Research Article

Some Results on Preconditioned Mixed-Type Splitting Iterative Method

Guangbin Wang¹ and Fuping Tan²

¹ Department of Mathematics, Qingdao University of Science and Technology, Qingdao 266061, China

² Department of Mathematics, Shanghai University, Shanghai 200444, China

Correspondence should be addressed to Guangbin Wang; wguangbin750828@sina.com

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We present a preconditioned mixed-type splitting iterative method for solving the linear system $Ax = b$, where A is a Z-matrix. And we give some comparison theorems to show that the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method. Finally, we give one numerical example to illustrate our results.

1. Introduction

For solving linear system,

$$Ax = b, \quad (1)$$

where A is an $n \times n$ square matrix and x and b are n -dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b, \quad k = 0, 1, \dots, \quad (2)$$

where $A = M - N$ and M is nonsingular. Thus, (2) can be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \dots, \quad (3)$$

where $T = M^{-1}N$ and $c = M^{-1}b$.

Assuming that A has unit diagonal entries, let $A = I - L - U$, where I is the identity matrix and $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A , respectively.

Transform the original system (1) into the preconditioned form as follows:

$$PAx = Pb. \quad (4)$$

Then, we can define the basic iterative scheme as follows:

$$M_p x^{k+1} = N_p x^k + Pb, \quad k = 0, 1, \dots, \quad (5)$$

where $PA = M_p - N_p$ and M_p is nonsingular. Thus, the equation above can also be written as

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, \dots, \quad (6)$$

where $T = M_p^{-1}N_p$ and $c = M_p^{-1}Pb$.

In paper [1], Cheng et al. presented the mixed-type splitting iterative method as follows:

$$(D + D_1 + L_1 - L)x^{k+1} = (D_1 + L_1 + U)x^k + b, \quad k = 0, 1, 2, \dots, \quad (7)$$

with the following iterative matrix:

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U), \quad (8)$$

where D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix, and $0 \leq L_1 \leq L$.

In this paper, we will establish the preconditioned mixed-type splitting iterative method with the preconditioners $P_\alpha = I + \alpha L$, $P_\beta = I + \beta U$, and $P_{\alpha\beta} = I + \alpha L + \beta U$ for solving linear systems. And we obtain some comparison results which show that the rate of convergence of the preconditioned mixed-type splitting iterative method with $P_{\alpha\beta}$ is faster than that of the preconditioned mixed-type splitting iterative method with P_α or P_β . Finally, we give one numerical example to illustrate our results.

2. Preconditioned Mixed-Type Splitting Iterative Method

For the linear system (1), we consider its preconditioned form as follows:

$$P_{\alpha\beta}Ax = P_{\alpha\beta}b, \quad (9)$$

with the preconditioner $P_{\alpha\beta} = I + \alpha L + \beta U$; that is,

$$A_{\alpha\beta}x = b_{\alpha\beta}. \quad (10)$$

We apply the mixed-type splitting iterative method to it and have the corresponding preconditioned mixed-type splitting iterative method as follows:

$$\begin{aligned} (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})x^{k+1} &= (D_1 + L_1 + U_{\alpha\beta})x^k + b_{\alpha\beta}, \\ k &= 0, 1, 2, \dots, \end{aligned} \quad (11)$$

that is,

$$\begin{aligned} x^{k+1} &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} (D_1 + L_1 + U_{\alpha\beta})x^k \\ &\quad + (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} b_{\alpha\beta}, \quad k = 0, 1, 2, \dots, \end{aligned} \quad (12)$$

So, the iterative matrix is

$$\tilde{T} = (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} (D_1 + L_1 + U_{\alpha\beta}), \quad (13)$$

where $D_{\alpha\beta}$, $-L_{\alpha\beta}$, and $-U_{\alpha\beta}$ are the diagonal, strictly lower, and strictly upper triangular matrices obtained from $A_{\alpha\beta}$, D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix, and $0 \leq L_1 \leq L_{\alpha\beta}$.

If we choose $\beta = 0$, we have the following corresponding iterative matrix:

$$\bar{T} = (D_{\alpha} + D_1 + L_1 - L_{\alpha})^{-1} (D_1 + L_1 + U_{\alpha}). \quad (14)$$

And if we choose $\alpha = 0$, we have the following corresponding iterative matrix:

$$\hat{T} = (D_{\beta} + D_1 + L_1 - L_{\beta})^{-1} (D_1 + L_1 + U_{\beta}). \quad (15)$$

If we choose certain auxiliary matrices, we can get the classical iterative methods as follows.

