Research Article

An Element Weakly Primary to Another Element

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We introduce the concept “An element weakly primary to another element” and using this concept we have generalized some result proved by Manjarekar and Chavan (2004). It is shown that if $\{b_\alpha\}$ is a family of elements weakly primary to a in L , then $\bigvee_\alpha b_\alpha$ is weakly primary to a .

1. Introduction

Multiplicative lattice is a complete lattice provided with commutative, associative, and join distributive multiplication for which the largest element 1 acts as a multiplicative identity. A proper element p of L is called prime element if $ab \leq p \Rightarrow a \leq p$ or $b \leq p$ for $a, b \in L$ and is called primary element if $ab \leq p$ implies $a \leq p$ or $b^n \leq p$ for some $n \in \mathbb{Z}_+$. An element a of L is called compact if $a \leq \bigvee X$, and $X \subseteq L$ implies the existence of finite number of elements $X_1, X_2, X_3, \dots, X_n$ of L such that $a \leq X_1 \vee X_2 \vee X_3 \vee \dots \vee X_n$. Throughout this paper, L denotes compactly generated multiplicative lattice with 1 compact and every finite product of compact elements is compact. Let L_* be the set of all compact elements in L . Also, $(a : b)$ is the greatest element c in L such that $cb \leq a$. An element $a \in L$ is join principal if $x \vee (y : a) = (xa \vee y) : a$ and meet principle if $x \wedge ya = ((x : a) \wedge y)a$, for all $x, y \in L$.

An element is principle if it is both join and meet principle. For $a \in L$, $\sqrt{a} = \bigvee \{x \in L_* \mid x^n \leq a \text{ for some } n \in \mathbb{Z}_+\}$. An element $a \in L$ is called semiprimary if $\sqrt{a}$ is primary element. L is said to satisfy the condition (*) if every semiprimary element is primary element.

An element $a \in L$ is said to be strong join principle element if a is compact and join principle. An element $a \in L$ is p -primary if a is primary and $\sqrt{a} = p$ and $a \in L$ is semiprime if $\sqrt{a} = a$. An element a of L is called zero divisor if $\exists 0 \neq b \in L$ such that $ab = 0$, and if L has no zero divisor then L will be

called lattice domain or simply a domain. L_* denotes the set of compact elements of L .

The concept of weakly prime element is studied by Callialp et al. [1]. The concept of weakly primary element is introduced by Sachin and Vilas [2]. For other definitions and simple properties of multiplicative lattice, one can refer to Dilworth [3].

Definition 1. Weakly primary element is defined as follows.

An element $q \in L$ is said to be a weakly primary element if for $a, b \in L_*$, $0 \neq ab \leq q$ implies $a \leq q$ or $b^n \leq q$ for some $n \in \mathbb{Z}_+$.

Example 2. Lattice of ideals of ring $R = \langle \mathbb{Z}_{12}, +, \cdot \rangle$ (see Figure 1).

In the lattice of Example 2, an element a is weakly primary element. From Definition 1, it is clear that every weakly prime element is weakly primary element. Converse need not be true. Since in Example 2, a is weakly primary element but it is not weakly prime element. Further, if q is a weakly primary element, then $\sqrt{q}$ is a weakly prime element. Because if for compact element x and y such that $0 \neq xy \leq \sqrt{q}$ then $x^n y^n = (xy)^n \leq q$ for some $n \in \mathbb{Z}_+$. As q is a weakly primary element, either $x^n \leq q$ or $(y^n)^m = y^{nm} \leq q$ for some $n \in \mathbb{Z}_+$. Consequently, $x \leq \sqrt{q}$ or $y \leq \sqrt{q}$. Thus, $\sqrt{q}$ is a weakly prime

FIGURE 1

element. This implies that every weakly primary element is a weakly semiprimary element. It need not be true that a is always weakly prime or a is always weakly semiprimary. In Example 2, the least element 0 is not semiprimary as $\sqrt{0} = b$ is not a weakly prime element. The concept of “An element primeto another element” is introduced in [4]. An element $b \in L$ is primeto an element a if for $x \in L$, $xb \leq a$ implies $x \leq a$.

Now, we define the following.

Definition 3. An element weakly prime to another element is defined as follows.

An element, $b \in L$, is called weakly prime to an element $a \in L$ if for any $x \in L$, $0 \neq xb \leq a$ implies $x \leq a$.

In Example 2, the element d is weakly prime to an element c , but d is not weakly prime to any other element of L . This follows directly from the fact that an element y is weakly prime to an element x if and only if $x : y = x$.

2. An Element Weakly Primary to Another Element

Now we introduce the following main concept which is a generalization of the concept introduced by Manjarekar and Chavan [5].

