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Research Article

Linear Isometries between Real Banach Algebras of Continuous Complex-Valued Functions

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Let X and Y be compact Hausdorff spaces, and let τ and η be topological involutions on X and Y, respectively. In 1991, Kulkarni and Arundhathi characterized linear isometries from a real uniform function algebra A on (X, τ) onto a real uniform function algebra B on (Y, η) applying their Choquet boundaries and showed that these mappings are weighted composition operators. In this paper, we characterize all onto linear isometries and certain into linear isometries between $C(X, \tau)$ and $C(Y, \eta)$ applying the extreme points in the unit balls of $C(X, \tau)^*$ and $C(Y, \eta)^*$.

1. Introduction and Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. The symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{F}$ are called scalars. We also denote by $\mathbb{S}_{\mathbb{F}}$ the set of all $\lambda \in \mathbb{F}$ with $|\lambda| = 1$.

Let $\mathfrak{X}$ be a normed space over $\mathbb{F}$. We denote by $\mathfrak{X}^*$ and $B_{\mathfrak{X}}$ the dual space of $\mathfrak{X}$ and the closed unit ball of $\mathfrak{X}$, respectively. For a subset E of $\mathfrak{X}$, let $\mathrm{Ext}(E)$ denote the set of all extreme points of E. Kulkarni and Limaye showed [1, Theorem 2] that if A is a nonzero linear subspace of $\mathfrak{X}$ and $\varphi \in \mathrm{Ext}(B_{A^*})$, then φ has an extension to some $\psi \in \mathrm{Ext}(B_{\mathfrak{X}^*})$.

We know that if $\mathfrak{X}$ and $\mathcal{Y}$ are normed spaces over $\mathbb{F}$ and $T : \mathfrak{X} \to \mathcal{Y}$ is a linear isometry from $\mathfrak{X}$ onto $\mathcal{Y}$ over $\mathbb{F}$, then T is a bijection mapping between $\mathrm{Ext}(B_{\mathfrak{X}})$ and $\mathrm{Ext}(B_{\mathcal{Y}})$.

Let X be a compact Hausdorff space. We denote by $C_{\mathbb{F}}(X)$ the unital commutative Banach algebra of all continuous functions from X into $\mathbb{F}$, with the uniform norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$, $f \in C_{\mathbb{F}}(X)$. We write $C(X)$ instead as $C_{\mathbb{C}}(X)$. For $x \in X$, we consider the linear functional $e_{C_{\mathbb{F}}(X),x}$ on $C_{\mathbb{F}}(X)$ defined by $e_{C_{\mathbb{F}}(X),x}(f) = f(x)$ $(f \in C_{\mathbb{F}}(X))$, which is called the evaluation functional on $C_{\mathbb{F}}(X)$ at x. Clearly, $\lambda e_{C_{\mathbb{F}}(X),x} \in B_{C_{\mathbb{F}}(X)^*}$ for all $(x, \lambda) \in X \times \mathbb{F}$. It is known [2, page 441] that

$$\mathrm{Ext}\left(B_{C_{\mathbb{F}}(X)^*}\right) = \left\{\lambda e_{C_{\mathbb{F}}(X),x} : (x, \lambda) \in X \times \mathbb{S}_{\mathbb{F}}\right\}. \tag{1}$$

Let A be a real or complex linear subspace of $C(X)$. A nonempty subset S of X is called a *boundary* for A (with respect to X), if for each $f \in A$ the function $|f|$ assumes its maximum on X at some $x \in S$. We denote by $\Gamma(A, X)$ the intersection of all closed boundaries for A. If $\Gamma(A, X)$ is a boundary for A, it is called the *Shilov boundary* for A (with respect to X).

Let A be linear subspace of $C_{\mathbb{F}}(X)$ containing 1_X, the constant function with value 1 on X. A *representing measure* for $\varphi \in A^*$ is an $\mathbb{F}$-valued regular Borel measure μ on X such that $\varphi(f) = \int_X f\, d\mu$ for all $f \in A$. Let $K_{\mathbb{F}}(A) = \{\varphi \in A^* : \|\varphi\| = \varphi(1_X) = 1\}$. If $x \in X$, then $e_{C_{\mathbb{F}}(X),x}|_A \in K_{\mathbb{F}}(A)$ and δ_x, the point mass measure on X at x, is a representing measure for $e_{C_{\mathbb{F}}(X),x}|_A$. We denote by $\mathrm{Ch}(A, X)$ the set of all $x \in X$ for which δ_x is the only representing measure for $e_{C_{\mathbb{F}}(X),x}|_A$. If $\mathrm{Ch}(A, X)$ is a boundary for A, it is called the *Choquet boundary* for A (with respect to X). We know that

$$\Gamma\left(C_{\mathbb{F}}(X), X\right) = \mathrm{Ch}\left(C_{\mathbb{F}}(X), X\right) = X. \tag{2}$$

Let A be a real linear subspace of $C(X)$, and let S be nonempty subset of X. We say that A is *extremely regular* at S if for every open neighborhood U of S and for each $\varepsilon > 0$ there is a function $f \in A$ with $\|f\|_X = 1$ such that $f(x) = 1$ for all $x \in S$ and $|f(y)| < 1$ for all $y \in X \setminus U$.

Let X be a nonempty set. A self-map $\tau : X \to X$ is called an *involution* on X if $\tau(\tau(x)) = x$ for all $x \in X$. A subset S of X is called *τ-invariant* if $\tau(S) \subseteq S$. Clearly, if $S \subseteq X$ is τ-invariant, then $\tau(S) = S$. A *τ-invariant measure* on E is a measure μ on E such that $\mu \circ \tau = \mu$.

Let X be a topological space. An involution τ on X is called a *topological involution* on X, if τ is continuous.

Let X be a compact Hausdorff space, and let τ be a topological involution on X. We define

$$C(X,\tau) = \left\{ f \in C(X) : f \circ \tau = \overline{f} \right\}. \tag{3}$$

Then, $C(X,\tau)$ is a unital uniformly closed self-adjoint real subalgebra of $C(X)$ which separates the points of X and does not contain i_X, the constant function with value i on X. Note that $\operatorname{Re} f \in C(X,\tau)$ for all $f \in C(X,\tau)$. Moreover, $C(X) = C(X,\tau) \oplus iC(X,\tau)$ and

$$\max\left\{\|f\|_X, \|g\|_X\right\} \le \|f + ig\|_X \le 2\max\left\{\|f\|_X, \|g\|_X\right\}, \tag{4}$$

for all $f, g \in C(X,\tau)$. In fact, the complex Banach algebra $(C(X), \|\cdot\|_X)$ can be regarded as the complexification of the real Banach algebra $(C(X,\tau), \|\cdot\|_X)$. Note that $C(X,\tau) = C_{\mathbb{R}}(X)$ if and only if τ is the identity self-map on X. Hence, the class of real Banach algebras $C(X,\tau)$ is, in fact, larger than the class of real Banach algebras $C_{\mathbb{R}}(X)$.

Let X be a compact Hausdorff space, τ a topological involution on X, and E a subset of $C(X,\tau)$. We say that E separates the points of X/τ if for each $x, y \in X$ with $y \notin \{x, \tau(x)\}$ there exists a function f in E such that $f(x) \neq f(y)$. It is known [3, Lemma 1.3.7] that if $\operatorname{Re} C(X,\tau) = \{\operatorname{Re} f : f \in C(X,\tau)\}$, then $\operatorname{Re} C(X,\tau)$ separates the points of X/τ.