(1) The PSOR method is

$$\begin{aligned} D_1 &= \frac{1}{r} (1 - r) D_{\alpha\beta}, \quad L_1 = 0, \\ \tilde{L}_r &= (D_{\alpha\beta} - rL_{\alpha\beta})^{-1} [(1 - r) D_{\alpha\beta} + rU_{\alpha\beta}]. \end{aligned} \quad (16)$$

(2) The PAOR method is

$$\begin{aligned} D_1 &= \frac{1}{w} (1 - w) D_{\alpha\beta}, \quad L_1 = \frac{1}{w} (w - r) L_{\alpha\beta}, \\ \tilde{L}_{rw} &= (D_{\alpha\beta} - rL_{\alpha\beta})^{-1} [(1 - w) D_{\alpha\beta} + (w - r) L_{\alpha\beta} + wU_{\alpha\beta}]. \end{aligned} \quad (17)$$

We need the following definitions and results.

Definition 1 (see [2]). A matrix A is a Z -matrix if $a_{ij} \leq 0$, for all $i, j = 1, 2, \dots, n$, such that $i \neq j$. A matrix A is an L -matrix if $a_{ii} > 0$, $i = 1, 2, \dots, n$, and $a_{ij} \leq 0$, for all $i, j = 1, 2, \dots, n$, such that $i \neq j$.

Definition 2 (see [2]). A matrix A is an M -matrix if A is a nonsingular Z -matrix, and $A^{-1} \geq 0$.

Definition 3 (see [2, 3]). Let $M, N \in R^{n,n}$. Then, $A = M - N$ is called a regular splitting if $M^{-1} \geq 0$ and $N \geq 0$; $A = M - N$ is called an M -splitting if M is an M -matrix, $N \geq 0$.

Lemma 4 (see [2]). Let $A \geq 0$ be an irreducible matrix. Then,

- (1) A has a positive real eigenvalue equal to its spectral radius;
- (2) to $\rho(A)$, there corresponds an eigenvector $x > 0$;
- (3) $\rho(A)$ is a simple eigenvalue of A .

Lemma 5 (see [4]). Let A be a nonnegative matrix. Then,

- (1) if $\alpha x \leq Ax$ for some nonnegative vector x , $x \neq 0$, then $\alpha \leq \rho(A)$;
- (2) if $Ax \leq \beta x$ for some positive vector x , then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vectors x , then

$$\alpha \leq \rho(A) \leq \beta. \quad (18)$$

Lemma 6 (see [5]). Let $A = M - N$ be an M -splitting of A . Then, $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M -matrix.

Lemma 7 (see [6, 7]). Let A be a Z -matrix. Then, A is a nonsingular M -matrix if and only if there is a positive vector x such that $Ax \geq 0$.

Lemma 8 (see [8]). Let $A = M - N$ be a regular splitting of A . Then, the splitting is convergent if and only if $A^{-1} \geq 0$.

Lemma 9 (see [9]). Let A and B be two $n \times n$ nonsingular lower triangular L -matrices. If $A \geq B$, then $B^{-1} \geq A^{-1} \geq 0$.

3. Convergence Analysis and Comparison Results

Theorem 10. Let A be a nonsingular Z -matrix. Assume that $D_1 \geq 0$, $0 \leq L_1 \leq L_{\alpha}$, $\alpha \in [0, 1]$, and $\bar{T}$ and T are the iterative

matrices given by (14) and (8), respectively. Consider the following.

- (i) If $\rho(T) < 1$, then $\rho(\bar{T}) < \rho(T) < 1$.
- (ii) Let A be irreducible. Assume that $1 - \alpha \sum_{i=1}^{i-1} a_{il}a_{ij} > 0$ and $a_{ij} + \sum_{k=i+1}^n a_{ik}a_{kj} \leq 0$;

then, one has

- (1) $\rho(\bar{T}) \geq \rho(T)$, if $\rho(T) \geq 1$,
- (2) $\rho(\bar{T}) \leq \rho(T)$, if $\rho(T) < 1$.

