} \frac{t^6 - 2t^4 + 1}{t - 1}.
\]

We have the following result.
Theorem 1. $T_{3\lambda}$ is defined in (1) and $\lambda \in \mathbb{R}$. Let $HD(J(T_{3\lambda}))$ be the Hausdorff dimension of $J(T_{3\lambda})$. Then $HD(J(T_{3\lambda}))$ is continuous at $\lambda \in \mathbb{R} \setminus \{\alpha, 0, \beta\}$.

2. Some Notations and Preliminary Results

Let $R : \mathbb{C} \to \mathbb{C}$ be a rational map with degree $\deg(R) \geq 2$. We denote by $R^n$ the $n$th iteration of $R$. A point $z$ is called critical if $R'(z) = 0$. A point $z$ is called periodic point if $R^n(z) = z$ for some $k \geq 1$; the minimal of such $k$ is called the period of $z$. For a periodic point $z_0$, denote the multiplier of $z_0$ by $(R^k)'(z_0)$; the periodic point $z_0$ is either attracting, indifferent, or repelling according to $|(R^k)'(z_0)| < 1$, $|(R^k)'(z_0)| = 1$ or $|(R^k)'(z_0)| > 1$. In the indifferent case, we say $z_0$ is parabolic if $(R^k)'(z_0)$ is a root of unity.

The Julia set, denoted by $J(R)$, is the closure of repelling periodic points. Its complement is called Fatou set, denoted by $F(R)$; a connected component of $F(R)$ is called a Fatou component. A rational map $R$ is called hyperbolic if $P(R) \cap J(R) = \emptyset$, and geometrically finite, if the set $P(R) \cap J(R)$ is finite; here the postcritical set $P(R)$ is the closure of the forward orbits of critical points. A geometrically finite map is subhyperbolic (resp. parabolic) if it has no (resp. some) parabolic periodic points. It is called critically non-recurrent with $c$ not $\omega(c)$ for each critical point $c \in J(R)$, where $\omega(c)$ is the omega limit set of $c$. A critically non-recurrent map is semihyperbolic if it has no parabolic periodic points. For the classical results in complex dynamics, see [12, 16, 17].

Definition 2. A domain $D \subset \mathbb{C}$ is called a John domain if there exists a $c > 0$ such that, for any $z_0 \in D$, there is an arc $\gamma$ joining $z_0$ to some fixed reference point $w_0 \in D$ satisfying

$$\text{dist}(z, \partial D) \geq c \|z - z_0\|, \quad z \in \gamma.$$  

If $\infty \in \partial D$, we use the spherical metric to measure the distance.

Lemma 3 (see [18]). Suppose $R$ is semihyperbolic rational map, then every Fatou component of $F(R)$ is a John domain.

Definition 4. A probability measure $\mu$ on the Julia set $J(R)$ is called $t$-conformal measure for a rational map $R : \mathbb{C} \to \mathbb{C}$ if $\mu(R(A)) = \int_A |R'|^t \mu(dA)$ for every Borel set $A \subset J(R)$ such that $R|_A$ is injective; $t$ is called the conformal exponent about $\mu$.  

Lemma 5 (see [19]). Let $h$ denote the Hausdorff dimension of $J(R)$ of a subhyperbolic rational map $R$, then there exists a unique invariant probability measure $\mu$ equivalent to the $h$-conformal measure; moreover, the normalized $h$-dimension Hausdorff measure is the only $h$-dimension conformal measure for $R$.

Lemma 6 (see [1]). Any normalized invariant conformal probability measure $\mu$ supported on the Julia set of a geometrically finite rational map $R$ is either the conformal measure of Hausdorff dimension of $J(R)$, or an atomic measure supported on the inverse orbits of parabolic points and critical points.