Let X be a compact Hausdorff space. We assume that $C(X)_r$ denotes $C(X)$, regarded as a real normed space by restricting the scalar multiplication to real scalars. Let τ be the topological involution on $X \times \{0,1\}$ defined by $\tau(x,0) = (x,1)$ $(x \in X)$ and $\tau(x,1) = (x,0)$ $(x \in X)$. Then the map $\Psi : C(X)_r \to C(X \times \{0,1\}, \tau)$ defined by

$$(\Psi f)(x,0) = f(x), \qquad (\Psi f)(x,1) = \overline{f(x)} \qquad (f \in C(X)_r; x \in X), \tag{5}$$

is an isometrical isomorphism from $C(X)_r$ onto $C(X \times \{0,1\}, \tau)$. Hence, the class of real Banach algebras $C(Y,\tau)$ is, in fact, larger than the class of real Banach algebras $C(X)_r$.

Let X be a compact Hausdorff space, and let τ be a topological involution on X. Let A be a real subspace of $C(X,\tau)$ containing 1_X. A *real part representing measure* for $\varphi \in A^*$ is a τ-invariant regular real Borel measure μ on X such that $\varphi(f) = \int_X \operatorname{Re} f \, d\mu$ for all $f \in A$. Let $K_{\mathbb{R}}(A) = \{\varphi \in A^* : \|\varphi\| = \varphi(1_X) = 1\}$. If $x \in X$, then $\operatorname{Re}(e_{C(X),x|_A}) \in K_{\mathbb{R}}(A)$ and the measure $m_x = (1/2)(\delta_x + \delta_{\tau(x)})$ is a real part representing measure for $\operatorname{Re}(e_{C(X),x}|_A)$. We denote by $\operatorname{Ch}(A,X,\tau)$ the set of all $x \in X$ for which m_x is the only real part representing measure for $\operatorname{Re}(e_{C(X),x}|_A)$. Clearly, $\operatorname{Ch}(A,X,\tau)$ is a τ-invariant subset of X. If $\operatorname{Ch}(A,X,\tau)$ is a boundary for A, it is called the *Choquet boundary* for A with respect to (X,τ). We know that $\operatorname{Ch}(C(X,\tau),X,\tau) = X$.

The real Banach algebra $C(X,\tau)$ and its real linear subspaces were first considered by Kulkarni and Limaye in [4]. For a detailed account of several properties of $C(X,\tau)$, we refer to [3].

Let X be a compact Hausdorff space. For each $(x,\lambda) \in X \times \mathbb{C}$, define the map $\psi_{C(X),x,\lambda} : C(X)_r \to \mathbb{R}$ by

$$\psi_{C(X),x,\lambda}(f) = \operatorname{Re}(\lambda f(x)), \quad (f \in C(X)_r), \tag{6}$$

in fact, $\psi_{C(X),x,\lambda} = \operatorname{Re}(\lambda e_{C(X),x})$. Clearly, $\psi_{C(X),x,\lambda} \in B_{(C(X)_r)^*}$.

Let τ be a topological involution on X. For each $(x,\lambda) \in X \times \mathbb{C}$ define the map $\varphi_{C(X,\tau),x,\lambda} : C(X,\tau) \to \mathbb{R}$ by $\varphi_{C(X,\tau),x,\lambda} = \psi_{C(X),x,\lambda}|_{C(X,\tau)}$. Kulkarni and Limaye showed [1, Proposition 3] that

$$\operatorname{Ext}\left(B_{(C(X)_r)^*}\right) = \left\{\psi_{C(X),x,\lambda} : (x,\lambda) \in X \times S_{\mathbb{C}}\right\}. \tag{7}$$

Applying this result, they obtained the following theorem.

Theorem 1 (see [1, Corollary 5]). *Let X be a compact Hausdorff space, and let τ be a topological involution on X. Suppose that*

$$P_{X,\tau} = \{(x,\lambda) \in X \times S_{\mathbb{C}} : \tau(x) \neq x\} \cup \{(x,\lambda) \in X \times S_{\mathbb{R}} : \tau(x) = x\}. \tag{8}$$

Then

$$\operatorname{Ext}\left(B_{C(X,\tau)^*}\right) = \left\{\varphi_{C(X,\tau),x,\lambda} : (x,\lambda) \in P_{X,\tau}\right\}. \tag{9}$$

The classical Banach-Stone theorem states that if T is a linear isometry from $C_{\mathbb{F}}(X)$ onto $C_{\mathbb{F}}(Y)$, then there exists a homeomorphism h from Y onto X and a continuous function $a \in C_{\mathbb{F}}(Y)$ with $a(Y) \subseteq \mathbb{S}_{\mathbb{F}}$ such that

$$(Tf)(y) = a(y) f(h(y)), \quad (f \in C_{\mathbb{F}}(X), y \in Y). \tag{10}$$

This well-known theorem has been generalized in several directions. In 1966, an important generalization was given by Holsztyński [5] by considering into linear isometries as the following.

Theorem 2. *Let X and Y be compact Hausdorff spaces. If T is a linear isometry from $C_{\mathbb{F}}(X)$ into $C_{\mathbb{F}}(Y)$, then there exists a closed subset Y_0 of Y, a continuous map φ from Y_0 onto X and a function $a \in C_{\mathbb{F}}(Y)$ with $\|a\|_Y = 1$ and $a(Y_0) \subseteq \mathbb{S}_{\mathbb{F}}$ such that*

$$(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C_{\mathbb{F}}(X), \ \forall y \in Y_0. \tag{11}$$

In 1975, another important generalization of Banach-Stone theorem was given by Novinger [6] for certain complex subalgebras of $C(X)$ and $C(Y)$ considering the concept of their Choquet boundaries. In 1991, Kulkarni and Arundhathi [7] generalized the given results by Novinger for certain real subalgebras of $C(X,\tau)$ and $C(Y,\eta)$ considering the concept of their Choquet boundaries. We can cite some other generalizations of Banach-Stone theorem, for example, the generalization obtained by Cambern [8] for space of vector-valued continuous functions, by Jeang and Wong [9] for spaces of scalar-valued continuous functions vanishing

at infinity, by Araujo and Font [10] for certain subspaces of scalar-valued continuous functions considering concepts of their Choquet and Shilov boundaries, and by Jiménez-Vargas and Villegas-Vallecillos [11] for spaces of scalar-valued Lipschitz functions.

Throughout the rest of this paper, X and Y are compact Hausdorff spaces and τ and η are topological involutions on X and Y, respectively.

In this paper, we characterize all linear isometries from $C(X,\tau)$ onto $C(Y,\eta)$ and certain linear isometries from $C(X,\tau)$ into $C(Y,\eta)$, applying the extreme points in $B_{C(X,\tau)^*}$ and $B_{C(Y,\eta)^*}$. Our results in Section 2 are some of the given results by Kulkarni and Arundhathi in [7] that we obtain by applying the extreme points in $B_{C(X,\tau)^*}$ and $B_{C(Y,\eta)^*}$. The main result in Section 3 is a generalization of the given result by Holsztyński in [5] for certain into isometries.