Proof. Let

$$\begin{aligned} M_\alpha &= D_\alpha + D_1 + L_1 - L_\alpha, \\ N_\alpha &= D_1 + L_1 + U_\alpha, \\ M &= I + D_1 + L_1 - L, \\ N &= D_1 + L_1 + U, \\ E_\alpha &= (I + \alpha L)(I + D_1 + L_1 - L), \\ F_\alpha &= (I + \alpha L)(D_1 + L_1 + U). \end{aligned} \quad (19)$$

Then, we have

$$A = M - N, \quad A_\alpha = M_\alpha - N_\alpha = E_\alpha - F_\alpha. \quad (20)$$

(i) Since A is a nonsingular Z -matrix and $D_1 \geq 0$, $0 \leq L_1 \leq L_\alpha$, it is clear that $M = I + D_1 + L_1 - L$ is a nonsingular M -matrix and the splitting

$$A = M - N = (I + D_1 + L_1 - L) - (D_1 + L_1 + U) \quad (21)$$

is an M -splitting. Since $\rho(T) < 1$, it follows from Lemma 6 that A is a nonsingular M -matrix. Then, by Lemma 7, there is a positive vector x such that $Ax \geq 0$, so $A_\alpha x = (I + \alpha L)Ax \geq 0$.

By Lemma 7, A_α is also a nonsingular M -matrix.

Obviously, we can get that D_α is a positive diagonal matrix. And from L_α is nonnegative, we know that M_α being a Z -matrix. Since $D_\alpha^{-1}L_\alpha \geq 0$ is a strictly lower triangular matrix, so that $\rho(D_\alpha^{-1}L_\alpha) = 0 < 1$.

So, we have $(I + D_\alpha^{-1}D_1 + D_\alpha^{-1}L_1 - D_\alpha^{-1}L_\alpha)^{-1} \geq 0$.

Then, $M_\alpha^{-1} = (I + D_\alpha^{-1}D_1 + D_\alpha^{-1}L_1 - D_\alpha^{-1}L_\alpha)^{-1}D_\alpha^{-1} \geq 0$; hence, M_α is a nonsingular M -matrix.

For $j \neq i+1$, it is obvious that $(U_\alpha)_{ij} = -a_{ij} + \alpha a_{ii+1}a_{i+1j} \geq 0$. And for $j = i+1$, we have $(U_\alpha)_{ij} = (\alpha - 1)a_{ij} \geq 0$. Thus, $U_\alpha \geq 0$ and $N_\alpha \geq 0$.

We have proven that $A_\alpha = M_\alpha - N_\alpha$ and $A = M - N$ are both M -splittings and $E_\alpha^{-1}F_\alpha = M^{-1}N$, two splittings $A_\alpha = M_\alpha - N_\alpha = E_\alpha - F_\alpha$ are nonnegative.

On the other hand, since $L_\alpha = D_\alpha - I + \alpha\bar{D} + L - \alpha L + \alpha L^2 + \alpha\bar{E}$, we get

$$\begin{aligned} E_\alpha - M_\alpha &= (I + \alpha L)(I + D_1 + L_1 - L) \\ &\quad - (D_\alpha + D_1 + L_1 - L_\alpha) \\ &= (I + D_1 + L_1 - L) + \alpha L(I + D_1 + L_1 - L) \\ &\quad - (D_\alpha + D_1 + L_1 - L_\alpha) \\ &= I - L + \alpha L + \alpha LD_1 + \alpha LL_1 - \alpha L^2 - D_\alpha + L_\alpha \\ &= I - L + \alpha L + \alpha LD_1 + \alpha LL_1 - \alpha L^2 - D_\alpha + D_\alpha \\ &\quad - I + \alpha\bar{D} + L - \alpha L + \alpha L^2 + \alpha\bar{E} \\ &= \alpha(LD_1 + LL_1 + \bar{D} + \bar{E}) \geq 0, \end{aligned} \quad (22)$$

which implies that

$$A_\alpha^{-1}E_\alpha - A_\alpha^{-1}M_\alpha = A_\alpha^{-1}(E_\alpha - M_\alpha) \geq 0. \quad (23)$$