Definition 4. An element b is said to be weakly primary to another element a in L if for $x \in L_*$, $0 \neq xb \leq a$ implies $x^n \leq a$ for some $n \in \mathbb{Z}_+$.

In Example 2, the element b is weakly primary to a , but note that b is not weakly prime to a . This follows directly from Corollary 9, and note that $(a : b) = c = \sqrt{a}$ and $a < c$. Evidently, if b is weakly prime to a , then b is weakly primary to a in L . Now if p is a weakly prime element and $a \not\leq p$, then a is weakly prime to p , and if q is a weakly primary element and a compact element $a \not\leq q$, then a is a weakly primary element to q .

Thus, from this, it is clear that elements weakly primary to another element exist in the lattice L . Since L is compactly generated multiplicative lattice with 1 compact, weakly prime element and hence weakly primary element exists in L . Hereafter, L will be a domain. We prove some interesting results including characterizations.

Theorem 5. No proper nonzero element is weakly prime or weakly primary to itself in L .

Proof. If a is a proper nonzero element in L and a is weakly primary to a itself, then $0 \neq a.1 = a$ implies that $1 = a$, a contradiction. Therefore, no proper nonzero element is weakly prime or weakly primary to itself in L . $\square$

Now we prove some characterizations of an element weakly primary to a .

Theorem 6. Let $a \in L$ be a semiprime element. Then $0 \neq b$ is weakly primary to a if and only if b is weakly prime to a .

Proof. Assume that b is weakly primary to a semiprime element a . Let $0 \neq xb \leq a$ for some $x \in L_*$. Then $x^n \leq a$ for some $n \in \mathbb{Z}_+$. Consequently, $x \leq \sqrt{a}$. As a is semiprime, $x \leq a$. Thus, b is weakly prime to a . The converse part is obvious. $\square$

Theorem 7. Let L be a lattice domain. Let $a, b \in L$; then $0 \neq b$ is weakly primary to a if and only if $(a : b) \leq \sqrt{a}$.

Proof. Assume that $0 \neq b$ is weakly primary to a . Let $0 \neq x \in L_*$ such that $0 \neq x \leq (a : b)$; then, $0 \neq xb \leq a$. As b is weakly primary to a , $x^n \leq a$ for some $n \in \mathbb{Z}_+$. Hence, $x \leq \sqrt{a}$. This shows that $(a : b) \leq \sqrt{a}$.

Conversely, assume that $(a : b) \leq \sqrt{a}$. Let $0 \neq xb \leq a$ for some $x \in L_*$. Then, we have $x \leq (a : b) \leq \sqrt{a}$. This implies that $x^n \leq a$ for some $n \in \mathbb{Z}_+$. Thus, b is weakly primary to a . $\square$

Theorem 8. Let L be a lattice domain. Let $a, b \in L$; then $0 \neq b$ is weakly prime to a if and only if $a = (a : b)$.

Proof. Assume that $0 \neq b$ is weakly prime to a in L . Let $0 \neq x \in L$ such that $x \leq (a : b)$. Then $0 \neq xb \leq a$. As b is weakly prime to a , we have $x \leq a$. This shows that $(a : b) \leq a$. But $a \leq (a : b)$. Therefore, we get $a = (a : b)$.

Conversely, assume that $(a : b) = a$. Let, $0 \neq xb \leq a$ for $x \in L$. Then, we get $x \leq (a : b) = a$. Thus b is weakly prime to a . $\square$

Corollary 9. Let L be a lattice domain. Let, $a, b \in L$. Then, $0 \neq b$ is weakly primary to a but it is nonweakly prime to a if and only if $a < (a : b) \leq \sqrt{a}$.

Proof. It follows from the fact that $a \leq (a : b)$ and from Theorems 7 and 8. $\square$

Corollary 10. Let $a, b \in L$. If a is weakly semiprimary element and $b \not\leq \sqrt{a}$, then $0 \neq b$ is weakly primary to a .

Proof. Assume that $a \in L$ is a weakly semiprimary element and $b \not\leq \sqrt{a}$. Let $x \in L^*$ such that $0 \neq xb \leq a$. Then $0 \neq xb \leq \sqrt{a}$. As $\sqrt{a}$ is a weakly prime element and $b \not\leq \sqrt{a}$, we have $x \leq \sqrt{a}$. This implies that $x^n \leq a$ for some $n \in \mathbb{Z}_+$. Thus, b is weakly primary to a . $\square$

Theorem 11. Let L be a lattice domain. Let $a, b \in L$. Then, $0 \neq b$ is weakly prime to a if and only if b is weakly prime to $(a : x)$ for every $0 \neq x \in L$.