2. Onto Linear Isometries

We first determine unit-preserving linear isometries from $C(X,\tau)$ onto $C(Y,\eta)$. For this purpose, we need the following lemmas.

Lemma 3 (see [3, Lemma 5.1.1]). *Let A and B be real linear subspaces of $C(X,\tau)$ and $C(Y,\eta)$, respectively. Suppose that $x \in X$, $y \in Y$, A is extremely regular at $\{x,\tau(x)\}$ and B is extremely regular at $\{y,\eta(y)\}$. Let $T : A \to B$ be a linear isometry from A onto B and $\mathrm{Re}((Tf)(y)) = \mathrm{Re}(f(x))$ for all $f \in A$. Then, $(Tf)(y) = f(x)$ for all $f \in A$ or $(Tf)(y) = f(\tau(x))$ for all $f \in A$.*

Lemma 4. *Let S be a nonempty τ-invariant closed subset of X. Then, $C(X,\tau)$ is extremely regular at S. In particular, $C(X,\tau)$ is extremely regular at $\{x,\tau(x)\}$ for all x in X.*

Proof. Assume that $\varepsilon > 0$ and U is an open neighborhood of S in X. By Urysohn's lemma, there exists a function $g \in C(X)$ with $0 \le g(x) \le 1$ for all $x \in X$, $g(x) = 1$ for all $x \in S$, and $g(x) = 0$ for all $x \in X \setminus U$. Define the function $f : X \to \mathbb{C}$ by $f = (g \circ \tau)g$. Then, $f \in C(X,\tau)$, $\|f\|_X = 1$, $f(x) = 1$ for all $x \in S$ and $|f(x)| < \varepsilon$ for all $x \in X \setminus U$. Hence, $C(X,\tau)$ is extremely regular at S. □

Theorem 5. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ with $T1_X = 1_Y$. Then, there exists a homeomorphism h from Y onto X with $h \circ \eta = \tau \circ h$ on Y such that*

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y. \tag{12}$$

Proof. We claim that

$$Y = \left\{ y \in Y : T^*\left(\varphi_{C(Y,\eta),y,1}\right) \in \mathrm{Ext}\left(B_{C(X,\tau)^*}\right) \right\}. \tag{13}$$

Let $y \in Y$. By Theorem 1, $\varphi_{C(Y,\eta),y,1} \in \mathrm{Ext}(B_{C(Y,\eta)^*})$. Since T is a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$, we conclude that T^* is a linear isometry from $C(Y,\eta)^*$ onto $C(X,\tau)^*$ and so

$$T^*\left(\mathrm{Ext}\left(B_{C(Y,\eta)^*}\right)\right) = \mathrm{Ext}\left(B_{C(X,\tau)^*}\right). \tag{14}$$

Therefore, $T^*(\varphi_{C(Y,\eta),y,1}) \in \mathrm{Ext}(B_{C(X,\tau)^*})$. Hence, our claim is justified.

We now show that for each $y \in Y$ there exists a unique $x \in X$ such that

$$(Tf)(y) = f(x), \quad \forall f \in C(X,\tau). \tag{15}$$

Let $y \in Y$. By (13), we have $T^*(\varphi_{C(Y,\eta),y,1}) \in \mathrm{Ext}(B_{C(X,\tau)^*})$. Hence, by Theorem 1, there exists $(x_0,\lambda_0) \in X \times S_{\mathbb{C}}$ such that

$$T^*\left(\varphi_{C(Y,\eta),y,1}\right) = \varphi_{C(X,\tau),x_0,\lambda_0}. \tag{16}$$

Since $1_X \in C(X,\tau)$ and $T1_X = 1_Y$, by (16), we have

$$\begin{aligned}
\mathrm{Re}(\lambda_0) &= \mathrm{Re}(\lambda_0 1_X(x_0)) = \varphi_{C(X,\tau),x_0,\lambda_0}(1_X) \\
&= T^*\left(\varphi_{C(Y,\eta),y,1}\right)(1_X) = \varphi_{C(Y,\eta),y,1}(T1_X) \\
&= \varphi_{C(Y,\eta),y,1}(1_Y) = \mathrm{Re}\left(1_Y(y)\, e_{C(Y),y}(1_Y)\right) \\
&= \mathrm{Re}(1_Y(y)\, 1_Y(y)) = \mathrm{Re}(1) \\
&= 1.
\end{aligned} \tag{17}$$

Since $|\lambda_0| = 1$ and $\mathrm{Re}(\lambda_0) = 1$, we have $\mathrm{Im}(\lambda_0) = 0$ and so $\lambda_0 = 1$. Thus, by (16), we have

$$T^*\left(\varphi_{C(Y,\eta),y,1}\right) = \varphi_{C(X,\tau),x_0,1}. \tag{18}$$

From (18) we deduce that

$$\mathrm{Re}((Tf)(y)) = \mathrm{Re}(f(x_0)), \quad \forall f \in C(X,\tau). \tag{19}$$

Since T is a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ and by Lemma 4, $C(X,\tau)$ is extremely regular at $\{x_0,\tau(x_0)\}$, and $C(Y,\eta)$ is extremely regular at $\{y,\eta(y)\}$, we conclude that $(Tf)(y) = f(x_0)$ for all f in $C(X,\tau)$ or $(Tf)(y) = f(\tau(x_0))$ for all $f \in C(X,\tau)$ by Lemma 3. We assume that $x = x_0$ whenever $(Tf)(y) = f(x_0)$ for all f in $C(X,\tau)$ and $x = \tau(x_0)$ whenever $(Tf)(y) = f(\tau(x_0))$ for all f in $C(X,\tau)$. Then, $x \in X$ and we have

$$(Tf)(y) = f(x), \quad \forall f \in C(X,\tau). \tag{20}$$

This proves the existent of x. To show uniqueness, assume that there exists $x' \in X$ such that

$$(Tf)(y) = f(x'), \quad \forall f \in C(X,\tau). \tag{21}$$

Then, $f(x) = f(x')$ for all f in $C(X,\tau)$. This implies that $x = x'$ since $C(X,\tau)$ separates the points of X.

Now we define the map $h : Y \to X$ by $h(y) = x$ whenever $(Tf)(y) = f(x)$ for all f in $C(X,\tau)$. In fact, we have

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau) \ \forall y \in Y. \tag{22}$$

Now we prove that $h \circ \eta = \tau \circ h$ on Y. Let $y \in Y$. Then, $\eta(y) \in Y$. From (22), for each $f \in C(X,\tau)$ we have

$$\begin{aligned}
f((h \circ \eta)(y)) &= f(h(\eta(y))) = (Tf)(\eta(y)) \\
&= \overline{(Tf)(y)} = \overline{f(h(y))} \\
&= f(\tau(h(y))) = f((\tau \circ h)(y)).
\end{aligned} \tag{23}$$

This implies that $(h \circ \eta)(y) = (\tau \circ h)(y)$ since $C(X,\tau)$ separates the points of X. Therefore, $h \circ \eta = \tau \circ h$ on Y.

