Linear Isometries between Real Banach Algebras of Continuous Complex-Valued Functions

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1. Introduction and Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers, respectively. The symbol $\mathbb{F}$ denotes a field that can be either $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{F}$ are called scalars. We also denote by $S_\mathbb{F}$ the set of all $\lambda \in \mathbb{F}$ with $|\lambda| = 1$.

Let $\mathcal{X}$ be a normed space over $\mathbb{F}$. We denote by $\mathcal{X}^*$ and $B_\mathcal{X}$ the dual space of $\mathcal{X}$ and the closed unit ball of $\mathcal{X}$, respectively. For a subset $E$ of $\mathcal{X}$, let $\operatorname{Ext}(E)$ denote the set of all extreme points of $\mathcal{X}$. Kulkarni and Limayesh showed [1, Theorem 2] that if $A$ is a nonzero linear subspace of $\mathcal{X}$ and $\varphi \in \operatorname{Ext}(B_A^*)$, then $\varphi$ has an extension to some $\psi \in \operatorname{Ext}(B_{\mathcal{X}}^*)$.

We know that if $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces over $\mathbb{F}$ and $T : \mathcal{X} \to \mathcal{Y}$ is a linear isometry from $\mathcal{X}$ onto $\mathcal{Y}$ over $\mathbb{F}$, then $T$ is a bijection mapping between $\operatorname{Ext}(B_\mathcal{X})$ and $\operatorname{Ext}(B_\mathcal{Y})$.

Let $X$ be a compact Hausdorff space. We denote by $\mathcal{C}_X(X)$ the unital commutative Banach algebra of all continuous functions from $X$ into $\mathbb{F}$, with the uniform norm $\|f\|_X = \sup \{|f(x)| : x \in X\}$, $f \in \mathcal{C}_X(X)$. We write $\mathcal{C}_X$ instead as $\mathcal{C}_X(X)$. For $x \in X$, we consider the linear functional $e_{\mathcal{C}_X(x)}$ on $\mathcal{C}_X(X)$ defined by $e_{\mathcal{C}_X(x)}(f) = f(x)$ ($f \in \mathcal{C}_X(X)$), which is called the evaluation functional on $\mathcal{C}_X(X)$ at $x$. Clearly, $\lambda e_{\mathcal{C}_X(x)} \in B_{\mathcal{C}_X(X)^*}$ for all $(x, \lambda) \in X \times \mathbb{F}$. It is known [2, page 441] that

$$\operatorname{Ext}(B_{\mathcal{C}_X(X)^*}) = \{\lambda e_{\mathcal{C}_X(x)} : (x, \lambda) \in X \times S_\mathbb{F}\}. \tag{1}$$

Let $A$ be a real or complex linear subspace of $\mathcal{C}_X(X)$. A nonempty subset $S$ of $X$ is called a boundary for $A$ (with respect to $X$), if for each $f \in A$ the function $|f|$ assumes its maximum on $S$ at some $x \in S$. We denote by $\Gamma(A, X)$ the intersection of all closed boundaries for $A$. If $\Gamma(A, X)$ is a boundary for $A$, it is called the Shilov boundary for $A$ (with respect to $X$).

Let $A$ be linear subspace of $\mathcal{C}_X(X)$ containing $1_X$, the constant function with value 1 on $X$. A representing measure for $\varphi \in A^*$ is an $\mathbb{F}$-valued regular Borel measure $\mu$ on $X$ such that $\varphi(f) = \int_X f \, d\mu$ for all $f \in A$. Let $K_\varphi(A) = \{\varphi \in A^* : \|\varphi\| = \varphi(1_X) = 1\}$. If $x \in X$, then $e_{\mathcal{C}_X(x), A} \in K_\varphi(A)$ and $\delta_x$, the point mass measure on $X$ at $x$, is a representing measure for $e_{\mathcal{C}_X(x), A}$. We denote by $\operatorname{Ch}(A, X)$ the set of all $x \in X$ for which $\delta_x$ is the only representing measure for $e_{\mathcal{C}_X(x), A}$. If $\operatorname{Ch}(A, X)$ is a boundary for $A$, it is called the Choquet boundary for $A$ (with respect to $X$). We know that

$$\Gamma(\mathcal{C}_X(X), X) = \operatorname{Ch}(\mathcal{C}_X(X), X) = X. \tag{2}$$

Let $A$ be a real linear subspace of $\mathcal{C}(X)$, and let $S$ be a nonempty subset of $X$. We say that $A$ is extremely regular at $S$ if for every open neighborhood $U$ of $S$ and for each $\varepsilon > 0$ there is a function $f \in A$ with $\|f\|_X = 1$ such that $f(x) = 1$ for all $x \in S$ and $|f(y)| < 1$ for all $y \in X \setminus U$. 


Let $X$ be a nonempty set. A self-map $\tau : X \to X$ is called an involution on $X$ if $\tau(\tau(x)) = x$ for all $x \in X$. A subset $S$ of $X$ is called $\tau$-invariant if $\tau(S) \subseteq S$. Clearly, if $S \subseteq X$ is $\tau$-invariant, then $\tau(S) = S$. A $\tau$-invariant measure on $E$ is a measure $\mu$ on $E$ such that $\mu \circ \tau = \mu$.

Let $X$ be a topological space. An involution $\tau$ on $X$ is called a topological involution on $X$, if $\tau$ is continuous.

Let $X$ be a compact Hausdorff space, let $\sigma$ be a topological involution on $X$. Then, $(X, \sigma)$ is a unital uniformly closed self-adjoint real subalgebra of $C(X)$ which separates the points of $X$ and does not contain $1_X$, the constant function with value $1$ on $X$. Note that $\mathcal{R} f \in C(X, \sigma)$ for all $f \in C(X, \sigma)$. Moreover, $C(X, \sigma) = C(X, \sigma) \oplus iC(X, \sigma)$ and $\max \{ \| f \|_{L_p}, \| g \|_{L_p} \} \leq 2 \max \{ \| f \|_{L_p}, \| g \|_{L_p} \}$.

The real Banach algebra $C(X, \tau)$ and its real linear subspaces were first considered by Kulkarni and Limaye in [4]. For a detailed account of several properties of $C(X, \tau)$, we refer to [3].

Let $X$ be a compact Hausdorff space. For each $(x, \lambda) \in X \times \mathbb{C}$, define the map $\psi_{C(X), x, \lambda} : C(X, \tau) \to \mathbb{R}$ by

$$\psi_{C(X), x, \lambda}(f) = \Re(f(x)),$$

in fact, $\psi_{C(X), x, \lambda} = \Re(\lambda e_{(x, \lambda)})$. Clearly, $\psi_{C(X), x, \lambda} \in B(C(X))$.