For simplicity, $T_{\lambda} = T_{3\lambda}$, and $A \sim B$ $(A, B \in \mathbb{R})$ means that $C^{-1}B < A < CB$ for some implicit constant $C$. By (1), for $\lambda \neq 0$, we have

$$T'_{\lambda}(z) = \frac{6(z - 1)(z + \lambda - 1)(z^2 + \lambda - 1)^2}{(2z + \lambda - 2)^4}.$$  

So, $T_{\lambda}$ has ten critical points: $1, 1 - \lambda, \pm \sqrt{\lambda - 1}i$ (with the multiplicity 2), $(1 - \lambda)/2$ (with the multiplicity 3), $\infty$. It is easy to see that $z = 1$ and $\infty$ are two superattracting fixed points.

Lemma 7 (see [6]). $\alpha \in (-2, 0), \beta \in (2, 3)$, and

1. $T_{\lambda}$ has only two real fixed points $q, 1 (q < -1)$ for $\lambda \in (-\infty, \alpha)$;
2. $T_{\lambda}$ has only two real fixed points $q, 1 (q > 1)$ for $\lambda \in (\beta, +\infty)$;
3. $T_{\lambda}$ has only three real fixed points $q_1, q_2, 1 (q_1 < -1, 0 < q_2 < 1)$ for $\lambda = \alpha$ or $\lambda = 0$;
4. $T_{\lambda}$ has only three real fixed points $q_1, 1, q_2 (q_1 < -1, q_2 > 1)$ for $\lambda = \beta$;
5. $T_{\lambda}$ has only four real fixed points $q_1, 0, 1, q_2 (q_1 < -1, q_2 > 1)$ for $\lambda = 1$;
6. $T_{\lambda}$ has only four real fixed points $q_1, 1, q_2, 1, q_3 (q_1 < 0, q_3 > 1)$ for $\lambda \in (1, \beta)$;
7. $T_{\lambda}$ has only four real fixed points $q_1, q_2, 1, q_3 (q_1 < -1, 0 < q_2 < 1, q_3 > 1)$ for $\lambda \in (0, 1)$;
8. $T_{\lambda}$ has only four real fixed points $q_1, 0, 1, q_3 (q_1 < -1, q_2, q_3 \in (0, 1))$ for $\lambda \in (\alpha, 0)$.

Lemma 8 (see [10]). $T_{\lambda}$ is hyperbolic for $\lambda \in \mathbb{R} \setminus [\alpha, \beta, 3 \pm \sqrt{2}]$, $T_{3\lambda \sqrt{2}}$ is subhyperbolic, and $T_{\alpha}$ and $T_{\beta}$ are parabolic.

3. The Proof of Theorem 1

In the following, we denote $T_{\lambda}^3(\pm \sqrt{\lambda - 1}i) = T_{\lambda}(0) = 0, \lambda_1 = q_1$ is the repelling fixed point for $\lambda$ close but not equal to $3 - \sqrt{2}$, and $\lambda_2 = q$ is also the repelling fixed point for $\lambda$ close but not equal to $3 + \sqrt{2}$. It is easy to see that $\lambda_2 - q_1 \to \lambda_1 - q_1 = 0$ when $\lambda \to \lambda_0, \lambda_0 \in [3 - \sqrt{2}, 3 + \sqrt{2}]$.

Proposition 9. Consider

$$q_1 = \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right)^3 + k(\lambda - \lambda_0) + O((\lambda - \lambda_0)^2)$$  

as $\lambda \to \lambda_0$; here $k = (78 + 36\sqrt{2})/97$ for $\lambda_0 = 3 - \sqrt{2}$ and $k = (78 - 36\sqrt{2})/97$ for $\lambda_0 = 3 + \sqrt{2}$.