Therefore, $A_\alpha^{-1}E_\alpha \geq A_\alpha^{-1}M_\alpha \geq 0$. So, we have $\rho(M_\alpha^{-1}N_\alpha) \leq \rho(E_\alpha^{-1}F_\alpha)$; that is,

$$\rho(\bar{T}) \leq \rho(T) < 1. \quad (24)$$

(ii) Let $A = I - L - U$ be irreducible. Since $L + U$ is a non-negative and irreducible matrix, and according to the proof of Lemma 4 in paper [9], we can obtain that $\bar{T}$ and T are nonnegative and irreducible matrices. Thus, from Lemma 4, we know that there exists a positive vector $x = (x_1, x_2, \dots, x_n)^T$ such that $Tx = \lambda x$, where we denote $\lambda = \rho(T)$, which is equivalent to

$$\begin{aligned} (D_1 + L_1 + U)x &= \lambda(I + D_1 + L_1 - L)x, \\ (U - \lambda I + \lambda L)x &= [(\lambda - 1)D_1 + (\lambda - 1)L_1]x. \end{aligned} \quad (25)$$

Let $LU = \bar{D} + \bar{E} + \bar{F}$, where $\bar{D}$, $\bar{E}$, and $\bar{F}$ are the diagonal, lower triangular, and upper triangular parts of LU , respectively. So,

$$\begin{aligned} A_\alpha &= D_\alpha - L_\alpha - U_\alpha \\ &= (I - \alpha\bar{D}) - (L - \alpha L + \alpha L^2 + \alpha\bar{E}) - (U + \alpha\bar{F}), \end{aligned} \quad (26)$$

where $D_\alpha = I - \alpha\bar{D}$, $L_\alpha = L - \alpha L + \alpha L^2 + \alpha\bar{E}$, $U_\alpha = U + \alpha\bar{F}$.

Now, we consider

$$\begin{aligned} \bar{T}x - Tx &= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\ &\quad \times (D_1 + L_1 + U_\alpha)x - \lambda x \\ &= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\ &\quad \times [(D_1 + L_1 + U_\alpha) - \lambda(D_\alpha + D_1 + L_1 - L_\alpha)]x \\ &= (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \end{aligned}$$

$$\begin{aligned}
& \times \left[(D_1 + L_1 + U + \alpha \bar{F}) \right. \\
& \quad \left. - \lambda (I - \alpha \bar{D} + D_1 + L_1 \right. \\
& \quad \quad \left. - L + \alpha L - \alpha L^2 - \alpha \bar{E}) \right] x \\
& = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
& \quad \times \left\{ [(D_1 + L_1 + U) - \lambda (I + D_1 + L_1 - L)] x \right. \\
& \quad \quad \left. + [\alpha \bar{F} + \lambda (\alpha \bar{D} - \alpha L + \alpha L^2 + \alpha \bar{E})] x \right\} \\
& = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
& \quad \times [\alpha \bar{F} + \lambda (\alpha \bar{D} - \alpha L + \alpha L^2 + \alpha \bar{E})] x \\
& = (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
& \quad \times [(\lambda - 1) \alpha L (D_1 + L_1) + (\lambda - 1) \alpha (\bar{D} + \bar{E})] x \\
& = (\lambda - 1) (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\
& \quad \times \alpha [LD_1 + LL_1 + \bar{D} + \bar{E}] x.
\end{aligned} \tag{27}$$

Since $D_\alpha + D_1 + L_1 - L_\alpha$ is an M -matrix, and $LD_1 + LL_1 + \bar{D} + \bar{E} \geq 0$, we have the following.

- (1) If $\lambda \geq 1$, then $\bar{T}x \geq Tx = \lambda x$. By Lemma 5, we get $\rho(\bar{T}) \geq \rho(T)$.
- (2) If $\lambda < 1$, then $\bar{T}x \leq Tx = \lambda x$. By Lemma 5, we get $\rho(\bar{T}) \leq \rho(T)$. $\square$

Theorem 11. Let A be a nonsingular Z -matrix. Assume that $D_1 \geq 0$, $0 \leq L_1 \leq L_{\alpha\beta}$, $\alpha, \beta \in [0, 1]$, and $\tilde{T}$ and T are the iterative matrices given by (13) and (8), respectively. Consider the following.