Let $\tau$ be a topological involution on $X$. For each $(x, \lambda) \in X \times \mathbb{C}$ define the map $\psi_{C(X), x, \lambda} : C(X, \tau) \to \mathbb{R}$ by $\psi_{C(X), x, \lambda}(f) = \psi_{C(X), x, \lambda}(f(x))$. Kulkarni and Limaye showed [1, Proposition 3] that

$$\psi_{C(X), x, \lambda} = \psi_{C(X), x, \lambda}(f) \in X \times S(C).$$

Applying this result, they obtained the following theorem.

**Theorem 1 (see [1, Corollary 5]).** Let $X$ be a compact Hausdorff space, and let $\tau$ be a topological involution on $X$. Suppose that

$$P_{X, \tau} = \{ (x, \lambda) \in X \times S(C) : \tau(x) \neq x \} \cup \{ (x, \lambda) \in X \times S(C) : \tau(x) = x \}.$$

Then

$$\psi_{C(X), x, \lambda} = \psi_{C(X), x, \lambda}(f) \in X \times S(C).$$

The classical Banach-Stone theorem states that if $T$ is a linear isometry from $C_T(X)$ onto $C_T(Y)$, then there exists a homeomorphism $h$ from $Y$ onto $X$ and a continuous function $a \in C_T(Y)$ with $a(Y) \subseteq S_T$ such that

$$(Tf)(y) = a(y)f(h(y)), \quad (f \in C_T(X), y \in Y).$$

This well-known theorem has been generalized in several directions. In 1966, an important generalization was given by Holsztynski [5] by considering into linear isometries as the following.

**Theorem 2.** Let $X$ and $Y$ be compact Hausdorff spaces. If $T$ is a linear isometry from $C_T(X)$ onto $C_T(Y)$, then there exists a closed subset $Y_0$ of $Y$, a continuous map $q$ from $Y_0$ onto $X$ and a function $a \in C_T(Y)$ with $\| a \| = 1$ and $a(Y_0) \subseteq S_T$ such that

$$(Tf)(y) = a(y)f(h(y)), \quad (f \in C_T(X), y \in Y_0).$$

In 1975, another important generalization of Banach-Stone theorem was given by Novinger [6] for certain complex subalgebras of $C(X)$ and $C(Y)$ considering the concept of their Choquet boundaries. In 1991, Kulkarni and Arundhati [7] generalized the given results by Novinger for certain real subalgebras of $C(X, \tau)$ and $C(Y, \eta)$ considering the concept of their Choquet boundaries. We can cite some other generalizations of Banach-Stone theorem, for example, the generalization obtained by Cambern [8] for space of vector-valued continuous functions, by Jeang and Wong [9] for spaces of scalar-valued continuous functions vanishing.

Throughout the rest of this paper, X and Y are compact Hausdorff spaces and r and η are topological involutions on X and Y, respectively.

In this paper, we characterize all linear isometries from $C(X, r)$ onto $C(Y, η)$ and certain linear isometries from $C(X, r)$ into $C(Y, η)$, applying the extreme points in $B_{C(X,r)'}$ and $B_{C(Y,η)'}$. Our results in Section 2 are some of the given results by Kulkarni and Arundhathi in [7] that we obtain by applying the extreme points in $B_{C(X,r)'}$ and $B_{C(Y,η)'}$. The main result in Section 3 is a generalization of the given result by Holstynski in [5] for certain into isometries.

2. Onto Linear Isometries

We first determine unit-preserving linear isometries from $C(X, r)$ onto $C(Y, η)$. For this purpose, we need the following lemmas.

Lemma 3 (see [3, Lemma 5.1.1]). Let A and B be real linear subspaces of C(X, r) and C(Y, η), respectively. Suppose that $x ∈ X$, $y ∈ Y$, A is extremely regular at $[x, r(x)]$ and B is extremely regular at $[y, η(y)]$. Let $T : A → B$ be a linear isometry from A onto B and $Re(Tf)(y) = Re(f(x))$ for all $f ∈ A$. Then, $(Tf)(y) = f(x)$ for all $f ∈ A$ or $(Tf)(y) = f(r(x))$ for all $f ∈ A$.

Lemma 4. Let S be a nonempty r-invariant closed subset of X. Then, C(X, r) is extremely regular at $[x, r(x)]$ for all x in X.

Proof. Assume that $ε > 0$ and U is an open neighborhood of S in X. By Urysohn’s lemma, there exists a function $g ∈ C(X)$ with $0 ≤ g(x) ≤ 1$ for all $x ∈ X$, $g(x) = 1$ for all $x ∈ S$, and $g(x) = 0$ for all $x ∈ X \setminus U$. Define the function $f : X → C$ by $f = (g ∗ r)g$. Then, $f ∈ C(X, r)$, $∥f∥_X = 1$, $f(x) = 1$ for all $x ∈ S$ and $|f(x)| < ε$ for all $x ∈ X \setminus U$. Hence, C(X, r) is extremely regular at S.

Theorem 5. Let $T : C(X, r) → C(Y, η)$ be a linear isometry from $C(X, r)$ onto $C(Y, η)$ with $T1_X = 1_Y$. Then, there exists a homeomorphism h from Y onto X with $h ∗ η = r ∗ h$ on Y such that

$$(Tf)(y) = f(h(y)), \quad ∀f ∈ C(X, r), \quad ∀y ∈ Y. \quad (20)$$

Proof. We claim that

$$Y = \{y ∈ Y : T^*(φ_{C(Y,η), y, 1}) ∈ Ext(B_{C(X,r)})\}. \quad (13)$$

Let $y ∈ Y$. By Theorem 1, $φ_{C(Y,η), y, 1} ∈ Ext(B_{C(Y,η)'}).$ Since $T$ is a linear isometry from $C(X, r)$ onto $C(Y, η)$, we conclude that $T^*$ is a linear isometry from $C(Y, η)^*$ onto $C(X, r)^*$ and so

$$T^*(Ext(B_{C(Y,η)'}) = Ext(B_{C(X,r)^*}). \quad (14)$$

Therefore, $T^*(φ_{C(Y,η), y, 1}) ∈ Ext(B_{C(X,r)})$. Hence, our claim is justified.