Continually, we prove that h is bijective. Since T^{-1} is a linear isometry from $C(Y,\eta)$ onto $C(X,\tau)$ with $T^{-1}1_Y = 1_X$, by the above argument, we deduce that there exists a map $k : X \to Y$ such that

$$\left(T^{-1}g\right)(x) = g(k(x)), \quad \forall g \in C(Y,\eta), \ \forall x \in X. \tag{24}$$

To prove the injectivity of h, let $y \in Y$. For each $g \in C(Y,\eta)$, by (22) and (24) we have

$$\begin{aligned} g((k \circ h)(y)) &= g(k(h(y))) = \left(T^{-1}g\right)(h(y)) \\ &= \left(T\left(T^{-1}g\right)\right)(y) = g(y). \end{aligned} \tag{25}$$

This implies that $(k \circ h)(y) = y$ since $C(Y,\eta)$ separates the points of Y. Therefore, $k \circ h = I_Y$, the identity mapping on Y, and so h is injective.

To prove that h is onto, let $x \in X$. For each $f \in C(X,\tau)$, by (24) and (22) we have

$$\begin{aligned} f((h \circ k)(x)) &= f(h(k(x))) = (Tf)(k(x)) \\ &= \left(T^{-1}(Tf)\right)(x) = fx. \end{aligned} \tag{26}$$

This implies that $(h \circ k)(x) = x$ since $C(X,\tau)$ separates the points of X. Therefore $k \circ h = I_X$, the identity mapping on X, and so h is onto.

We now check that $h : Y \to X$ is continuous. Let $y \in Y$, and let $\{y_\alpha\}_{\alpha \in J}$ be a net in Y such that $\lim_\alpha y_\alpha = y$ in Y. Then we have

$$\lim_\alpha (Tf)(y_\alpha) = (Tf)(y), \quad \forall f \in C(X,\tau). \tag{27}$$

By (27) and definition of h, we deduce that

$$\lim_\alpha f(h(y_\alpha)) = f(h(y)), \quad \forall f \in C(X,\tau). \tag{28}$$

Since $C(X,\tau)$ separates the points of X, by (28), we have $\lim_\alpha h(y_\alpha) = h(y)$ in X. Therefore, $h : Y \to X$ is continuous.

Since $h : Y \to X$ is a bijective continuous mapping, Y is a compact space, and X is a Hausdorff space, we conclude that $h^{-1} : X \to Y$ is continuous, and so h is a homeomorphism. Hence, the proof is complete. □

Corollary 6. *Let $T : C(X,\tau) \to C(Y,\eta)$ be an isometry mapping from $C(X,\tau)$ onto $C(Y,\eta)$ with $T(0) = 0$ and $T1_X = 1_Y$. Then T is an isomorphism.*

Proof. Since $(C(X,\tau), \|\cdot\|_X)$ and $(C(Y,\eta), \|\cdot\|_Y)$ are real Banach algebras and T is an isometry from $C(X,\tau)$ onto $C(Y,\eta)$ with $T(0) = 0$, we conclude that T is a linear isometry by Mazur-Ulam theorem (see [12]). Since $T1_X = 1_Y$, applying Theorem 5, we deduce that there exists a homeomorphism h from Y onto X such that

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y. \tag{29}$$

It follows that T is a homomorphism. Hence, the proof is complete. □

We now study the onto case (not necessarily unit preserving).

Lemma 7. *Let $f \in C(X,\tau)$ with $\|f\|_X = 1$. Then, $|f(x)| = 1$ for all $x \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $g \in C(x,\tau)$ with $\|g\|_X \geq \varepsilon$ implies that*

$$\max\{\|f+g\|_X, \|f-g\|_X\} \geq 1 + \delta. \tag{30}$$

Proof. Assume that there exists $x_0 \in X$ such that $|f(x_0)| \neq 1$. Then, $|f(x_0)| < 1$ since $\|f\|_X = 1$. Let $\varepsilon = (1/3)(1 - |f(x_0)|)$. Then, $0 < \varepsilon < 1/3$ and $|f(x_0)| < 1 - 2\varepsilon$. Take $V = \{x \in X : |f(x)| < 1 - \varepsilon\}$. Since $f \in C(X,\tau)$, we deduce that V is a τ-invariant open subset of X and $\{x_0, \tau(x_0)\} \subseteq V$. By Urysohn's lemma, there exists $g_1 \in C(X)$ such that $0 \leq g_1(x) \leq 1$ for all $x \in X$, $g_1(x_0) = g_1(\tau(x_0)) = 1$, and $g_1(x) = 0$ for all $x \in X \backslash V$. Let $g = \varepsilon g_1(g_1 \circ \tau)$. Then, $g \in C(X,\tau)$, $0 \leq g(x) \leq \varepsilon$ for all $x \in X$, $g(x_0) = g(\tau(x_0)) = \varepsilon$, and $g(x) = 0$ for all $x \in X \setminus V$. Therefore,

$$\|f+g\|_X \leq 1, \qquad \|f-g\|_X \leq 1. \tag{31}$$

Hence, for each $\delta > 0$ we have

$$\max\{\|f+g\|_X, \|f-g\|_Y\} < 1 + \delta. \tag{32}$$

This completes the proof. □

Lemma 8. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$, and let $a = T1_X$. Then, $|a(y)| = 1$ for all $y \in Y$.*

Proof. Since $a \in C(Y,\eta)$ and $\|a\|_Y = \|T1_X\|_Y = \|1_X\|_X = 1$, by Lemma 7, it is sufficient to show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $g \in C(Y,\eta)$ with $\|g\|_Y \geq \varepsilon$ implies that

$$\max\{\|a+g\|_Y, \|a-g\|_X\} \geq 1 + \delta. \tag{33}$$

Let $\varepsilon > 0$, and let $g \in C(Y,\eta)$ with $\|g\|_Y \geq \varepsilon$. The surjectivity of T implies that there exists $f \in C(X,\tau)$ such that $Tf = g$. Since T is an isometry, we have $\|f\|_X = \|g\|_Y \geq \varepsilon$. This implies that there exists $x_0 \in X$ such that $|f(x_0)| \geq \varepsilon$. Therefore,

$$\begin{aligned} &\max\left\{|1 + f(x_0)|^2, |1 - f(x_0)|^2\right\} \\ &\quad \geq \frac{1}{2}\left[|1 + f(x_0)|^2 + |1 - f(x_0)|^2\right] \\ &\quad = 1 + |f(x_0)|^2 \\ &\quad \geq 1 + \varepsilon^2. \end{aligned} \tag{34}$$

It follows that

$$\max\{|1 + f(x_0)|, |1 - f(x_0)|\} \geq \sqrt{1+\varepsilon^2}. \tag{35}$$

Hence,

$$\max\{\|1_X + f\|_X, \|1_X - f\|_X\} \geq \sqrt{1+\varepsilon^2}. \tag{36}$$

Since $\|a+g\|_Y = \|1_X + f\|_X$ and $\|a-g\|_Y = \|1_X - f\|_X$, we have

$$\max\{\|a+g\|_Y, \|a-g\|_Y\} \geq \sqrt{1+\varepsilon^2}. \tag{37}$$

It is enough that we choose $\delta = \sqrt{1+\varepsilon^2} - 1$. □

In the following result, we show that every linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$ is a weighted composition operator.