- (i) If $\rho(T) < 1$, then

$$\rho(\tilde{T}) < \rho(T) < 1. \tag{28}$$

- (ii) Let A be irreducible. Assume that

$$\begin{aligned}
1 - \alpha \sum_{l=1}^{i-1} a_{il} a_{lj} - \beta \sum_{k=i+1}^n a_{ik} a_{kj} & > 0, \\
a_{ij} + \sum_{k=i+1}^n a_{ik} a_{kj} & \leq 0;
\end{aligned} \tag{29}$$

Then, one has

- (1) $\rho(\tilde{T}) \geq \rho(T)$ if $\rho(T) \geq 1$,
- (2) $\rho(\tilde{T}) \leq \rho(T)$ if $\rho(T) < 1$.

Proof. Let

$$\begin{aligned}
M_{\alpha\beta} &= D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}, \\
N_{\alpha\beta} &= D_1 + L_1 + U_{\alpha\beta}, \\
M &= I + D_1 + L_1 - L, \\
N &= D_1 + L_1 + U, \\
E_\alpha &= (I + \alpha L + \beta U) (I + D_1 + L_1 - L), \\
F_\alpha &= (I + \alpha L + \beta U) (D_1 + L_1 + U).
\end{aligned} \tag{30}$$

Then, we have

$$A = M - N, \quad A_{\alpha\beta} = M_{\alpha\beta} - N_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta}. \tag{31}$$

(i) By a similar proof of Theorem 10, we can prove that $A_{\alpha\beta} = M_{\alpha\beta} - N_{\alpha\beta}$ and $A = M - N$ are both M -splitting and $E_{\alpha\beta}^{-1} F_{\alpha\beta} = M^{-1} N$, two splittings $A_{\alpha\beta} = M_{\alpha\beta} - N_{\alpha\beta} = E_{\alpha\beta} - F_{\alpha\beta}$, are nonnegative.

On the other hand, since $L_{\alpha\beta} = D_{\alpha\beta} - I + \alpha \bar{D} + \beta \hat{D} + L - \alpha L + \alpha L^2 + \alpha \bar{E} + \beta \hat{E}$, we get

$$\begin{aligned}
E_{\alpha\beta} - M_{\alpha\beta} &= (I + \alpha L + \beta U) (I + D_1 + L_1 - L) \\
&\quad - (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}) \\
&= (I + D_1 + L_1 - L) + \alpha L (I + D_1 + L_1 - L) \\
&\quad + \beta U (I + D_1 + L_1 - L) \\
&\quad - (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}) \\
&= I - L + \alpha L (I + D_1 + L_1 - L) \\
&\quad + \beta U (I + D_1 + L_1 - L) - D_{\alpha\beta} + L_{\alpha\beta} \\
&= I - L + \alpha L (I + D_1 + L_1 - L) \\
&\quad + \beta U (I + D_1 + L_1 - L) - D_{\alpha\beta} + D_{\alpha\beta} \\
&\quad - I + \alpha \bar{D} + \beta \hat{D} + L - \alpha L \\
&\quad + \alpha L^2 + \alpha \bar{E} + \beta \hat{E} \\
&= \beta (U - \hat{F}) + \beta U (D_1 + L_1) \\
&\quad + \alpha (\bar{D} + \bar{E}) + \alpha L (D_1 + L_1) \geq 0,
\end{aligned} \tag{32}$$

which implies that

$$A_{\alpha\beta}^{-1} E_{\alpha\beta} - A_{\alpha\beta}^{-1} M_{\alpha\beta} = A_{\alpha\beta}^{-1} (E_{\alpha\beta} - M_{\alpha\beta}) \geq 0. \tag{33}$$

Therefore, $A_{\alpha\beta}^{-1} E_{\alpha\beta} \geq A_{\alpha\beta}^{-1} M_{\alpha\beta} \geq 0$. So, we have $\rho(M_{\alpha\beta}^{-1} N_{\alpha\beta}) \leq \rho(E_{\alpha\beta}^{-1} F_{\alpha\beta})$; that is,

$$\rho(\tilde{T}) \leq \rho(T) < 1. \tag{34}$$

(ii) Let $A_{\alpha\beta} = P_{\alpha\beta}A = (I + \alpha L + \beta U)A = I - L - U + \alpha L(I - L - U) + \beta U(I - L - U)$

$$= I - \alpha\bar{D} - \beta\hat{D} - (L - \alpha L + \alpha L^2 + \alpha\bar{E} + \beta\hat{E}) - (U + \alpha\bar{F} - \beta U + \beta\hat{F} + \beta U^2), \quad (35)$$

where $LU = \bar{D} + \bar{E} + \bar{F}$, $UL = \hat{D} + \hat{E} + \hat{F}$, and $\bar{D}$, $\bar{E}$, $\bar{F}$, $\hat{D}$, $\hat{E}$, and $\hat{F}$ are the diagonal, strictly lower, and strictly upper triangular matrices of LU and UL , respectively.