We now show that for each $y ∈ Y$ there exists a unique $x ∈ X$ such that

$$(Tf)(y) = f(x), \quad ∀f ∈ C(X, r). \quad (15)$$

Let $y ∈ Y$. By (13), we have $T^*(φ_{C(Y,η), y, 1}) ∈ Ext(B_{C(X,r)})$. Hence, by Theorem 1, there exists $(x_0, λ_0) ∈ X × S_C$ such that

$$T^*(φ_{C(Y,η), y, 1}) = φ_{C(X,r), x_0, 1}. \quad (16)$$

Since $1_X ∈ C(X, r)$ and $T1_X = 1_Y$, by (16), we have

$$Re(λ_0) = Re(λ_0 1_X(x_0)) = φ_{C(X,r), x_0, 1}(1_X) = T^*(φ_{C(Y,η), y, 1})(1_X) = φ_{C(Y,η), y, 1}(1_Y) = Re(1_Y(y) φ_{C(Y,η), y, 1}(1_Y)) = Re(1_Y(y) 1_Y(y)) = Re(1) = 1.$$

Since $|λ_0| = 1$ and $Re(λ_0) = 1$, we have $Im(λ_0) = 0$ and so $λ_0 = 1$. Thus, by (16), we have

$$T^*(φ_{C(Y,η), y, 1}) = φ_{C(X,r), x_0, 1}. \quad (18)$$

From (18) we deduce that

$$Re((Tf)(y)) = Re(f(x_0)), \quad ∀f ∈ C(X, r). \quad (19)$$

Since $T$ is a linear isometry from $C(X, r)$ onto $C(Y, η)$ and by Lemma 4, $C(X, r)$ is extremely regular at $[x_0, τ(x_0)]$, and $C(Y, η)$ is extremely regular at $[y, η(y)]$, we conclude that $(Tf)(y) = f(x_0)$ for all $f ∈ C(X, r)$ or $(Tf)(y) = f(r(x_0))$ for all $f ∈ C(X, r)$ by Lemma 3. We assume that $x = x_0$ whenever $(Tf)(y) = f(x_0)$ for all $f ∈ C(X, r)$ and $x = τ(x_0)$ whenever $(Tf)(y) = f(r(x_0))$ for all $f ∈ C(X, r)$. Then, $x ∈ X$ and we have

$$(Tf)(y) = f(x), \quad ∀f ∈ C(X, r). \quad (20)$$

This proves the existence of x. To show uniqueness, assume that there exists $x' ∈ X$ such that

$$(Tf)(y) = f(x'), \quad ∀f ∈ C(X, r). \quad (21)$$

Then, $f(x) = f(x')$ for all $f ∈ C(X, r).$ This implies that $x = x'$ since $C(X, r)$ separates the points of X.

Now we define the map $h : Y → X$ by $h(y) = x$ whenever $(Tf)(y) = f(x)$ for all $f ∈ C(X, r)$. In fact, we have

$$(Tf)(y) = f(h(y)), \quad ∀f ∈ C(X, r) \forall y ∈ Y. \quad (22)$$

Now we prove that $h ∗ η = r ∗ h$ on Y. Let $y ∈ Y$. Then, $h(y) ∈ Y$. From (22), for each $f ∈ C(X, r)$ we have

$$f ((h ∗ η)(y)) = f (h(η(y))) = (Tf)(η(y)) = (Tf)(η(y)) = f (h(y)) = f (h(y)) = f (r(h(y))) = f ((r ∗ h)(y)). \quad (23)$$
This implies that \((h \circ \eta)(y) = (\tau \circ h)(y)\) since \(C(X, \tau)\) separates the points of \(X\). Therefore, \(h \circ \eta = \tau \circ h\) on \(Y\).

Continuously, we prove that \(h\) is bijective. Since \(T^{-1}\) is a linear isometry from \(C(Y, \eta)\) onto \(C(X, \tau)\) with \(T^{-1}1_Y = 1_X\), by the above argument, we deduce that there exists a map \(k : X \to Y\) such that
\[
(T^{-1} g)(x) = g(k(x)), \quad \forall g \in C(Y, \eta), \quad \forall x \in X. \tag{24}
\]
To prove the injectivity of \(h\), let \(y \in Y\). For each \(g \in C(Y, \eta)\), by (22) and (24) we have
\[
g((k \circ h)(y)) = g(k(h(y))) = (T^{-1}g)(h(y)) = (T(T^{-1}g))(y) = g(y). \tag{25}
\]
This implies that \((k \circ h)(y) = y\) since \(C(Y, \eta)\) separates the points of \(Y\). Therefore, \(k \circ h = I_Y\), the identity mapping on \(Y\), and so \(h\) is injective.

To prove that \(h\) is onto, let \(x \in X\). For each \(f \in C(X, \tau)\), by (24) and (22) we have
\[
f((h \circ k)(x)) = f(h(k(x))) = (Tf)(k(x)) = (T^{-1}(Tf))(x) = fx. \tag{26}
\]
This implies that \((h \circ k)(x) = x\) since \(C(X, \tau)\) separates the points of \(X\). Therefore \(k \circ h = I_X\), the identity mapping on \(X\), and so \(h\) is onto.

We now check that \(h : Y \to X\) is continuous. Let \(y \in Y\), and let \(\{y_n\}_{n \in \mathbb{N}}\) be a net in \(Y\) such that \(\lim_n y_n = y\) in \(Y\). Then we have
\[
\lim_{\alpha} (Tf)(y_\alpha) = (Tf)(y), \quad \forall f \in C(X, \tau). \tag{27}
\]
By (27) and definition of \(h\), we deduce that
\[
\lim_{\alpha} f(h(y_\alpha)) = f(h(y)), \quad \forall f \in C(X, \tau). \tag{28}
\]
Since \(C(X, \tau)\) separates the points of \(X\), by (28), we have \(\lim_{\alpha} h(y_\alpha) = h(y)\) in \(X\). Therefore, \(h : Y \to X\) is continuous.

Since \(h : Y \to X\) is a bijective continuous mapping, \(Y\) is a compact space, and \(X\) is a Hausdorff space, we conclude that \(h^{-1} : X \to Y\) is continuous, and so \(h\) is a homeomorphism. Hence, the proof is complete.

**Corollary 6.** Let \(T : C(X, \tau) \to C(Y, \eta)\) be an isometry mapping from \(C(X, \tau)\) onto \(C(Y, \eta)\) with \(T(1) = 0\) and \(T1_X = 1_Y\). Then \(T\) is an isomorphism.