Theorem 9. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$. Then, there exist a function $a \in C(Y,\eta)$ with $|a(y)| = 1$ for all $y \in Y$ and a homeomorphism h from Y onto X with $h \circ \eta = \tau \circ h$ on Y such that*

$$(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y. \quad (38)$$

Proof. Assume that $a = T1_X$. Then $a \in C(Y,\eta)$. By Lemma 8, we have $|a(y)| = 1$ for all $y \in Y$. Clearly, $(\overline{a})(Tf) \in C(Y,\eta)$ for all $f \in C(X,\tau)$ and $(\overline{a})(T1_X) = 1_Y$. We define the map $\overline{a}\cdot T : C(X,\tau) \to C(Y,\eta)$ by $(\overline{a}\cdot T)(f) = (\overline{a})(Tf)$. Then, $\overline{a}\cdot T$ is a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$, and $(\overline{a}\cdot T)(1_X) = 1_Y$. By Theorem 5, there exists a homeomorphism h from Y onto X with $h \circ \eta = \tau \circ h$ on Y such that

$$((\overline{a}\cdot T)(f))(y) = f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y. \quad (39)$$

This implies that

$$(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y. \quad (40)$$

Hence, the proof is complete. □

3. Into Linear Isometries

We formulate our main result in this section which is a version for into linear isometries of $C(X,\tau)$-spaces of a known Holsztyńskis's theorem (Theorem 2) for into linear isometries of $C_{\mathbb{F}}(X)$-spaces. We first study unit-preserving linear isometries from $C(X,\tau)$ into $C(Y,\eta)$.

Theorem 10. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ into $C(Y,\eta)$ satisfying the following conditions:*

(i) $T1_X = 1_Y$,

(ii) *if $y \in Y$ and there is a point $x \in X$ such that $\mathrm{Re}((Tf)(y)) = \mathrm{Re}(f(x))$ for all $f \in C(X,\tau)$, then $T(C(X,\tau))$ is extremely regular at $\{y, \tau(y)\}$.*

Then, there exist a η-invariant closed boundary Y_0 for $T(C(X,\tau))$ and a continuous map h from Y_0 onto X with $h \circ \eta = \tau \circ h$ on Y_0 such that

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y_0. \quad (41)$$

Proof. Let $E = T(C(X,\tau))$. Since $T : C(X,\tau) \to C(Y,\eta)$ is a linear isometry, we deduce that E is a uniformly closed linear subspace of $C(Y,\eta)$. Moreover, $T(1_X) = 1_Y$ implies that $1_Y \in E$. We define the map $S : C(X,\tau) \to E$ by $Sf = Tf$. Then, S is a linear isometry from the real Banach space $(C(X,\tau), \|\cdot\|_X)$ onto the real Banach space $(E, \|\cdot\|_Y)$. Therefore, S^* is a linear isometry from E^* onto $C(X,\tau)^*$ and so

$$S^*(\mathrm{Ext}(B_{E^*})) = \mathrm{Ext}(B_{C(X,\tau)^*}). \quad (42)$$

Now we define

$$Y_0 = \left\{ y \in Y : S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right) \in \mathrm{Ext}(B_{C(X,\tau)^*}) \right\}. \quad (43)$$

We first show that Y_0 is nonempty. To prove this fact, we show that the following statement, namely (I), holds.

(I) For each $x \in X$ there exists $y_0 \in Y$ such that $S^*(\varphi_{C(Y,\eta),y,1}|_E) = \varphi_{C(X,\tau),x,1}$.

Let $x \in X$. Then $\varphi_{C(X,\tau),x,1} \in \mathrm{Ext}(B_{C(X,\tau)^*})$ by Theorem 1. From (42), there exists $\Lambda \in \mathrm{Ext}(B_{E^*})$ such that

$$S^*\Lambda = \varphi_{C(X,\tau),x,1}. \quad (44)$$

Since E is a linear subspace of $C(Y,\eta)$, there exists $\widetilde{\Lambda} \in \mathrm{Ext}(B_{C(Y,\eta)^*})$ such that

$$\widetilde{\Lambda}|_E = \Lambda. \quad (45)$$

By Theorem 1, there exists $(y_0, \gamma_0) \in Y \times \mathbb{S}_{\mathbb{C}}$ such that

$$\widetilde{\Lambda} = \varphi_{C(X),y_0,\gamma_0}. \quad (46)$$

From (44), (45), and (46), we have

$$S^*\left(\varphi_{C(Y,\eta),y_0,\gamma_0}\big|_E\right) = \varphi_{C(X,\tau),x,1}. \quad (47)$$

Since $1_X \in C(X,\tau)$ and $T1_X = 1_Y$, by (47) we have

$$\begin{aligned}
\mathrm{Re}(\gamma_0) &= \mathrm{Re}(\gamma_0 1_Y(y_0)) = \varphi_{C(Y,\eta),y_0,\gamma_0}(1_Y) \\
&= \varphi_{C(Y,\eta),y_0,\gamma_0}(T1_X) = \varphi_{C(Y,\eta),y_0,\gamma_0}(S1_X) \\
&= S^*\left(\varphi_{C(Y,\eta),y_0,\gamma_0}\big|_E\right)(1_X) = \varphi_{C(X,\tau),x,1}(1_X) \\
&= \mathrm{Re}(1e_{C(X),x}(1_X)) = \mathrm{Re}(1_X(x)) \\
&= \mathrm{Re}(1) = 1.
\end{aligned} \quad (48)$$

This implies that $\gamma_0 = 1$ since $|\gamma_0| = 1$. Hence, by (47) we have

$$S^*\left(\varphi_{C(Y,\eta),y_0,1}\big|_E\right) = \varphi_{C(X,\tau),x,1}, \quad (49)$$

and so statement (I) holds.

Since $\varphi_{C(X,\tau),x,1} \in \mathrm{Ext}(B_{C(X,\tau)^*})$ for all x in X, statement (I) implies that Y_0 is nonempty.

We next show that the following statement, namely, (II), holds.

(II) For each $y \in Y_0$ there exists a unique point $x \in X$ such that $(Tf)(y) = f(x)$ for all f in $C(X,\tau)$.

Let $y \in Y_0$. Then, $S^*(\varphi_{C(Y,\eta),y,1}|_E) \in \mathrm{Ext}(B_{C(X,\tau)^*})$. Hence, by Theorem 1, there exists $(x_0, \lambda_0) \in X \times \mathbb{S}_{\mathbb{C}}$ such that

$$S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right) = \varphi_{C(X,\tau),x_0,\lambda_0}. \quad (50)$$

Since $1_X \in C(X,\tau)$ and $T1_X = 1_Y$, by (50) we have

$$\begin{aligned}
\mathrm{Re}(\lambda_0) &= \mathrm{Re}(\lambda_0 \cdot 1_X(x_0)) = \varphi_{C(X,\tau),x_0,\lambda_0}(1_X) \\
&= S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right)(1_X) = \varphi_{C(Y,\eta),y,1}(S1_X) \\
&= \varphi_{C(Y,\eta),y,1}(T1_X) = \varphi_{C(Y,\eta),y,1}(1_Y) \\
&= \mathrm{Re}\left(1e_{C(Y,\eta),y}(1_Y)\right) = \mathrm{Re}\left(e_{C(Y,\eta),y}(1_Y)\right) \\
&= \mathrm{Re}(1_Y(y)) = \mathrm{Re}(1) \\
&= 1.
\end{aligned} \quad (51)$$

This implies that $\lambda_0 = 1$ since $|\lambda_0| = 1$. Thus, by (50) we have

$$S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right) = \varphi_{C(X,\tau),x_0,1}. \tag{52}$$