Proof. Considering the real fixed points of $T_{\lambda}$ and taking $t = \sqrt{\lambda}$, from the equation $T'_{\lambda}(x) = x$, it follows that

$$\lambda = 2 + \frac{t^6 - 2t^4 + 1}{t - 1}. $$  


When $\lambda$ is close but not equal to $\lambda_0$, denote that
\[ q_\lambda = \left( \frac{\lambda_0 - 1}{\lambda_0 - 2} \right)^3 + k(\lambda - \lambda_0) + O\left( (\lambda - \lambda_0)^2 \right). \] (7)

(1) If $\lambda_0 = 3 - \sqrt{2}$, $q_{\lambda_0} = -2\sqrt{2}$. By the continuity, $q_\lambda < 0$. By (6) and $\lambda \in \mathbb{R}$, it satisfies
\[ (\lambda - \lambda_0 + \lambda_0 - 2)(\sqrt{q_\lambda} - 1) = q_\lambda^2 - 2q_\lambda \sqrt{q_\lambda} + 1. \] (8)
Substituting (8) with (7), by a calculation, we can deduce that
\[ (\lambda - \lambda_0 + 1 - \sqrt{2}) \left( -\sqrt{2} - 1 + \frac{k(\lambda - \lambda_0)}{6} \right) + O \left( (\lambda - \lambda_0)^2 \right) = \frac{2\sqrt{2} + k(\lambda - \lambda_0)^2}{2} + 2 - \sqrt{2} + k(\lambda - \lambda_0) \times \left( -\frac{\sqrt{2}}{6} + \frac{k(\lambda - \lambda_0)}{6} \right) + 1 + O \left( (\lambda - \lambda_0)^2 \right), \] (9)
then $k = (78 + 36\sqrt{2})/97$.

(2) If $\lambda_0 = 3 + \sqrt{2}$, $q_{\lambda_0} = 2\sqrt{2}$. By the similar method used in Case (1), we can deduce that $k = (78 - 36\sqrt{2})/97$.

\[ \Box \]

**Proposition 10.** $HD(J(T_3^i))$ is continuous for $\lambda \in \{ 3 + \sqrt{2}, 3 - \sqrt{2} \}$.

**Proof.** By Lemma 8, $T_3^i$ is hyperbolic for $\lambda$ close but not equal to $\lambda_0$. Then there exists a unique conformal probability measure $\mu_3$ for $T_3^i$ supported in $J(T_3^i)$; $\mu_3$ has exponent $d_3 = HD(J(T_3^i))$. This means that, for every measurable set $\mathcal{V} \subset J(T_3^i)$ where $T_3^i$ is injective, $\mu_3(T_3^i(\mathcal{V})) = \int_{\mathcal{V}} |(T_3^i)'|d\mu_3$. Furthermore the measure of a point is zero for $\mu_3$, that is, $\mu_3$ is not atomic.

Since $T_3^i$ is subhyperbolic, by Lemma 5, there exists a unique conformal probability measure for $T_3^i$ supported in $J(T_3^i)$, by cases (6) and (10) in the proof of Theorem 1 of the paper [10], we know that $1 - \lambda_0 \in F(T_3^i)$ for $\lambda_0 = 3 \pm \sqrt{2}$. By Lemma 6, the unique conformal probability measure has exponent $d_3 = HD(J(T_3^i))$, or is atomic, supported in $\{ T_3^{i}\lambda_0(\pm \sqrt{\lambda_0 - 1}i) \}_{\lambda_0 \neq 20}$. By a similar discussion used in [4], in order to prove that
\[ \lim_{\lambda \to \lambda_0} HD(J(T_3^i)) = HD(J(T_3^{i}\lambda_0)), \] (10)
it is enough to prove that
\[ \lim_{r \to 0^+ - \lambda_0} \mu_3 \left( B_r \left( \pm \sqrt{\lambda_0 - 1}i \right) \right) = 0; \] (11)
here $B_r(x) = \{ z \mid |z - x| < r \}$. Noting that $J(T_3^i)$ and $F(T_3^i) \ (\lambda \in \mathbb{R})$ are symmetry with the real axis, it suffices to prove that
\[ \lim_{r \to 0^+} \mu_3 \left( B_r \left( \sqrt{\lambda_0 - 1}i \right) \right) = 0. \]
(12)
In fact, if $\mu_{\lambda_0}$ is any weak limit of $\{ \mu_3 \}$, then $\mu_{\lambda_0}$ is a conformal probability measure for $T_3^{i\lambda_0}$ supported in $J(T_3^{i\lambda_0})$. The previous limit implies that the measure $\mu_{\lambda_0}$ is not atomic at $\sqrt{\lambda_0 - 1}i$, so, it has exponent $d_{\lambda_0} = HD(J(T_3^{i\lambda_0}))$. Noting that $\mu_{\lambda_0}(T_3^{i\lambda_0}(\mathcal{V})) = \int_{\mathcal{V}} |(T_3^{i\lambda_0})'|d\mu_{\lambda_0}$ and $\mu_{\lambda_0}(T_3^{i\lambda_0}(\mathcal{V})) \to \mu_{\lambda_0}(T_3^{i\lambda_0}(\mathcal{V}))$ as $\lambda \to \lambda_0$ for any measurable set $\mathcal{V}$, it follows that $d_{\lambda_0} \to d_{\lambda_0}$. Next we set that $\lambda$ is close but not equal to $\lambda_0$.