**Proof.** Since \((C(X, \tau), \|\cdot\|_X)\) and \((C(Y, \eta), \|\cdot\|_Y)\) are real Banach algebras and \(T\) is an isometry from \(C(X, \tau)\) onto \(C(Y, \eta)\) with \(T(0) = 0\), we conclude that \(T\) is a linear isometry by Mazur-Ulam theorem (see [12]). Since \(T(1)_X = 1_Y\), applying Theorem 5, we deduce that there exists a homeomorphism \(h\) from \(Y\) onto \(X\) such that
\[
(Tf)(y) = f(h(y)), \quad \forall f \in C(X, \tau), \quad \forall y \in Y. \tag{29}
\]
It follows that \(T\) is a homomorphism. Hence, the proof is complete.

We now study the onto case (not necessarily unit preserving).

**Lemma 7.** Let \(f \in C(X, \tau)\) with \(|\|f\|_X = 1\). Then, \(|f(x)| = 1\) for all \(x \in X\), if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(g \in C(X, \tau)\) with \(|\|g\|_X \geq \varepsilon\) implies that
\[
\max \{||f + g||_X, ||f - g||_X\} \geq 1 + \delta. \tag{30}
\]
**Proof.** Assume that there exists \(x_0 \in X\) such that \(|f(x_0)| \neq 1\). Then, \(|f(x_0)| < 1\) since \(|\|f\|_X = 1\). Let \(\varepsilon = (1/3)(1 - |f(x_0)|)\). Then, \(0 < \varepsilon < 1/3\) and \(|f(x_0)| < 1 - 2\varepsilon\). Take \(V = \{x \in X : |f(x)| < 1 - \varepsilon\}. Since f \in C(X, \tau), we deduce that \(V\) is a \(T\)-invariant open subset of \(X\) and \(\{x_0, \tau(x_0)\} \in V\). By Urysohn's lemma, there exists \(g_1 \in C(X, \tau)\) such that \(0 \leq g_1(x) \leq 1\) for all \(x \in X, g_1(x_0) = g_1(\tau(x_0)) = 1,\) and \(g_1(x) = 0\) for all \(x \in X \setminus V\). Let \(g = \varepsilon g_1(\tau \circ \tau)\). Then, \(g \in C(X, \tau), 0 \leq g(x) \leq \varepsilon\) for all \(x \in X, g(x_0) = g(\tau(x_0)) = \varepsilon,\) and \(g(\tau(x_0)) = 0\) for all \(x \in X \setminus V\). Therefore,
\[
\|f + g\|_X \leq 1, \quad \|f - g\|_X \leq 1. \tag{31}
\]
Hence, for each \(\delta > 0\) we have
\[
\max \{||f + g||_X, ||f - g||_Y\} < 1 + \delta. \tag{32}
\]
This completes the proof.

**Lemma 8.** Let \(T : C(X, \tau) \to C(Y, \eta)\) be a linear isometry from \(C(X, \tau)\) onto \(C(Y, \eta)\), and let \(a = T1_X\). Then, \(|a(y)| = 1\) for all \(y \in Y\).

**Proof.** Since \(a \in C(Y, \eta)\) and \(|\|a\|_Y = \|T1_X\|_Y = \|1_X\|_X = 1\), by Lemma 7, it is sufficient to show that for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(g \in C(Y, \eta)\) with \(|\|g\|_Y \geq \varepsilon\) implies that
\[
\max \{||a + g||_Y, ||a - g||_Y\} \geq 1 + \delta. \tag{33}
\]
Let \(\varepsilon > 0\), and let \(g \in C(Y, \eta)\) with \(|\|g\|_Y \geq \varepsilon\). The surjectivity of \(T\) implies that there exists \(f \in C(X, \tau)\) such that \(Tf = g\). Since \(T\) is an isometry, we have \(|\|f\|_X = \|g\|_Y \geq \varepsilon\). This implies that there exists \(x_0 \in X\) such that \(|f(x_0)| \geq \varepsilon\). Therefore,
\[
\max \{1 + f(x_0), 1 - f(x_0)\} \geq 1 + \varepsilon. \tag{34}
\]
It follows that
\[
\max \{1 + f(x_0), 1 - f(x_0)\} \geq \sqrt{1 + \varepsilon^2}. \tag{35}
\]
Hence,
\[
\|1_X + f\|_X \|1_X - f\|_X \geq \sqrt{1 + \varepsilon^2}. \tag{36}
\]
Since \(|a + g|_Y = \|1_X + f\|_X\) and \(|a - g|_Y = \|1_X - f\|_X\), we have
\[
\max \{||a + g||_Y, ||a - g||_Y\} \geq \sqrt{1 + \varepsilon^2}. \tag{37}
\]
It is enough that we choose \(\delta = \sqrt{1 + \varepsilon^2} - 1\).
In the following result, we show that every linear isometry from \( C(X, \tau) \) onto \( C(Y, \eta) \) is a weighted composition operator.

**Theorem 9.** Let \( T : C(X, \tau) \rightarrow C(Y, \eta) \) be a linear isometry from \( C(X, \tau) \) onto \( C(Y, \eta) \). Then, there exist a function \( a \in C(Y, \eta) \) with \( |a(y)| = 1 \) for all \( y \in Y \) and a homeomorphism \( h \) from \( Y \) onto \( X \) with \( h \circ \eta = \tau \circ h \) on \( Y \) such that

\[
(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X, \tau), \quad \forall y \in Y.
\]

**Proof.** Assume that \( a = T1_X \). Then \( a \in C(Y, \eta) \). By Lemma 8, we have \( |a(y)| = 1 \) for all \( y \in Y \). Clearly, \( \overline{a(Tf)} \in C(Y, \eta) \) for all \( f \in C(X, \tau) \) and \( \overline{a(T1_X)} = 1_Y \). We define the map \( \overline{a \cdot T} : C(X, \tau) \rightarrow C(Y, \eta) \) by \( \overline{a \cdot T}(f) = \overline{a(Tf)} \). Then, \( \overline{a \cdot T} \) is a linear isometry from \( C(X, \tau) \) onto \( C(Y, \eta) \), and \( \overline{a \cdot T}(1_X) = 1_Y \). By Theorem 5, there exists a homeomorphism \( h \) from \( Y \) onto \( X \) with \( h \circ \eta = \tau \circ h \) on \( Y \) such that

\[
(\overline{a \cdot T})(f)(y) = f(h(y)), \quad \forall f \in C(X, \tau), \quad \forall y \in Y.
\]

This implies that

\[
(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X, \tau), \quad \forall y \in Y.
\]

Hence, the proof is complete. \( \Box \)

### 3. Into Linear Isometries

We formulate our main result in this section which is a version for into linear isometries of \( C(X, \tau) \)-spaces of a known Holsztynskii’s theorem (Theorem 2) for into linear isometries of \( C_F(X) \)-spaces. We first study unit-preserving linear isometries from \( C(X, \tau) \) into \( C(Y, \eta) \).