From (52) we give

$$\operatorname{Re}\left((Sf)(y)\right) = \operatorname{Re}\left(f(x_0)\right), \quad \forall f \in C(X,\tau). \tag{53}$$

It follows that

$$\operatorname{Re}\left((Tf)(y)\right) = \operatorname{Re}\left(f(x_0)\right), \quad \forall f \in C(X,\tau). \tag{54}$$

According to (54) and condition (ii), E is extremely regular at $\{y, \eta(y)\}$. On the other hand, $C(X,\tau)$ is extremely regular at $\{x_0, \tau(x_0)\}$ by Lemma 4. Hence, by Lemma 3, we deduce that $(Sf)(y) = f(x_0)$ for all f in $C(X,\tau)$ or $(Sf)(y) = f(\tau(x_0))$ for all f in $C(X,\tau)$. We assume that $x = x_0$ whenever $(Sf)(y) = f(x_0)$ for all f in $C(X,\tau)$ and $x = \tau(x_0)$ whenever $(Sf)(y) = f(\tau(x_0))$ for all f in $C(X,\tau)$. Then, $x \in X$ and we have

$$(Tf)(y) = f(x), \quad \forall f \in C(X,\tau). \tag{55}$$

This proves the existent of x. To show uniqueness, assume that there exists $x' \in X$ such that

$$(Tf)(y) = f\left(x'\right), \quad \forall f \in C(X,\tau). \tag{56}$$

From (55) and (56) we have $f(x) = f(x')$ for all f in $C(X,\tau)$. This implies that $x = x'$ since $C(X,\tau)$ separates the points of X. Thus, statement (II) holds.

Now we define the map $h : Y_0 \to X$ by $h(y) = x$ whenever $(Tf)(y) = f(x)$ for all f in $C(X,\tau)$. The statement (II) implies that h is well-defined. By definition of h, we have

$$(Tf)(y) = f(h(y)), \quad \forall f \in C(X,\tau) \ \forall y \in Y_0. \tag{57}$$

We prove that $h \circ \eta = \tau \circ h$ in Y_0. Let $y \in Y_0$. For each $f \in C(X,\tau)$, by (57) we have

$$\begin{aligned} f\left((h\circ\eta)(y)\right) &= f\left(h(\eta(y))\right) = (Tf)(\eta(y)) \\ &= \overline{(Tf)(y)} = \overline{f(h(y))} \\ &= f\left(\tau(h(y))\right) = f\left((\tau\circ h)(y)\right). \end{aligned} \tag{58}$$

This implies that $(h\circ\eta)(y) = (\tau\circ h)(y)$ since $C(X,\tau)$ separates the points of X. Hence, $h \circ \eta = \tau \circ h$ on Y_0.

Continually, we show that Y_0 is a η-invariant. Let $y \in Y_0$. For each $f \in C(X,\tau)$ we have

$$\begin{aligned} S^*\left(\varphi_{C(Y,\eta),\eta(y),1}\big|_E\right)(f) &= \varphi_{C(Y,\eta),\eta(y),1}(Sf) \\ &= \varphi_{C(Y,\eta),\eta(y),1}(Tf) \\ &= \operatorname{Re}\left(1e_{C(Y),\eta(y)),1}(Tf)\right) \\ &= \operatorname{Re}\left((Tf)(\eta(y))\right) = \operatorname{Re}\left(\overline{(Tf)(y)}\right) \\ &= \operatorname{Re}\left(\overline{(f(h(y)))}\right) \\ &= \operatorname{Re}\left(1f(\tau(h(y)))\right) \\ &= \operatorname{Re}\left(1e_{C(X),\tau(h(y))}\big|_{C(X,\tau)}(f)\right) \\ &= \left(\operatorname{Re}\left(1e_{C(X),\tau(h(y))}\big|_{C(X,\tau)}\right)\right)(f) \\ &= \varphi_{C(X,\tau),\tau(h(y)),1}(f). \end{aligned} \tag{59}$$

This implies that

$$S^*\left(\varphi_{C(Y,\eta),\eta(y),1}\big|_E\right) = \varphi_{C(X,\tau),\tau(h(y)),1}. \tag{60}$$

Hence, $S^*(\varphi_{C(Y,\eta),y,1}|_E) \in \operatorname{Ext}(B_{C(X,\tau)^*})$ by Theorem 1. Therefore, $\eta(y) \in Y_0$ and so Y_0 is η-invariant.

Now we show that h is surjective. Let $x \in X$. According to statement (I), we deduce that there exists a point $y_0 \in Y_0$ such that

$$S^*\left(\varphi_{C(Y,\eta),y_0,1}\right) = \varphi_{C(X,\tau),\tau(x),1}. \tag{61}$$

It follows that $y_0 \in Y$ and we have

$$\operatorname{Re}\left((Tf)(y_0)\right) = \operatorname{Re}\left(f(x)\right), \quad \forall f \in C(X,\tau). \tag{62}$$

Statement (II) implies that there exists a point $x' \in X$ such that

$$(Tf)(y_0) = f\left(x'\right), \quad \forall f \in C(X,\tau). \tag{63}$$

From (62) and (63), we have $\operatorname{Re}(f(x)) = \operatorname{Re}(f(x'))$ for all f in $C(X,\tau)$. This implies that $x' \in \{x, \tau(x)\}$ since Re $C(X,\tau)$ separates the points of X/τ. If $x' = x$, then by (63) and definition of h we have $x = h(y_0)$. If $x' = \tau(x)$, then by (63) and definition of h we have

$$\begin{aligned} (Tf)(\eta(y_0)) &= \overline{(Tf)(y_0)} = \overline{f(x')} \\ &= f\left(\tau\left(x'\right)\right) = f(x), \end{aligned} \tag{64}$$

for all f in $C(X,\tau)$, and so $x = h(\eta(y_0))$. Therefore, h is surjective.

Continually, we show that Y_0 is a boundary for E. Let $g \in E$. Then, there exists a function $f \in C(X,\tau)$ such that $Tf = g$. Since f is a continuous complex-valued function on the compact space X, there exists a point $x_0 \in X$ such that $\|f\|_X =$

$|f(x_0)|$. The surjectivity of $h : Y_0 \to X$ implies that there exists a point $y_0 \in Y_0$ such that $h(y_0) = x_0$. Hence, $(Tf)(y_0) = f(x_0)$, and so

$$\begin{aligned}\|g\|_Y = \|Tf\|_Y = \|f\|_X &= |f(x_0)| \\ &= |(Tf)(y_0)| = |g(y_0)|.\end{aligned} \tag{65}$$

Therefore, Y_0 is a boundary for E.

We now check that $h : Y_0 \to X$ is continuous. Let $y \in Y_0$, and let $\{y_\alpha\}_{\alpha\in J}$ be a net in Y_0 such that $\lim_\alpha y_\alpha = y$ in Y_0. Then, for each $f \in C(X,\tau)$ we have $\lim_\alpha (Tf)(y_\alpha) = (Tf)(y)$ in X. This implies that for each $\lim_\alpha f(h(y_\alpha)) = f(h(y))$ in X. Since $C(X,\tau)$ separates the points of X, we have $\lim_\alpha h(y_\alpha) = h(y)$ in X. Therefore, $h : Y_0 \to X$ is continuous.