Since $q_{\lambda_0}$ and $q_\lambda$ are the real repelling fixed points of $T_{\lambda_0}$ and $T_\lambda$, respectively, by the continuity, $q_\lambda \to q_{\lambda_0}$ as $\lambda \to \lambda_0$.

By the Koenig’s Theorem [16], there exist a neighborhood $U_0$ of $q_{\lambda_0}$ with diameter not more than a $\delta > 0$ and a conformal map $\phi_{\lambda_0} : U_0 \to B_{\delta}(0)$ for some $\delta_1 > 0$ such that $\phi_{\lambda_0}$ conjugates $T_{\lambda_0}$ on $U_0$ to the scaling function $z \to T_{\lambda_0}^i(q_{\lambda_0})z$ on $B_{\delta}(0)$. Similarly, there exists a conformal map $\phi_\lambda : U_0 \to B_{\delta}(0)$ which conjugates $T_\lambda$ to the scaling function $z \to T_\lambda^i(q_\lambda)z$. It is easy to construct a quasiconformal map $\phi : A_{\delta_1} = \{ z \mid \delta_1 < |z| < \delta \} \to A_{\delta_2} = \{ z \mid |\delta_2 < |z| < \delta_2 \}$; here $\delta_2 = \{ T_\lambda^i(q_\lambda)L_1 \}$ and $\delta = \{ T_\lambda^i(q_\lambda)L_2 \}$, such that $\phi(T_\lambda^i(q_\lambda)L_1) = T_{\lambda_0}^i(q_{\lambda_0})L_1$ for $|z| = \delta$. Pull back by the scaling function; we can extend $\phi$ to a quasiconformal map $\phi : B_{\delta}(0) \to B_{\delta}(0)$ which conjugates $z \to T_{\lambda_0}^i(q_{\lambda_0})z$ to $z \to T_{\lambda_0}^i(q_\lambda)z$. For every $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, define
\[ j_\lambda = \phi_\lambda^{-1} \circ \phi \circ \phi_{\lambda_0} : U_0 \to U_0. \]

Hence, $j_\lambda$ is a conjugation between $T_{\lambda_0}$ on $U_0$ and $T_\lambda$ on $U_0$. Let $z(\lambda) = j_\lambda(q_{\lambda_0})$, by definition, $z(\lambda_0) = q_{\lambda_0}$ and $z(\lambda_0) = q_\lambda$. Reducing $\epsilon > 0$ if necessary, there are constants $C_0 > 0$ and $\delta_0 \in (0, 1)$ such that, for all $m \geq 1$, all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, and all $q_\lambda$,
\[ \left| \left( T_\lambda^m \right)' \left( q_\lambda \right) \right|^{-1} \leq C_0 \delta_0^m. \] (14)
On the other hand, for every $k \geq 1$, let $U_k$ be the preimage of $B_\delta(q_{\lambda_0})$ under $T_{\lambda_0}^i$ containing $q_{\lambda_0}$, and let $V_k$ be the pullback of $U_k$ by $T_\lambda^i$ containing $\sqrt{\lambda_0 - 1}i$. Moreover, we denote $j_k(U_k) \to U_k$ by $V_k$ containing $q_{\lambda_0}$, and let $V_k$ be the pullback of $U_k$ by $T_\lambda^i$ containing $\sqrt{\lambda_0 - 1}i$. By Koebe Distortion Theorem, reducing $\delta > 0$ if necessary, there is an implicit constant $K > 1$ such that, for all $\omega \in U_k \subset B_\delta(q_{\lambda_0})$ and all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$,
\[ \frac{1}{K} \leq \left| \frac{U_k'}{V_k'}(\omega) \right| \leq K. \] (15)
So, $\left| \left( U_k^m \right)'(\omega) \right|^{-1} \leq K C_0 \delta_0^m$; that is, the distortion of $T_{\lambda_0}^i$ in $U_k$ is bounded by $K$; denote this property as the Uniform Bounded Distortion Property.