**Theorem 10.** Let \( T : C(X, \tau) \rightarrow C(Y, \eta) \) be a linear isometry from \( C(X, \tau) \) into \( C(Y, \eta) \) satisfying the following conditions:

(i) \( T1_X = 1_Y \),

(ii) if \( y \in Y \) and there is a point \( x \in X \) such that \( \text{Re}((Tf)(y)) = \text{Re}(f(x)) \) for all \( f \in C(X, \tau) \), then \( T(C(X, \tau)) \) is extremely regular at \( (y, \tau(y)) \).

Then, there exists a \( \eta \)-invariant closed boundary \( Y_0 \) for \( T(C(X, \tau)) \) and a continuous map \( h \) from \( Y_0 \) onto \( X \) with \( h \circ \eta = \tau \circ h \) on \( Y_0 \) such that

\[
(Tf)(y) = f(h(y)), \quad \forall f \in C(X, \tau), \quad \forall y \in Y_0.
\]

**Proof.** Let \( E = T(C(X, \tau)) \). Since \( T : C(X, \tau) \rightarrow C(Y, \eta) \) is a linear isometry, we deduce that \( E \) is a uniformly closed linear subspace of \( C(Y, \eta) \). Moreover, \( T1_X = 1_Y \) implies that \( 1_Y \in E \). We define the map \( S : C(X, \tau) \rightarrow E \) by \( Sf = T\overline{f} \). Then, \( S \) is a linear isometry from the real Banach space \( (C(X, \tau), \| \cdot \|_X) \) onto the real Banach space \( (E, \| \cdot \|_Y) \). Therefore, \( S^* \) is a linear isometry from \( E^* \) onto \( C(X, \tau) \) and so

\[
S^*(\text{Ext}(B_{E^*})) = \text{Ext}(B_{C(X, \tau)^*}).
\]

Now we define

\[
Y_0 = \{ y \in Y : S^* \left( \phi_{C(Y, \eta), y, 1}_{E^*} \right) \in \text{Ext}(B_{C(X, \tau)^*}) \}.
\]

We first show that \( Y_0 \) is nonempty. To prove this fact, we show that the following statement, namely (I), holds.

(I) For each \( x \in X \) there exists \( y_0 \in Y \) such that

\[
S^* \left( \phi_{C(Y, \eta), y, 1}_{E^*} \right) = \phi_{C(X, \tau), x, 1}_{E^*}.
\]

Let \( x \in X \). Then \( \phi_{C(Y, \eta), y, 1}_{E^*} \in \text{Ext}(B_{C(X, \tau)^*}) \) by Theorem 1. From (42), there exists \( \Lambda \in \text{Ext}(B_{E^*}) \) such that

\[
S^* \Lambda = \phi_{C(X, \tau), x, 1}_{E^*}.
\]

Since \( E \) is a linear subspace of \( C(Y, \eta) \), there exists \( \tilde{\Lambda} \in \text{Ext}(B_{C(Y, \eta)^*}) \) such that

\[
\Lambda |_{E} = \tilde{\Lambda}.
\]

By Theorem 1, there exists \( (y_0, y_0) \in Y \times S_Y \) such that

\[
\tilde{\Lambda} = \phi_{C(X, \tau), x_0, y_0}. \quad (46)
\]

From (44), (45), and (46), we have

\[
S^* \left( \phi_{C(Y, \eta), y_0, 1}_{E^*} \right) = \phi_{C(X, \tau), x_0, 1}_{E^*}. \quad (47)
\]

Since \( 1_X \in C(X, \tau) \) and \( T1_X = 1_Y \), by (47) we have

\[
\text{Re}(y_0) = \text{Re}(y_0 1_Y) = \phi_{C(Y, \eta), y_0, 1}_{E^*} (1_Y) = \phi_{C(Y, \eta), y_0, 1}_{E^*} (S1_X) = S^* \left( \phi_{C(Y, \eta), y_0, 1}_{E^*} (1_X) \right) = \phi_{C(X, \tau), x_0, 1} (1_X) = \phi_{C(X, \tau), x_0, 1} (1_X) \quad (48)
\]

\[
= \text{Re} (1 e_{C(Y, \eta), x_0} (1_X)) = \text{Re} (1 (1_X)) = (1) = 1.
\]

This implies that \( y_0 = 1 \) since \( |y_0| = 1 \). Hence, by (47) we have

\[
S^* \left( \phi_{C(Y, \eta), y_0, 1}_{E^*} \right) = \phi_{C(X, \tau), x_0, 1} \quad (49)
\]

and so statement (I) holds.

Since \( \phi_{C(X, \tau), x_0, 1} \in \text{Ext}(B_{C(X, \tau)^*}) \) for all \( x \) in \( X \), statement (I) implies that \( Y_0 \) is nonempty.

We next show that the following statement, namely, (II), holds.

(II) For each \( y \in Y_0 \) there exists a unique point \( x \in X \) such that \( (Tf)(y) = f(x) \) for all \( f \in C(X, \tau) \).

Let \( y \in Y_0 \). Then, \( S^* \left( \phi_{C(Y, \eta), y, 1}_{E^*} \right) \in \text{Ext}(B_{C(X, \tau)^*}) \). Hence, by Theorem 1, there exists \( (x_0, \lambda_0) \in X \times S_X \) such that

\[
S^* \left( \phi_{C(Y, \eta), y, 1}_{E^*} \right) = \phi_{C(X, \tau), x_0, \lambda_0} \quad (50)
\]

Since \( 1_X \in C(X, \tau) \) and \( T1_X = 1_Y \), by (50) we have

\[
\text{Re}(1) = \text{Re}(1 1_Y) = \phi_{C(Y, \eta), y_0, 1}_{E^*} (1_Y) = \phi_{C(Y, \eta), y_0, 1}_{E^*} (S1_X) = S^* \left( \phi_{C(Y, \eta), y_0, 1}_{E^*} (1_X) \right) = \phi_{C(X, \tau), x_0, 1} (1_Y) \quad (51)
\]

\[
= \text{Re} (1 e_{C(Y, \eta), y_0} (1_Y)) = \text{Re} (1 (1_Y)) = \text{Re} (1 (1_Y)) = (1) = 1.
\]
This implies that \( \lambda_0 = 1 \) since \(|\lambda_0| = 1\). Thus, by (50) we have

\[
S^*\left(\varphi_{C(X,\tau)}|_{E}\right)(f) = \varphi_{C(X,\tau),x_0,1}.
\] (52)