Finally, we show that Y_0 is closed in Y. Let $y \in \overline{Y_0}$, the closure of Y_0 in Y. Then, there exists a net $\{y_\alpha\}_{\alpha\in J}$ in Y_0 such that $\lim_\alpha y_\alpha = y$ in Y. Then, for each $f \in C(X,\tau)$ we have $\lim_\alpha (Tf)(y_\alpha) = (Tf)(y)$, and so $\lim_\alpha f(h(y_\alpha)) = (Tf)(y)$ in X. Since the net $\{f(h(y_\alpha))\}_{\alpha\in J}$ converges for all f in $C(X,\tau)$, we conclude that the net $\{h(y_\alpha)\}_{\alpha\in J}$ converges in X to a point $x \in X$. Hence, for each $f \in C(X,\tau)$ we have $\lim_\alpha f(h(y_\alpha)) = f(x)$. Therefore, $(Tf)(y) = f(h(y))$ for all f in $C(X,\tau)$. This implies that for each $f \in C(X,\tau)$ we have

$$\begin{aligned}S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right)(f) &= \left(\varphi_{C(Y,\eta),y,1}\big|_E\right)(Sf) \\ &= \left(\varphi_{C(Y,\eta),y,1}\right)(Tf) \\ &= \operatorname{Re}\left(1e_{C(Y),y}(Tf)\right) \\ &= \operatorname{Re}\left((Tf)(y)\right) \\ &= \operatorname{Re}\left(f(h(y))\right) \\ &= \operatorname{Re}\left(e_{C(X),h(y)}(f)\right) \\ &= \left(\operatorname{Re}\left(1e_{C(X),h(y)}\big|_{C(X,\tau)}\right)\right)(f) \\ &= \varphi_{C(X,\tau),h(y),1}(f).\end{aligned} \tag{66}$$

Hence,

$$S^*\left(\varphi_{C(Y,\eta),y,1}\big|_E\right) = \varphi_{C(X,\tau),h(y),1}. \tag{67}$$

This implies that $S^*(\varphi_{C(Y,\eta),y,1}|_E) \in \operatorname{Ext}(B_{C(X,\tau)^*})$, by Theorem 1. Therefore, $y \in Y_0$. So Y_0 is closed in Y. □

Note that if $T : C(X,\tau) \to C(Y,\eta)$ is a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$, then $T(C(X,\tau)) = C(Y,\eta)$ and so $T(C(X,\tau))$ is extremely regular at $\{y,\eta(y)\}$ for all y in Y. Therefore, if $T : C(X,\tau) \to C(Y,\eta)$ is a linear isometry from $C(X,\tau)$ on $C(Y,\eta)$ with $T(1_X) = 1_Y$, then conditions (i) and (ii) in Theorem 10 hold.

We now study the into case (not necessarily unit-preserving).

Theorem 11. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ into $C(Y,\eta)$ satisfying the following conditions:*

(i) $|T1_X(y)| = 1$ *for all* $y \in Y$,

(ii) *if $y \in Y$ and there is a point $x \in X$ such that $\operatorname{Re}(\overline{T1_X(y)}(Tf)(y)) = \operatorname{Re}(f(x))$ for all f in $C(X,\tau)$, then $(\overline{T1_X})T(C(X,\tau))$ is extremely regular at $\{y,\tau(y)\}$.*

Then, there exist a η-invariant closed boundary Y_0 for $T(C(X,\tau))$, a function $a \in C(Y,\eta)$ with $|a(y)| = 1$ for all $y \in Y$, and a continuous map h from Y_0 onto X with $h\circ\eta = \tau\circ h$ on Y_0 such that

$$(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X,\tau),\ \forall y \in Y_0. \tag{68}$$

Proof. Assume that $a = T1_X$. Then, $a \in C(Y,\eta)$, and by (i) we have $|a(y)| = 1$ for all $y \in Y$. Clearly, $(\overline{a})(Tf) \in C(Y,\eta)$ for all $C(Y,\eta)$, and $(\overline{a})(T1_X) = 1_Y$. We define the map $\overline{a}\cdot T : C(X,\tau) \to C(Y,\eta)$ by $(\overline{a}\cdot T)(f) = (\overline{a})(Tf)$. It is easy to see that $\overline{a}\cdot T$ is a linear isometry from $C(X,\tau)$ into $C(Y,\eta)$ such that $(\overline{a}\cdot T)(1_X) = 1_Y$.

Applying (ii), we can easily show that if $y \in Y$ and there exists a point $x \in X$ such that $\operatorname{Re}(((\overline{a}\cdot T)f)(y)) = \operatorname{Re}(f(x))$ for all f in $C(X,\tau)$, then $(\overline{a}\cdot T)(C(X,\tau))$ is extremely regular at $\{y,\eta(y)\}$. Let $E = T(C(X,\tau))$. Then, $\overline{a}E = (\overline{a}\cdot T)(C(X,\tau))$. We define $S : C(X,\tau) \to E$ by $Sf = Tf$ and $S_1 : C(X,\tau) \to \overline{a}E$ by $S_1 f = (\overline{a}\cdot T)f$. Clearly, S is a linear isometry from $C(X,\tau)$ onto E, and S_1 is a linear isometry from $C(X,\tau)$ onto $\overline{a}E$. Now, we define

$$Y_0 = \left\{y \in Y : S^*\left(\varphi_{C(Y,\eta),y,\overline{a(y)}}\big|_E\right) \in \operatorname{Ext}\left(B_{C(X,\tau)^*}\right)\right\}. \tag{69}$$

It is easy to see that

$$Y_0 = \left\{y \in Y : {S_1}^*\left(\varphi_{C(Y,\eta),y,1}\big|_{\overline{a}E}\right) \in \operatorname{Ext}\left(B_{C(X,\tau)^*}\right)\right\}. \tag{70}$$

By given arguments in the proof of Theorem 10, Y_0 is a η-invariant closed boundary for $\overline{a}E$, and there exists a continuous function h from Y_0 onto X with $h\circ\eta = \tau\circ h$ such that

$$((\overline{a}\cdot T)(f))(y) = f(h(y)), \quad \forall f \in C(X,\tau),\ \forall y \in Y_0. \tag{71}$$

We now show that Y_0 is a boundary for E. Let $g \in E$. Then, $\overline{a}g \in \overline{a}E$. Hence, there exists a point $y_0 \in Y_0$ such that

$$\|\overline{a}g\|_Y = |(\overline{a}g)(y_0)|. \tag{72}$$

Since $|a(y)| = 1$ for all $y \in Y$, we conclude that

$$\|\overline{a}g\|_Y = \|g\|_Y. \tag{73}$$

From (72) and (73), we have

$$\begin{aligned}\|g\|_Y = |\overline{a}g(y_0)| = |\overline{a}(y_0)g(y_0)| &= \left|\overline{a(y_0)}g(y_0)\right| \\ = \left|\overline{a(y_0)}\right||g(y_0)| = |a(y_0)||g(y_0)| &= |g(y_0)|.\end{aligned} \tag{74}$$

Therefore, Y_0 is a boundary for E.

