Research Article

Some Properties of Quasilinearly Subharmonic Functions and Maximal Theorem for Bergman Type Spaces

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Received 17 July 2013; Accepted 24 August 2013

Academic Editors: O. Miyagaki and A. M. Peralta

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Let $\text{QNS}(\mathbf{B})$ denote the class of quasilinearly subharmonic functions in unit ball $\mathbf{B}$. We provide, following result: if $0 < p < \infty$ and if $u \in \text{QNS}(\mathbf{B})$, then $\int_{\mathbf{S}} (u^*)^p d\sigma \leq C \int_{\mathbf{S}} (u^+)^p d\sigma$, where u^+ is the radial maximal function and $u^*(\xi) = \sup_{z \in D_c(\xi)} |u(z)|$, $D_c(\xi) = \{z \in \mathbf{B} : |\xi - z| < c(1 - |z|)\}$, and $\xi \in \mathbf{S}$, $c > 1$. Also, we prove a maximal theorem for Bergman type spaces.

1. Introduction and Preliminaries

Let $\mathbb{R}^n$ ($n \geq 2$) denote the n -dimensional Euclidean space. Let $\mathbf{B} = \mathbf{B}_n$ be the unit ball centered at the origin. The boundary of $\mathbf{B}$ will be denoted by $\mathbf{S}$.

The Hardy space $H^p(\mathbf{B})$ ($0 < p < \infty$) consists of functions u harmonic in $\mathbf{B}$ for which

$$\|u\|_{H^p} := \|u^+\|_{L^p} = \left(\int_{\mathbf{S}} |u^+(y)|^p d\sigma(y) \right)^{1/p} < \infty, \quad (1)$$

where $d\sigma$ denotes the normalized surface measure on $\mathbf{S}$ and u^+ is the radial maximal function

$$u^+(y) = \sup_{0 \leq r < 1} |u(ry)|, \quad |y| = 1. \quad (2)$$

Also, define a function u^* by

$$u^*(\xi) = \sup_{z \in D_c(\xi)} |u(z)|, \quad (3)$$

where

$$D_c(\xi) = \{z \in \mathbf{B} : |\xi - z| < c(1 - |z|)\}, \quad \xi \in \mathbf{S}, \quad c > 1. \quad (4)$$

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

The Class $\text{QNS}(\mathbf{B})$. Let $\text{QNS}(\mathbf{B})$ denote the class of nonnegative measurable functions u on $\mathbf{B}$ for which a constant $Q = Q(u)$ exists such that

$$u(a) \leq Q \varepsilon^{-N} \int_{\mathbf{B}(a, \varepsilon)} u dv \quad (5)$$

whenever

$$\mathbf{B}(a, \varepsilon) := \{x : |x - a| < \varepsilon\} \subset \mathbf{B}. \quad (6)$$

Here dv denotes the normalized Lebesgue measure on $\mathbf{B}$.

Members of $\text{QNS}(\mathbf{B})$ are called quasilinearly subharmonic functions (see [1, 2]). The class QNS contains nonnegative subharmonic functions.

We need the following results.

Theorem A (see [2, 3]). *Let $p > 0$. If $u \in \text{QNS}(\mathbf{B})$, then $u^p \in \text{QNS}(\mathbf{B})$, and $Q(u^p) \leq C_{p,N} Q(u)$.*

For a function $\varphi : \mathbf{S} \rightarrow \mathbb{R}$, $\varphi \in L^1(\mathbf{S})$, let

$$(M\varphi)(\xi) = \sup_{r>0} \frac{1}{\sigma(\mathbf{S}_r(\xi))} \int_{\mathbf{S}_r(\xi)} |\varphi(\eta)| d\sigma(\eta), \quad (7)$$

where

$$\mathbf{S}_r(\xi) = \{\eta \in \mathbf{S} : |\eta - \xi| < r\}. \quad (8)$$

Theorem B (Hardy-Littlewood maximal theorem). *If $1 < p < \infty$, then*

$$\int_S |M\varphi|^p d\sigma \leq C_p \int_S |\varphi|^p d\sigma. \quad (9)$$

The following theorem is well known in the case of nonnegative subharmonic functions and is due to Fefferman and Stein (see [4]).

Theorem 1. *Let $0 < p < \infty$. If $u \in \text{QNS}(\mathbf{B})$, then*

$$\int_S (u^*)^p d\sigma \leq C \int_S (u^+)^p d\sigma, \quad (10)$$

where C depends only on p , N , and $Q(u)$.

Proof. In view of Theorem A, we can assume that $p > 1$. Let u be a QNS function on $\mathbf{B}$. Then

$$\sup_{z \in D_\varepsilon(\xi)} u(z) \leq K \sup_{z \in D_\varepsilon(\xi)} \frac{1}{(1 - |z|)^n} \int_{\mathbf{B}_\varepsilon(z)} u(w) dv(w), \quad (11)$$

where $\mathbf{B}_\varepsilon(z) = \{w : |w - z| < \varepsilon(1 - |z|)\}$, $0 < \varepsilon < 1$.

To continue the proof, we need the following lemma.

Lemma 2. *If $z \in D_\varepsilon(\xi)$ and $w \in \mathbf{B}_\varepsilon(z)$, $0 < \varepsilon < 1$, then*

$$|\rho - r| < \varepsilon(1 - r), \quad |\eta - \xi| < k|1 - r|, \quad (12)$$

where $\rho = |w|$, $r = |z|$.