We also denote the largest $k = p$ such that $B_\delta(\sqrt{\lambda_0 - 1}i) \subset V_k^i$ for $r > 0$ small enough and all $\lambda$ sufficiently close to $\lambda_0$. It follows that, for $r \to 0$, $p = p(r) \to \infty$. The following suffices to prove that
\[ \lim_{p \to \infty} \mu_\lambda \left( V_{p}^i \right) = 0. \] (16)
Step 1. Let $D$ be a disc containing $\sqrt{\lambda_0 - 1}i$, small enough such that $\deg T_\lambda|_D = 3$, since $\sqrt{\lambda_0 - 1}i$ is a critical point with the multiplicity 2. Reducing $\epsilon > 0$ if necessary, such that $U_m^\lambda \in T_\lambda^2(D), T_\lambda$ is hyperbolic when $\lambda$ is close to $\lambda_0$, then the probability measure $\mu_\lambda$ is not atomic; we have
\[
\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) = \sum_{m \geq p} \mu_\lambda (V_m^\lambda - V_{m+1}^\lambda)
\]
(17) for all $p \geq 1$. By the construction of the t-conformal measure $\mu$ of a rational map $R$ ([2]), we know that $\mu(\mathcal{A}_{-1}) = \int_{\mathcal{A}_{-1}} |(R^{-1})'|^t d\mu$ for every Borel set $A \subset J(R)$ such that $R : A_{-1} \rightarrow A$ is conformal. For $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, we have
\[
\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) \leq 3\mu_\lambda (U_m^\lambda - U_{m+1}^\lambda)
\]
\[
\times \inf_{z \in (V_m^\lambda - V_{m+1}^\lambda) \cap (V_1)} \left|\left(\frac{V_1}{\lambda}\right)'(z) - \frac{V_1}{\lambda}\right|^{-d_1}.
\]
By the uniform Bounded Distortion Property, note that $\lambda(\lambda) = \frac{1}{2}$ and $\mu_\lambda$ is a probability measure, then
\[
\mu_\lambda (U_m^\lambda - U_{m+1}^\lambda) \leq K_4 \|T_m^\lambda\| (\lambda(\lambda))^{-d_1}.
\]
(19) Furthermore, we claim that there exists $C_1 > 0$ such that, for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ and $z \in V_1$,
\[
\left|\left(\frac{V_1}{\lambda}\right)'(z) - \frac{V_1}{\lambda}\right| \geq C_1 \left|\left(\frac{V_1}{\lambda}\right)'(z) - \frac{V_1}{\lambda}\right|^{2/3}.
\]
(20) In fact, (20) is obvious for $z = \sqrt{\lambda - 1}i$, since $T_m^\lambda(\sqrt{\lambda - 1}i) = 0$ and $V_1 = T_m^\lambda(\sqrt{\lambda - 1}i)$. Suppose $z \neq \sqrt{\lambda - 1}i$, by the uniform Bounded Distortion Property and Koebe Distortion Theorem, it follows that
\[
dist (v_\lambda, \partial V_1^\lambda) \sim \text{diam} (U_1^\lambda),
\]
\[
dist \left(\sqrt{\lambda - 1}i, \partial V_1^\lambda\right) \sim \text{diam} (V_1^\lambda) \sim \left(\text{diam} (T_\lambda (V_1^\lambda))\right)^{1/3}
\]
\[
\sim \left(\text{diam} (U_1^\lambda)\right)^{1/3},
\]
(21) since $\deg T_\lambda|_{V_1} = 3$ and $\deg T_\lambda|_{T_\lambda(V_1^\lambda)} = 1$. Then
\[
\left|\left(\frac{V_1}{\lambda}\right)'(z) - \left(\frac{V_1}{\lambda}\right)'(z)\right| \sim \left(\text{diam} (U_1^\lambda)\right)^{1/3} \sim \left(\text{diam} (V_1^\lambda)\right)^{2/3},
\]
(22) so, we get (20).