And denote $A_{\alpha\beta} = D_{\alpha\beta} - L_{\alpha\beta} - U_{\alpha\beta}$; then according to (35), we have

$$\begin{aligned} D_{\alpha\beta} &= I - \alpha\bar{D} - \beta\hat{D}, \\ L_{\alpha} &= L - \alpha L + \alpha L^2 + \alpha\bar{E} + \beta\hat{E}, \\ U_{\alpha\beta} &= U - \beta U + \beta U^2 + \alpha\bar{F} + \beta\hat{F}. \end{aligned} \quad (36)$$

By (25), we have

$$\begin{aligned} \tilde{T}x - \lambda x &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times (D_1 + L_1 + U_{\alpha\beta})x - \lambda x \\ &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [(D_1 + L_1 + U_{\alpha\beta}) \\ &\quad \quad - \lambda (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})] x \\ &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [(D_1 + L_1 + U + \alpha\bar{F} - \beta U + \beta\hat{F} + \beta U^2) \\ &\quad \quad - \lambda (I - \alpha\bar{D} - \beta\hat{D} + D_1 + L_1 - L \\ &\quad \quad \quad + \alpha L - \alpha L^2 - \alpha\bar{E} - \beta\hat{E})] x \\ &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times \{ [(D_1 + L_1 + U) - \lambda (I + D_1 + L_1 - L)] x \\ &\quad \quad + [\alpha\bar{F} - \beta U + \beta\hat{F} + \beta U^2 \\ &\quad \quad \quad - \lambda (-\alpha\bar{D} - \beta\hat{D} + \alpha L \\ &\quad \quad \quad \quad - \alpha L^2 - \alpha\bar{E} - \beta\hat{E})] x \} \\ &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [\alpha\bar{F} - \beta U + \beta\hat{F} + \beta U^2 \\ &\quad \quad - \lambda (-\alpha\bar{D} - \beta\hat{D} + \alpha L - \alpha L^2 - \alpha\bar{E} - \beta\hat{E})] x \\ &= (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [\alpha\bar{F} + \lambda (\alpha\bar{D} - \alpha L + \alpha L^2 + \alpha\bar{E}) \\ &\quad \quad - \beta U + \beta U^2 + \beta\hat{F} + \lambda (\beta\hat{D} + \beta\hat{E})] x \end{aligned}$$

$$\begin{aligned} &= (\lambda - 1) (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [\alpha L (D_1 + L_1) + \alpha (\bar{D} + \bar{E}) \\ &\quad \quad + \beta U (D_1 + L_1) + \beta (U - \hat{F})] x \\ &= (\lambda - 1) (D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta})^{-1} \\ &\quad \times [(\alpha L + \beta U) (D_1 + L_1) \\ &\quad \quad + \alpha (\bar{D} + \bar{E}) + \beta (U - \hat{F})] x. \end{aligned} \quad (37)$$

If $a_{ij} + \sum_{k=i+1}^n a_{ik}a_{kj} \leq 0$, then by the proof of Theorem 10, we have $U - \hat{F} \geq 0$.

Therefore, one has the following.

- (1) If $\lambda \geq 1$, then $\tilde{T}x - \lambda x \geq 0$ but not equal to 0. Therefore, $\tilde{T}x \geq \lambda x$. By Lemma 5, we get $\rho(\tilde{T}) \geq \lambda = \rho(T)$.
- (2) If $\lambda < 1$, then $\tilde{T}x - \lambda x \leq 0$ but not equal to 0. Therefore, $\tilde{T}x \leq \lambda x$. By Lemma 5, we get $\rho(\tilde{T}) \leq \lambda = \rho(T)$.

Remark. If we choose $\alpha = 0$ in Theorem 11, we have a similar result which is showed by the following corollary. $\square$

Corollary 12. Let A be a nonsingular Z -matrix. Assume that $D_1 \geq 0$, $0 \leq L_1 \leq L_\beta$, $\beta \in [0, 1]$, and $\hat{T}$ and T are the iterative matrices given by (15) and (8), respectively. Consider the following.