Finally, we show that

$$(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X,\tau),\ \forall y \in Y_0. \tag{75}$$

Let $f \in C(X,\tau)$ and $y \in Y_0$. From (73), we have

$$((\overline{a} \cdot T)(f))(y) = f(h(y)). \tag{76}$$

Applying (76), we deduce that

$$\begin{aligned}(Tf)(y) &= a(y)\overline{a(y_0)}(Tf)(y) = a(y)((\overline{a})(Tf))(y) \\ &= a(y)((\overline{a} \cdot T)f)(y) = a(y)f(h(y)).\end{aligned} \tag{77}$$

Hence, the proof is complete. □

Note that if $T : C(X,\tau) \to C(Y,\eta)$ is a linear isometry from $C(X,\tau)$ onto $C(Y,\eta)$, then $|T1_x(y)| = 1$ for all y in Y by Lemma 8 and $T(C(X,\tau))$ is extremely regular at $\{y, \eta(y)\}$ for all y in Y by Lemma 4, and so the conditions (i) and (ii) in Theorem 11 hold.

The following result shows that the triple $\{Y_0, a, \eta\}$ associated to isometry T in Theorem 11 possesses a universal property.

Corollary 12. *Let $T : C(X,\tau) \to C(Y,\eta)$ be a linear isometry from $C(X,\tau)$ into $C(Y,\eta)$ satisfying the following conditions:*

(i) $|T1_X(y)| = 1$ *for all* $y \in Y$,

(ii) *if* $y \in Y$ *and there is a point* $x \in X$ *such that* $\mathrm{Re}(\overline{T1_X(y)}(Tf)(y)) = \mathrm{Re}(f(x))$ *for all* f *in* $C(X,\tau)$, *then* $T(C(X,\tau))$ *is extremely regular at* $\{y, \tau(y)\}$.

Let Y_0 and h be as in Theorem 11. If Y_1 is a nonempty η-invariant subset (not necessarily closed) of Y and $b \in C(Y,\eta)$ with $|b(y)| = 1$ for all $y \in Y_1$ and $k : Y_1 \to X$ is a continuous map (not necessarily onto) with $h \circ \eta = \tau \circ k$ on Y_1 such that

$$(Tf)(y) = b(y)f(k(y)), \quad \forall f \in C(X,\tau), \ \forall y \in Y_1, \tag{78}$$

then $Y_1 \subseteq Y_0$, $b = a|_{Y_1}$ and $k = h|_{Y_1}$.

Proof. Let $y \in Y_1$. Since $1_X \in C(X,\tau)$, taking $f = 1_X$ in the above expression of Tf, we have $b(y) = (T1_X)(y) = a(y)$. Thus, $b = a|_{Y_1}$.

Let $E = T(C(X,\tau))$ and $S : C(X,\tau) \to E$ defined by $Sf = Tf$. Then,

$$Y_0 = \left\{y \in Y : S^*\left(\varphi_{C(Y,\eta),y,\overline{a(y)}}\Big|_E\right) \in \mathrm{Ext}\left(B_{C(X,\tau)^*}\right)\right\}. \tag{79}$$

Let $y \in Y_1$. Then, the above expression of Tf reads as follows:

$$(Tf)(y) = b(y)f(k(y)), \quad \forall f \in C(X,\tau). \tag{80}$$

From (80), for each $f \in C(X,\tau)$ we have

$$\begin{aligned}S^*\left(\varphi_{C(Y,\eta),y,\overline{a(y)}}\Big|_E\right)(f) &= \left(\varphi_{C(Y,\eta),y,\overline{a(y)}}\Big|_E\right)(Tf) \\ &= \mathrm{Re}\left(\overline{a(y)}e_{C(Y),y}(Tf)\right) \\ &= \mathrm{Re}\left(\overline{a(y)}(Tf)(y)\right) \\ &= \mathrm{Re}\left(\overline{a(y)}a(y)f(k(y))\right) \\ &= \mathrm{Re}\left(|a(y)|^2 f(k(y))\right) \\ &= \mathrm{Re}(f(k(y))) \\ &= \left(\mathrm{Re}\left(1e_{C(X),k(y)}\right)\right)(f) \\ &= \varphi_{C(X,\tau),k(y),1}(f).\end{aligned} \tag{81}$$

This implies that

$$S^*\left(\varphi_{C(Y,\eta),y,\overline{a(y)}}\Big|_E\right) = \varphi_{C(X,\tau),k(y),1}. \tag{82}$$

From (82), we deduce that $S^*(\varphi_{C(Y,\eta),y,\overline{a(y)}}|_E)$ is an extreme point of $\mathrm{Ext}\, B_{C(X,\tau)^*}$ by Theorem 1. Therefore, $y \in Y_0$. So $Y_1 \subseteq Y_0$.

We now show that $k = h|_{Y_1}$. Let $y \in Y_1$. By (80), we have

$$(Tf)(y) = a(y)f(k(y)), \quad \forall f \in C(X,\tau). \tag{83}$$

Since $Y_1 \subseteq Y_0$, we deduce that $y \in Y_0$, and so by Theorem 11 we have

$$(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X,\tau). \tag{84}$$

Since $|a(y)| = 1$, by (83) and (84) we have

$$f(k(y)) = f(h(y)), \quad \forall f \in C(X,\tau). \tag{85}$$

This implies that $k(y) = h(y)$ since $C(X,\tau)$ separates the points of X. Thus, $k = h|_{Y_1}$. Hence, the proof is complete. □

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References

[1] S. H. Kulkarni and B. V. Limaye, "Extreme points of the unit ball in the dual spaces of some real subspaces of $C(X)$," *Glasnik Matematički*, vol. 29, no. 49, pp. 333–340, 1994.

[2] N. Dunford and J. T. Schwartz, *Linear Operators, Part 1, General Theory*, Interscience, New York, NY, USA, 1958.

[3] S. H. Kulkarni and B. V. Limaye, *Real Function Algebras*, Marcel Dekker, New York, NY, USA, 1992.

[4] S. H. Kulkarni and B. V. Limaye, "Gleason parts of real function algebras," *Canadian Journal of Mathematics*, vol. 33, no. 1, pp. 181–200, 1981.

[5] W. Holsztyński, "Continuous mappings induced by isometries of spaces of continuous function," *Studia Mathematica*, vol. 26, pp. 133–136, 1966.

[6] W. Novinger, "Linear isometries of subspaces of continuous functions," *Studia Mathematica*, vol. 53, pp. 273–276, 1975.

[7] S. H. Kulkarni and S. Arundhathi, "Isometries of real function algebras," *Commentationes Mathematicae*, vol. 30, no. 2, pp. 343–356, 1991.

[8] M. Cambern, "A Holsztynski theorem for spaces of continuous vector-valued functions," *Studia Mathematica*, vol. 63, no. 3, pp. 213–217, 1978.

[9] J.-S. Jeang and N.-C. Wong, "Weighted composition operators of $C_0(X)'$," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 3, pp. 981–993, 1996.

[10] J. Araujo and J. J. Font, "Linear isometries between subspaces of continuous functions," *Transactions of the American Mathematical Society*, vol. 349, no. 1, pp. 413–428, 1997.

[11] A. Jiménez-Vargas and M. Villegas-Vallecillos, "Into linear isometries between spaces of Lipschitz functions," *Houston Journal of Mathematics*, vol. 34, no. 4, pp. 1165–1184, 2008.

[12] S. Mazur and S. Ulam, "Sur les transformations isometriques d'espace vectoriels normes," *Comptes Rendus de l'Académie des Sciences*, vol. 194, pp. 946–948, 1932.