Step 2. Let $k = k(\lambda)$ be the largest integer such that $v_\lambda \in U_k^\lambda$ and let $m \geq 1$. Then there are three cases.

Case 1. $k < m \leq k+1$. By the uniform Bounded Distortion Property, it follows that $|\left(T_m^\lambda\right)'(\lambda(\lambda))\|^{-1} \sim |z(\lambda) - v_\lambda|$, since $k \rightarrow \infty$ as $\lambda \rightarrow \lambda_0$. By Proposition 9, it follows that
\[
|z(\lambda) - v_\lambda| \sim \left|\frac{\lambda - 1}{\lambda - 2}\right|^3 \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right)^3
\]
\[
\sim \frac{\lambda - 1}{\lambda - 2} \left(\frac{\lambda_0 - 1}{\lambda_0 - 2}\right) \sim |\lambda - \lambda_0|,
\]
(23) since $\lambda_0 \neq 1 + (k/2)$. So, we get $|\left(T_m^\lambda\right)'(\lambda(\lambda))\|^{-1} \sim |\lambda - \lambda_0|$ with constant independent of $\lambda$; hence, $|\left(T_m^\lambda\right)'(\lambda(\lambda))\|^{-1} \leq C_2 |\lambda - \lambda_0|$ for some constant $C_2 > 0$ independent of $\lambda$, but on the other hand,
\[
dist (v_\lambda, (U_m^\lambda - U_{m+1}^\lambda) \cap J(T_\lambda)) \geq \dist (v_\lambda, J(T_\lambda)).
\]
(24) Then for all $z \in (U_m^\lambda - U_{m+1}^\lambda) \cap J(T_\lambda)$, by (20), it follows that
\[
\left|\left(T_m^\lambda\right)'(z)\right| > C_1 \dist (v_\lambda, J(T_\lambda))^2/3,
\]
(25) so,
\[
\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) \leq C_3 |\lambda - \lambda_0| d_1 \dist (v_\lambda, J(T_\lambda))^{-2/3} d_1,
\]
(26) where $C_3 = (K_4^2 C_1^2)^{-1} d_1$.

Case 2. $m < k - 1$. Noting that
\[
dist (v_\lambda, (U_m^\lambda - U_{m+1}^\lambda)) \geq \dist (\partial U_m^\lambda, U_{m+1}^\lambda),
\]
(27) then by the uniform Bounded Distortion Property, we have
\[
dist (v_\lambda, (U_m^\lambda - U_{m+1}^\lambda)) > C_4 \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-1}.
\]
(28) As in Case 1, we have
\[
\left|\left(T_m^\lambda\right)'(z)\right| > C_1 \left(\dist (v_\lambda, U_m^\lambda - U_{m+1}^\lambda)\right)^{2/3}
\]
\[
\geq C_1 C_2^{2/3} \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-2/3}.
\]
(29) It follows that
\[
\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) \leq 3K_4 d_1 \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-d_1}
\]
\[
\times \left(C_4 C_2^{1/3}\right)^{-d_1} \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-2/3 (d_1)}
\]
\[
= C_5 \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-d_1/3}.\]

By (14), $\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) \leq C_5 \theta_0^{m d_1/3}$, where $C_5 = K_4^3 (C_1 C_2^{2/3})^{-d_1} C_4 d_1/3$.