Proof. Assume that b is weakly prime to a in L . Let $0 \neq yb \leq (a : x)$ for some $y \in L$. Then, $0 \neq xyb \leq a$. As b is weakly prime to a , $xy \leq a$. Consequently, $y \leq (a : x)$. Thus, b is weakly prime to $(a : x)$ for every $0 \neq x \in L$. The converse is obvious, since, if b is weakly prime to $(a : x)$ for every $0 \neq x \in L$, b is weakly prime to $(a : 1) = a$. $\square$

Theorem 12. Let $a, b \in L$ and let L be a lattice domain. If b is weakly primary to a and a is a semiprime element in L , then b is weakly primary to $(a : x)$ for every $x \in L$.

Proof. It follows from Theorems 6 and 11. $\square$

Theorem 13. Let $a, b \in L$. Then $0 \neq b$ is weakly primary to a in L if and only if each $x \geq a$ is weakly primary to a .

Proof. Assume that $0 \neq b$ is weakly primary to a in L . Let $x \geq b$ and $yx \leq a$ for some $y \in L^*$. Then $0 \neq yb \leq a$. Therefore, by assumption, $y^n \leq a$ for some $n \in \mathbb{Z}_+$.

This shows that each $x \geq b$ is weakly prime to a . The converse part is obvious. $\square$

Theorem 14. If $\{b_\alpha\}$ is a family of a elements weakly primary to a in L , then $\bigvee_\alpha b_\alpha$ is weakly primary to a .

Proof. It follows from the fact that $b_\alpha \leq \bigvee_\alpha b_\alpha$ and from Theorem 13. $\square$

Theorem 15. Let $a, b \in L$. Then, $0 \neq b$ is nonweakly primary to a if and only if each $x \leq b$ is nonweakly primary to a .

Proof. Assume that b is nonweakly primary to a . Therefore, by Theorem 8, we have $a : b \not\leq \sqrt{a}$. Let x be an element of L such that $x \leq b$. Then $(a : b) \leq (a : x)$. This shows that $(a : x) \not\leq \sqrt{a}$. Thus, again by Theorem 8, each $x \leq b$ is nonweakly primary to a . $\square$

This lemma leads us to the following two obvious corollaries.

Corollary 16. If $\{b_\alpha\}$ is a family of elements nonweakly primary to a in L , then $\bigwedge_\alpha b_\alpha$ is nonweakly primary to a .

Corollary 17. If $\bigvee_\alpha b_\alpha$ is nonweakly primary to a in L , then each b_α is nonweakly primary to a .

Theorem 18. If y is compact and $0 \neq xy$ is nonweakly primary to a , then either x is nonweakly primary to a or y^n is nonweakly primary to a for some $n \in \mathbb{Z}_+$.

FIGURE 2

Definition 19. Completely meet semiprimary elements are defined as follows.

An element a is said to be completely meet semiprimary element if $\sqrt{a}$ is a completely meet prime element.

Example 20. Every element $x < d$ is not a completely meet prime element. But note that $\sqrt{x} = d$. Thus, each $x < d$ is a completely meet semiprimary element (see Figure 2).

Result 1. In any multiplicative lattice S , we have $\bigwedge_\alpha (a : x_\alpha) = a : (\bigvee_\alpha x_\alpha)$ for any $a, x_\alpha \in S$.

Theorem 21. Suppose that a is a completely meet semiprimary element of L and let $\{b_\alpha\} \subseteq L$. If each $0 \neq b_\alpha$ is nonweakly primary to a in L , then $\bigvee_\alpha b_\alpha$ is nonweakly primary to a .

Proof. By Result 1, we have $a : (\bigvee_\alpha b_\alpha) = \bigwedge_\alpha (a : b_\alpha)$. But each $0 \neq b_\alpha$ is nonweakly primary to a . Consequently, by Theorem 12, each $(a : b_\alpha) \not\leq \sqrt{a}$. As a is completely meet semiprimary element, $\bigwedge_\alpha (a : b_\alpha) = a : (\bigvee_\alpha b_\alpha) \not\leq \sqrt{a}$. Therefore, by Theorem 8, $\bigvee_\alpha b_\alpha$ is nonweakly primary to a . $\square$

Now we construct a new element as follows. Define $q_a = \bigvee \{x \in L \mid x \text{ is non weakly primary to } a\}$.

Theorem 22. If a is a completely meet semiprimary element, then q_a is weakly primary element and is nonweakly primary to a .

Proof. By Theorem 21, it follows that q_a is nonweakly primary element to a . Let x and y be compact elements such that, $0 \neq xy \leq q_a$. Then by Theorem 15, xy is nonweakly primary to a . Therefore, by Theorem 18, either x is nonweakly primary to a or y^n is nonweakly primary to a for some $n \in \mathbb{Z}_+$. Thus, $x \leq q_a$ or $y^n \leq q_a$ for some $n \in \mathbb{Z}_+$. This shows that q_a is a weakly primary element. $\square$

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