- (i) If $\rho(T) < 1$, then

$$\rho(\hat{T}) < \rho(T) < 1. \quad (38)$$

- (ii) Let A be irreducible. Assume that

$$1 - \beta \sum_{k=i+1}^n a_{ik}a_{kj} > 0, \quad a_{ij} + \sum_{k=i+1}^n a_{ik}a_{kj} \leq 0; \quad (39)$$

then, one has

- (1) $\rho(\hat{T}) \geq \rho(T)$ if $\rho(T) \geq 1$,
- (2) $\rho(\hat{T}) \leq \rho(T)$ if $\rho(T) < 1$.

Now, one will provide some results to show the relations among $\rho(\tilde{T})$, $\rho(\bar{T})$, and $\rho(\hat{T})$.

Theorem 13. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z -matrix. Let $\tilde{T}$ and $\bar{T}$ be iterative matrices given by (13) and (14), respectively. Assume that $\alpha, \beta \in [0, 1]$, $D_1 \geq 0$, $0 \leq L_1 \leq L_{\alpha\beta}$. If $1 - \alpha \sum_{l=1}^{i-1} a_{il}a_{lj} - \beta \sum_{k=i+1}^n a_{ik}a_{kj} > 0$ and $a_{ij} + \sum_{k=i+1}^n a_{ik}a_{kj} \leq 0$, then

- (1) $\rho(\tilde{T}) \geq \rho(\bar{T})$ if $\rho(T) \geq 1$;
- (2) $\rho(\tilde{T}) \leq \rho(\bar{T})$ if $\rho(T) < 1$.

Proof. Since $D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}$ and $D_\alpha + D_1 + L_1 - L_\alpha$ are two lower triangular L -matrices with $D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \leq D_\alpha + D_1 + L_1 - L_\alpha$, by Lemma 9, we have

$$\left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \geq \left(D_\alpha + D_1 + L_1 - L_\alpha\right)^{-1} \geq 0. \quad (40)$$

By the proof of Theorems 10 and 11, we consider

$$\begin{aligned} \tilde{T}x - \bar{T}x &= \tilde{T}x - \lambda x - (\bar{T}x - \lambda x) \\ &= (\lambda - 1) \left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \\ &\quad \times \left[(\alpha L + \beta U)(D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E} + \beta(U - \hat{F})\right] x \\ &\quad - (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\ &\quad \times \left[\alpha L(D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E}\right] x \\ &\geq (\lambda - 1) \left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \\ &\quad \times \left[\beta U(D_1 + L_1) + \beta(U - \hat{F})\right] x. \end{aligned} \quad (41)$$

In view of the proof of Theorem 11, we have $\beta U(D_1 + L_1) + \beta(U - \hat{F}) \geq 0$.

Therefore, one has the following.

- (1) If $\lambda \geq 1$, the right-hand side of the above inequality is more than zero. By Lemma 8, $\rho(\tilde{T}) \geq \rho(\bar{T})$.
- (2) If $\lambda < 1$, the right-hand side of the above inequality is more than zero. By Lemma 8, $\rho(\tilde{T}) \leq \rho(\bar{T})$. $\square$

Theorem 14. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z -matrix. Let $\tilde{T}$ and $\hat{T}$ be iterative matrices given by (13) and (15), respectively. Assume that $\alpha, \beta \in [0, 1]$, $D_1 \geq 0$, $0 \leq L_1 \leq L_{\alpha\beta}$. If $1 - \alpha \sum_{l=1}^{i-1} a_{il} a_{lj} - \beta \sum_{k=i+1}^n a_{ik} a_{kj} > 0$ and $a_{ij} + \sum_{k=i+1}^n a_{ik} a_{kj} \leq 0$, then

- (1) $\rho(\tilde{T}) \geq \rho(\hat{T})$ if $\rho(T) \geq 1$,
- (2) $\rho(\tilde{T}) \leq \rho(\hat{T})$ if $\rho(T) < 1$.