From (52) we give

\[
\text{Re} \left((Sf)(y)\right) = \text{Re} \left(f(x_0)\right), \quad \forall f \in C(X, \tau).
\] (53)

It follows that

\[
\text{Re} \left((Tf)(y)\right) = \text{Re} \left(f(x_0)\right), \quad \forall f \in C(X, \tau).
\] (54)

According to (54) and condition (ii), \( E \) is extremely regular at \((y, \eta(y))\). On the other hand, \( C(X, \tau) \) is extremely regular at \((x_0, \tau(x_0))\) by Lemma 4. Hence, by Lemma 3, we deduce that \((Sf)(y) = f(x_0)\) for all \( f \in C(X, \tau) \) or \((Sf)(y) = f(\tau(x_0))\) for all \( f \in C(X, \tau) \). We assume that \( x = x_0 \) whenever \((Sf)(y) = f(x_0)\) for all \( f \) in \( C(X, \tau) \) and \( x = \tau(x_0) \) whenever \((Sf)(y) = f(\tau(x_0))\) for all \( f \) in \( C(X, \tau) \). Then, \( x \in X \) and we have

\[
(Tf)(y) = f(x), \quad \forall f \in C(X, \tau).
\] (55)

This proves the existent of \( x \). To show uniqueness, assume that there exists \( x' \in X \) such that

\[
(Tf)(y) = f(x'), \quad \forall f \in C(X, \tau).
\] (56)

From (55) and (56) we have \( f(x) = f(x') \) for all \( f \) in \( C(X, \tau) \). This implies that \( x = x' \) since \( C(X, \tau) \) separates the points of \( X \). Thus, statement (II) holds.

Now we define the map \( h : Y_0 \rightarrow X \) by \( h(y) = x \) whenever \((Tf)(y) = f(x)\) for all \( f \) in \( C(X, \tau) \). The statement (II) implies that \( h \) is well-defined. By definition of \( h \), we have

\[
(Tf)(y) = f(h(y)), \quad \forall f \in C(X, \tau) \quad \forall y \in Y_0.
\] (57)

We prove that \( h \circ \eta = \tau \circ h \) in \( Y_0 \). Let \( y \in Y_0 \). For each \( f \in C(X, \tau) \), by (57) we have

\[
f \left((h \circ \eta)(y)\right) = f \left(h(\eta(y))\right) = (Tf)(\eta(y)) = \left((Tf)(y)\right) = \left(f(h(y))\right) = f \left(\tau(h(y))\right) = f \left((\tau \circ h)(y)\right).
\] (58)

This implies that \( h \circ \eta(y) = (\tau \circ h)(y) \) since \( C(X, \tau) \) separates the points of \( X \). Hence, \( h \circ \eta = \tau \circ h \) on \( Y_0 \).

Continually, we show that \( Y_0 \) is a \( \eta \)-invariant. Let \( y_0 \in Y_0 \). For each \( f \in C(X, \tau) \) we have

\[
S^*\left(\varphi_{C(Y,\eta)}|_{E}\right)(f) = \varphi_{C(Y,\tau),\eta,1}(Sf)
\]

\[
= \varphi_{C(Y,\tau),\eta,1}(Tf)
\]

\[
= \text{Re} \left(1e_{C(Y,\eta)}(Tf)\right)
\]

\[
= \text{Re} \left((Tf)(\eta(y))\right) = \text{Re} \left(\overline{(Tf)}(y)\right)
\]

\[
= \text{Re} \left(1f(\eta(h(y)))\right)
\]

\[
= \text{Re} \left(1e_{C(X,\tau)}(h(y))\right)\bigg|_{C(X,\tau)}(f)
\]

\[
= \varphi_{C(X,\tau),\tau(h(y)),1}(f).
\] (59)

This implies that

\[
S^*\left(\varphi_{C(Y,\eta)}|_{E}\right) = \varphi_{C(Y,\tau),\eta,1}.
\] (60)

Hence, \( S^*\left(\varphi_{C(Y,\eta)}|_{E}\right) \in \text{Ext}(B_{C(X,\tau)}) \) by Theorem 1. Therefore, \( \eta(y) \in Y_0 \) and so \( Y_0 \) is \( \eta \)-invariant.

Now we show that \( h \) is surjective. Let \( x \in X \). According to statement (I), we deduce that there exists a point \( y_0 \in Y_0 \) such that

\[
S^*\left(\varphi_{C(Y,\eta)}|_{E}\right)(f) = \varphi_{C(X,\tau),\tau(h(y)),1}.
\] (61)

It follows that \( y_0 \in Y \) and we have

\[
\text{Re} \left((Tf)(y_0)\right) = \text{Re} \left(f(x)\right), \quad \forall f \in C(X, \tau).
\] (62)

Statement (II) implies that there exists a point \( x' \in X \) such that

\[
(Tf)(y_0) = f(x'), \quad \forall f \in C(X, \tau).
\] (63)

From (62) and (63), we have \( \text{Re}(f(x)) = \text{Re}(f(x')) \) for all \( f \) in \( C(X, \tau) \). This implies that \( x' \in \{x, \tau(x)\} \) since \( \text{Re}(C(X, \tau)) \) separates the points of \( X/\tau \). If \( x' = x \), then by (63) and definition of \( h \) we have \( x = h(y_0) \). If \( x' = \tau(x) \), then by (63) and definition of \( h \) we have

\[
(Tf)(\eta(y_0)) = \overline{(Tf)}(y_0) = f(x'),
\] (64)

for all \( f \) in \( C(X, \tau) \), and so \( x = h(\eta(y_0)) \). Therefore, \( h \) is surjective.

Continually, we show that \( Y_0 \) is a boundary for \( E \). Let \( g \in E \). Then, there exists a function \( f \in C(X, \tau) \) such that \( Tf = g \). Since \( f \) is a continuous complex-valued function on the compact space \( X \), there exists a point \( x_0 \in X \) such that \( \|f\|_X =
The surjectivity of $h : Y_0 \to X$ implies that there exists a point $y_0 \in Y_0$ such that $h(y_0) = x_0$. Hence, $(Tf)(y_0) = f(x_0)$, and so
\[
\|g\|_Y = \|Tf\|_Y = \|f\|_X = |f(x_0)| = \|g(y_0)|. \tag{65}
\]
Therefore, $Y_0$ is a boundary for $E$.