Case 3. $m > k + 1$. We have
\[
dist (v_\lambda, (U_m^\lambda - U_{m+1}^\lambda)) \geq \dist (\partial U_m^\lambda, U_{m+1}^\lambda).
\]
(31) By a similar discussion as used in Case 2,
\[
dist (v_\lambda, (U_m^\lambda - U_{m+1}^\lambda)) \geq C_4 \left|\left(T_m^\lambda\right)'(\lambda(\lambda))\right|^{-1},
\]
(32) then $\mu_\lambda (V_m^\lambda - V_{m+1}^\lambda) \leq C_5 \theta_0^{m d_1/3}$.
Step 3. Since $T_\lambda$ is hyperbolic when $\lambda$ is close but not equal to $\lambda_0$, by Lemma 3, every Fatou component of $F(T_\lambda)$ is a John domain. Noting that $F(T_\lambda)$ is symmetry with the real axis $\mathbb{R}$ and $q_1 \in J(T_\lambda)$, then the angle at $q_1$ of two curves $y_1$ and $y_2$ of $\partial A(\infty)$ (or $\partial A(1)$) is positive. Since $y_1 \to q_3$ as $\lambda \to \lambda_0$, it follows that $\text{dist}(y_1, J(T_\lambda)) \sim \text{dist}(y_3, q_3)$ as $\lambda \to \lambda_0$. On the other hand, by Proposition 9, it follows that $\text{dist}(y_1, q_3) \sim |\epsilon(\lambda) - y_1| \sim |\lambda - \lambda_0|$. Thus, $\text{dist}(y_3, J(T_\lambda)) \sim |\lambda - \lambda_0|$ as $\lambda \to \lambda_0$.

By Steps 1 and 2, for $\rho \geq 1$, we have

$$\mu_\lambda \left(V^p_\rho\right) \leq 3C_3 |\lambda - \lambda_0|^{d_\lambda} \text{dist}(y_\lambda, J(T_\lambda))^{-2(3/d_\lambda)}$$

$$+ C_5 \sum_{m \leq p, m \neq k-1, k, k+1} \theta_0^{nm\gamma/3}.$$  \hspace{1cm} (33)

Since

$$\sum_{m \leq p} \theta_0^{nm\gamma/3} = \left(\frac{\theta_0^{d_\lambda/3}}{1 - \theta_0^{d_\lambda/3}}\right)^p$$ \hspace{1cm} (34)

we conclude that

$$\lim_{p \to \infty} \lim_{\lambda \to \lambda_0} \mu_\lambda \left(V^p_\rho\right) = 0.$$ \hspace{1cm} (35)

So, $\text{HD}(J(T_\lambda))$ is continuous at $\lambda \in \{3 - \sqrt{2}, 3 + \sqrt{2}\}$.

The Proof of Theorem 1. Since the Hausdorff dimension $\text{HD}(J(T_\lambda))$ varies continuously in $\text{Rat}_d$ if $T_\lambda$ is hyperbolic [1, Theorem 11.1] and $\deg(T_0) \neq \deg(T_\lambda) = 6$ for $\lambda \neq 0$, by Lemma 8 and Proposition 10, $\text{HD}(J(T_\lambda))$ is continuous for $\lambda \in \mathbb{R} \setminus \{0, \alpha, \beta\}$.

Acknowledgments

The first author was supported by the Construct Program of the Key Discipline in Hunan Province and Science Foundation of Hengyang Normal University of China (no. 12B335). The second author was supported by the Fundamental Research Funds of Central Universities of China (no. 2009QS15). The authors were supported by the NSFs of China (no. 11261002 and no. 11231009). The authors would like to thank the referees for their valuable comments for improving this paper.

References

Submit your manuscripts at
http://www.hindawi.com