Proof. Since $D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}$ and $D_\beta + D_1 + L_1 - L_\beta$ are two lower triangular L -matrices with $D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta} \leq D_\beta + D_1 + L_1 - L_\beta$, by Lemma 9, we have

$$\left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \geq \left(D_\beta + D_1 + L_1 - L_\beta\right)^{-1} \geq 0. \quad (42)$$

By the proof of Corollary 12 and Theorem 11, we consider

$$\begin{aligned} \tilde{T}x - \hat{T}x &= \tilde{T}x - \lambda x - (\hat{T}x - \lambda x) \\ &= (\lambda - 1) \left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \\ &\quad \times \left[(\alpha L + \beta U)(D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E} + \beta(U - \hat{F})\right] x \\ &\quad - (D_\alpha + D_1 + L_1 - L_\alpha)^{-1} \\ &\quad \times \left[\beta U(D_1 + L_1) + \beta(U - \hat{F})\right] x \\ &\geq (\lambda - 1) \left(D_{\alpha\beta} + D_1 + L_1 - L_{\alpha\beta}\right)^{-1} \\ &\quad \times \left(\alpha L(D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E}\right) x. \end{aligned} \quad (43)$$

Since $\alpha L(D_1 + L_1) + \alpha \bar{D} + \alpha \bar{E} \geq 0$, we get the following.

- (1) If $\lambda \geq 1$, the right-hand side of the above inequality is more than zero. By Lemma 8, $\rho(\tilde{T}) \geq \rho(\hat{T})$.
- (2) If $\lambda < 1$, the right-hand side of the above inequality is more than zero. By Lemma 8, $\rho(\tilde{T}) \leq \rho(\hat{T})$.

Remark. The results (theorems and corollaries) in Section 3 are in some sense the generalized Stein-Rosenberg-type theorems like those in the papers [10–13]. The results (theorems and corollaries) in Section 3 are the comparisons of spectral radius of iterative matrices between the mixed-type splitting method and the preconditioned mixed-type splitting method, while the results in the papers [10–13] are the comparisons of spectral radius of iterative matrices between the parallel decomposition-type relaxation method and its special case. $\square$

4. Numerical Example

Consider the following equation:

$$-\Delta u + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f, \quad (44)$$

in the unit square Ω with Dirichlet boundary conditions.

If we apply the central difference scheme on a uniform grid with $N \times N$ interior nodes ($N^2 = n$) to the discretization of the above equation, we can get a system of linear equations with the coefficient matrix

$$A = I \otimes P + Q \otimes I, \quad (45)$$

where $\otimes$ denotes the Kronecker product,

$$\begin{aligned} P &= \text{tridiag}\left(-\frac{2+h}{8}, 1, -\frac{2-h}{8}\right), \\ Q &= \text{tridiag}\left(-\frac{1+h}{4}, 1, -\frac{1-h}{4}\right) \end{aligned} \quad (46)$$

are $N \times N$ tridiagonal matrices, and the step size is $h = 1/N$.

TABLE 1

$\alpha(\beta)$	$\rho(T)$	$\rho(\bar{T})$	$\rho(\hat{T})$	$\rho(\tilde{T})$
0	0.563691	0.563691	0.563691	0.563691
0.05	0.563691	0.554915	0.554027	0.545537
0.1	0.563691	0.545958	0.544027	0.527441
0.15	0.563691	0.536827	0.533673	0.509401
0.2	0.563691	0.527531	0.522949	0.491418
0.25	0.563691	0.518081	0.511842	0.473488
0.3	0.563691	0.508493	0.500331	0.455608
0.35	0.563691	0.498782	0.488412	0.437775
0.4	0.563691	0.488967	0.476073	0.419983
0.45	0.563691	0.479065	0.463309	0.402227
0.5	0.563691	0.469097	0.450121	0.384541

We choose $N = 5$; then $A \in R^{25 \times 25}$.

If we choose

- (1) $D_1 = 0.45D_\alpha, L_1 = 0.4L_\alpha, \alpha \in [0, 0.5]$,
- (2) $D_1 = 0.45D_\beta, L_1 = 0.4L_\beta, \beta \in [0, 0.5]$,
- (3) $D_1 = 0.45D_{\alpha\beta}, L_1 = 0.4L_{\alpha\beta}, \alpha = \beta \in [0, 0.5]$, then we can obtain the following results by Theorems 10–14.

Table 1 shows that the rate of convergence of the preconditioned mixed-type splitting method is faster than that of the mixed-type splitting method. And it shows that the rate of convergence of the preconditioned mixed-type splitting method with $P_{\alpha\beta}$ is faster than that of the preconditioned mixed-type splitting method with P_α or P_β .

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