We now check that $h : Y_0 \to X$ is continuous. Let $y \in Y_0$, and let $\{y_n\}_{n \in \mathbb{N}}$ be a net in $Y_0$ such that $\lim_{n \to \infty} y_n = y$ in $Y_0$. Then, for each $f \in C(X, \tau)$ we have $\lim_{n \to \infty} (Tf)(y_n) = (Tf)(y)$ in $X$. This implies that for each $f \in C(X, \tau)$, we have $\lim_{n \to \infty} f(h(y_n)) = f(h(y))$ in $X$. Since $C(X, \tau)$ separates the points of $X$, we have $\lim_{n \to \infty} h(y_n) = h(y)$ in $X$. Therefore, $h : Y_0 \to X$ is continuous.

Finally, we show that $Y_0$ is closed in $Y$. Let $y \in \overline{Y_0}$, the closure of $Y_0$ in $Y$. Then, there exists a net $\{y_n\}_{n \in \mathbb{N}} \subseteq Y_0$ such that $\lim_{n \to \infty} y_n = y$ in $Y$. Then, for each $f \in C(X, \tau)$ we have $\lim_{n \to \infty} (Tf)(y_n) = (Tf)(y)$, and so $\lim_{n \to \infty} f(h(y_n)) = f(h(y))$ in $X$. Since the net $\{ f(h(y_n)) \}_{n \in \mathbb{N}}$ converges for all $f \in C(X, \tau)$, we conclude that the net $\{ f(h(y_n)) \}_{n \in \mathbb{N}}$ converges in $X$ to a point $x \in X$. Hence, for each $f \in C(X, \tau)$ we have $\lim_{n \to \infty} f(h(y_n)) = f(x)$. Therefore, $(Tf)(y) = f(h(y))$ for all $f \in C(X, \tau)$. This implies that for each $f \in C(X, \tau)$ we have
\[
S^* \left( \varphi_{C(Y,\eta),Y} \right)(f) = (\varphi_{C(Y,\eta),Y})(Tf) = Re(1_{C(Y,\tau)})(Tf) = \text{Re}((Tf)(y)) = \text{Re}(f(h(y))) = \text{Re}(e_{C(X,\tau)} h(y)) = \left( \text{Re}(1_{C(X,\tau)} h(y)) \right)(f) = \varphi_{C(X,\tau),Y}(y)(f). \tag{66}
\]
Hence,
\[
S^* \left( \varphi_{C(Y,\eta),Y} \right)(f) = \varphi_{C(X,\tau),Y}(y)(f). \tag{67}
\]
This implies that $S^* (\varphi_{C(Y,\eta),Y}(f)) \in \text{Ext}(B(C(X,\tau)^*))$, by Theorem 1. Therefore, $y \in Y_0$, So $Y_0$ is closed in $Y$.

Note that if $T : C(X, \tau) \to C(Y, \eta)$ is a linear isometry from $C(X, \tau)$ onto $C(Y, \eta)$, then $T(C(X, \tau)) = C(Y, \eta)$ and so $T(C(X, \tau))$ is extremely regular at $(y, \eta(y))$ for all $y \in Y$. Therefore, if $T : C(X, \tau) \to C(Y, \eta)$ is a linear isometry from $C(X, \tau)$ onto $C(Y, \eta)$ with $T(1_{X}) = 1_{Y}$, then conditions (i) and (ii) of Theorem 10 hold.

We now study the into case (not necessarily unit-preserving).

**Theorem 11.** Let $T : C(X, \tau) \to C(Y, \eta)$ be a linear isometry from $C(X, \tau)$ into $C(Y, \eta)$ satisfying the following conditions:

(i) $|T_1(X)(y)| = 1$ for all $y \in Y$,

(ii) if $y \in Y$ and there is a point $x \in X$ such that $\text{Re}(T_1(X)(y)) = \text{Re}(f(x))$ for all $f \in C(X, \tau)$, then $(T_1(X))T(C(X, \tau))$ is extremally regular at $(y, \tau(y))$.

Then, there exist an $\eta$-invariant closed boundary $Y_0$ for $T(C(X, \tau))$, a function $a \in C(Y, \eta)$ with $|a(y)| = 1$ for all $y \in Y$, and a continuous map $h$ from $Y_0$ onto $X$ with $h \cdot \eta = \tau \cdot h$ on $Y_0$ such that
\[
(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X, \tau), \forall y \in Y. \tag{68}
\]

**Proof.** Assume that $a = T_1(X)$. Then, $a \in C(Y, \eta)$, and by (i) we have $|a(y)| = 1$ for all $y \in Y$. Clearly, $\langle \eta \rangle(Tf) \in C(Y, \eta)$ for all $C(Y, \eta)$, and $\langle \tau \rangle(T_1(X)) = 1_Y$. We define the map $\langle \tau \rangle : C(X, \tau) \to C(Y, \eta)$ by $\langle \tau \rangle(Tf) = (\eta)(Tf)$. It is easy to see that $\langle \tau \rangle$ is a linear isometry from $C(X, \tau)$ into $C(Y, \eta)$ such that $\langle \tau \rangle(T_1(X))^{(1)} = 1_Y$.

Applying (ii), we can easily show that if $y \in Y$ and there exists a point $x \in X$ such that $\text{Re}(\langle \tau \rangle(T_1(X))(y)) = \text{Re}(f(x))$ for all $f \in C(X, \tau)$, then $\langle \tau \rangle(T_1(X))$ is extremally regular at $(y, \eta(y))$. Let $E = T(C(X, \tau))$. Then, $\langle \eta \rangle = \langle \tau \rangle(T_1(X))$. We define $S : C(X, \tau) \to E$ by $Sf = Tf$ and $S_1 : C(X, \tau) \to \overline{E}$ by $S_1f = \langle \tau \rangle(Tf)$. Clearly, $S$ is a linear isometry from $C(X, \tau)$ onto $E$, and $S_1$ is a linear isometry from $C(X, \tau)$ onto $\overline{E}$. Now, we define
\[
Y_0 = \left\{ y \in Y : S^* \left( \varphi_{C(Y,\eta),Y} \right)(y) \in \text{Ext}(B(C(X,\tau)^*)) \right\}. \tag{69}
\]

It is easy to see that
\[
Y_0 = \left\{ y \in Y : S_1^* \left( \varphi_{C(Y,\eta),Y} \right)(y) \in \text{Ext}(B(C(X,\tau)^*)) \right\}. \tag{70}
\]

By given arguments in the proof of Theorem 10, $Y_0$ is a $\eta$-invariant closed boundary for $\overline{E}$, and there exists a continuous function $h$ from $Y_0$ onto $X$ with $h \cdot \eta = \tau \cdot h$ such that
\[
\langle \tau \rangle(T_1(X))(y) = f(h(y)), \quad \forall f \in C(X, \tau), \forall y \in Y. \tag{71}
\]

We now show that $Y_0$ is a boundary for $E$. Let $g \in E$. Then, $\langle \eta \rangle g \in \overline{E}$. Hence, there exists a point $y_0 \in Y_0$ such that
\[
\|\langle \eta \rangle g\|_Y = \|g\|_Y. \tag{72}
\]

Since $|\langle \eta \rangle| = 1$ for all $y \in Y$, we conclude that
\[
\|\langle \eta \rangle g\|_Y = \|g\|_Y. \tag{73}
\]

From (72) and (73), we have
\[
\|g\|_Y = \|\langle \eta \rangle g\|_Y = \|\langle \tau \rangle g\|_Y = \|\langle \sigma \rangle g\|_Y = \|\langle \sigma \rangle g\|_Y = |\langle \sigma \rangle| \|g\|_Y = |\langle \sigma \rangle| \|g\|_Y = |g|_Y. \tag{74}
\]

Therefore, $Y_0$ is a boundary for $E$.

Finally, we show that
\[
(Tf)(y) = a(y) f(h(y)), \quad \forall f \in C(X, \tau), \forall y \in Y. \tag{75}
\]
Let \( f \in C(X, \tau) \) and \( y \in Y_0 \). From (73), we have
\[
((\overline{\alpha} \cdot T)(f))(y) = f(h(y)).
\] (76)
Applying (76), we deduce that
\[
(Tf)(y) = a(y)\overline{a(y_0)}(Tf)(y) = a(y)(\overline{a}(Tf))(y)
\]
\[
= a(y)((\overline{\alpha} \cdot T)f)(y) = a(y)f(h(y)).
\] (77)
Hence, the proof is complete.

Note that if \( T : C(X, \tau) \to C(Y, \eta) \) is a linear isometry from \( C(X, \tau) \) onto \( C(Y, \eta) \), then \( |T1_x(y)| = 1 \) for all \( y \) in \( Y \) by Lemma 8 and \( T(C(X, \tau)) \) is extremely regular at \((y, \eta(y))\) for all \( y \) in \( Y \) by Lemma 4, and so the conditions (i) and (ii) in Theorem II hold.

The following result shows that the triple \( \{Y_0, a, \eta\} \) associated to isometry \( T \) in Theorem II possesses a universal property.

**Corollary 12.** Let \( T : C(X, \tau) \to C(Y, \eta) \) be a linear isometry from \( C(X, \tau) \) into \( C(Y, \eta) \) satisfying the following conditions:

(i) \( |T1_X(y)| = 1 \) for all \( y \in Y \),

(ii) if \( y \in Y \) and there is a point \( x \in X \) such that \( \text{Re}(T1_X(y)(Tf)(y)) = \text{Re}(f(x)) \) for all \( f \) in \( C(X, \tau) \), then \( T(C(X, \tau)) \) is extremely regular at \((y, \tau(y))\).

Let \( Y_0 \) and \( h \) be as in Theorem II. If \( Y_1 \) is a nonempty \( \eta \)-invariant subset (not necessarily closed) of \( Y \) and \( b \in C(Y, \eta) \) with \( |b(y)| = 1 \) for all \( y \in Y_1 \) and \( k : Y_1 \to X \) is a continuous map (not necessarily onto) with \( h \circ \eta = \tau \circ k \) on \( Y_1 \), such that
\[
(Tf)(y) = b(y)f(k(y)), \quad \forall f \in C(X, \tau), \forall y \in Y_1,
\] (78)
then \( Y_1 \subseteq Y_0, b = a|Y_1, \) and \( k = h|Y_1 \).

**Proof.** Let \( y \in Y_1 \). Since \( 1_X \in C(X, \tau) \), taking \( f = 1_X \) in the above expression of \( Tf \), we have \( b(y) = (T1_X)(y) = a(y) \). Thus, \( b = a|Y_1 \).

Let \( E = T(C(X, \tau)) \) and \( S : C(X, \tau) \to E \) defined by \( Sf = Tf \). Then,
\[
Y_0 = \left\{ y \in Y : S^* \left( \varphi_{C(Y, \eta), y, \overline{a(y)}|E} \right) \in \text{Ext}(B_{C(X, \tau)}) \right\}.
\] (79)
Let \( y \in Y_1 \). Then, the above expression of \( Tf \) reads as follows:
\[
(Tf)(y) = b(y)f(k(y)), \quad \forall f \in C(X, \tau).
\] (80)
From (80), for each \( f \in C(X, \tau) \) we have
\[
S^* \left( \varphi_{C(Y, \eta), y, \overline{a(y)}|E} \right)(f) = (\varphi_{C(Y, \eta), y, \overline{a(y)}|E})(Tf)
\]
\[
= \text{Re} \left( a(y)e_{C(Y, \eta)}(Tf) \right)
\]
\[
= \text{Re} \left( \overline{a(y)}(Tf)(y) \right)
\]
\[
= \text{Re} \left( a(y)a(y)f(k(y)) \right)
\]
\[
= \text{Re} \left( \overline{f}(k(y)) \right)
\]
\[
= \varphi_{C(X, \tau), k(y), 1}(f).
\] (81)
This implies that
\[
S^* \left( \varphi_{C(Y, \eta), y, \overline{a(y)}|E} \right) = \varphi_{C(X, \tau), k(y), 1}.
\] (82)

From (82), we deduce that \( S^*(\varphi_{C(Y, \eta), y, \overline{a(y)}|E}) \) is an extreme point of \( \text{Ext}B_{C(X, \tau)} \) by Theorem I. Therefore, \( y \in Y_0 \). So \( Y_1 \subseteq Y_0 \).

We now show that \( k = h|Y_1 \). Let \( y \in Y_1 \). By (80), we have
\[
(Tf)(y) = a(y)f(k(y)), \quad \forall f \in C(X, \tau).
\] (83)
Since \( Y_1 \subseteq Y_0 \), we deduce that \( y \in Y_0 \), and so by Theorem II we have
\[
(Tf)(y) = a(y)f(h(y)), \quad \forall f \in C(X, \tau).
\] (84)
Since \( |a(y)| = 1 \), by (83) and (84) we have
\[
f(k(y)) = f(h(y)), \quad \forall f \in C(X, \tau).
\] (85)
This implies that \( k(y) = h(y) \) since \( C(X, \tau) \) separates the points of \( X \). Thus, \( k = h|Y_1 \). Hence, the proof is complete.

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**References**


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