Proof. Let

$$|z - \xi| < c(1 - |z|), \quad w \in B_\varepsilon(z). \quad (13)$$

That is,

$$|w - z| < \varepsilon(1 - |z|). \quad (14)$$

Hence,

$$||w| - |z|| < \varepsilon(1 - |z|). \quad (15)$$

That is,

$$|\rho - r| < \varepsilon(1 - r). \quad (16)$$

On the other hand,

$$\begin{aligned} |\eta - \xi| &\leq |\eta - z| + |z - \xi| \\ &\leq |\eta - z| + c(1 - r). \end{aligned} \quad (17)$$

Since

$$\begin{aligned} |\eta - z| &= |\eta - r\eta| \leq |\eta - \rho\eta| + |\rho\eta - r\eta| \\ &= (1 - \rho) + |w - z| \\ &\leq (1 - \rho) + \varepsilon(1 - r) \\ &\leq 1 - r + \varepsilon(1 - r) + \varepsilon(1 - r) \\ &= (1 + 2\varepsilon)(1 - r), \end{aligned} \quad (18)$$

from this and (17), we get

$$|\eta - \xi| \leq (c + 1 + 2\varepsilon)(1 - r). \quad (19)$$

The proof of the lemma is complete.

We continue the proof of the theorem.

From (11) and Lemma 2, we get

$$\begin{aligned} \sup_{z \in D_\varepsilon(\xi)} u(z) &\leq K \sup_{z \in D_\varepsilon(\xi)} \frac{1}{(1 - r)^n} \int_{|\eta - \xi| < k(1 - r)} d\sigma(\eta) \\ &\quad \times \int_{|\rho - r| \leq \varepsilon(1 - r)} u(\rho\eta) n\rho^{n-1} d\rho \\ &\leq K \sup_{z \in D_\varepsilon(\xi)} \frac{n}{(1 - r)^n} \\ &\quad \times \int_{|\eta - \xi| < k(1 - r)} d\sigma(\eta) 2\varepsilon(1 - r) u^+(\eta) \\ &= C \sup_{z \in D_\varepsilon(\xi)} \frac{1}{(1 - r)^{n-1}} \int_{|\eta - \xi| < k(1 - r)} u^+(\eta) d\sigma(\eta) \\ &= C \sup_{0 < r < 1} \frac{1}{(1 - r)^{n-1}} \int_{|\eta - \xi| < k(1 - r)} u^+(\eta) d\sigma(\eta). \end{aligned} \quad (20)$$

Hence, if $p > 1$, then, according to Theorem B,

$$\int_S u^*(\xi)^p d\sigma(\xi) \leq C \int_{\mathbf{B}} u^+(\eta)^p d\sigma(\eta). \quad (21)$$

The proof of the theorem is complete. $\square$

2. A Maximal Theorem for Bergman Type Spaces

The harmonic Bergman space $\mathcal{A}_\alpha^p = \mathcal{A}_\alpha^p(\mathbf{B})$ consists of functions u harmonic in $\mathbf{B}$ for which

$$\|u\|_{p,\alpha}^p := \int_{\mathbf{B}} |u(z)|^p (1 - |z|)^\alpha dv(z) < \infty. \quad (22)$$

Define now the maximal function u^* by

$$u^*(z) = \sup \left\{ |u(w)| : |w - z_0| < 2(1 - |w|), |w| < \frac{1 + |z|}{2} \right\}, \quad (23)$$

where $z_0 = z/|z|$, $z \neq 0$.

Theorem 3. *If $u \in \mathcal{A}_\alpha^p$, then $u^* \in \mathcal{A}_\alpha^p$, and $\|u^*\|_{p,\alpha} \leq C\|u\|_{p,\alpha}$, where C is independent of u .*

Since $|u| \in \text{QNS}$, this theorem is a special case of the following.

Theorem 4. *If $u \in \text{QNS}$ and*

$$K_{p,\alpha}(f) := \int_{\mathbf{B}} u(z)^p (1 - |z|^2)^\alpha dv(z) < \infty, \quad (24)$$

then $K_{p,\alpha}(u^*) < \infty$, and there is a constant C depending only on $Q(u)$, p , and α such that $K(u^*) \leq C K(u)$.

Proof. For $\rho = (3 + r)/4$, $r = |z|$, we have

$$\begin{aligned} |w - z_0\rho| &\leq |w - z_0| + 1 - \rho \leq 2(1 - |w|) + 1 - \rho \\ &\leq 2(\rho - |w|) + 3(1 - \rho) \leq 5(\rho - |w|) \end{aligned} \quad (25)$$

because $1 - \rho \leq \rho - |w|$, and we have

$$\begin{aligned} |w - z_0| &< 2(1 - |w|), \\ |w| < \frac{1 + |z|}{2} &\implies \left| w - z_0 \frac{3 + |z|}{4} \right| < 5 \left(\frac{3 + |z|}{4} - |w| \right). \end{aligned} \quad (26)$$

Further, we have

$$\begin{aligned} &\int_{\mathbf{B}} u^*(z) (1 - |z|^2)^\alpha dv(z) \\ &= n \int_0^1 (1 - r^2)^\alpha r^{n-1} dr \int_{\mathbf{S}} u^*(ry) d\sigma(y), \\ u^*(ry) &\leq \sup \left\{ |u(w)| : \left| w - \frac{3+r}{4} y \right| \right. \\ &\quad \left. < 5 \left(\frac{3+r}{4} - |w| \right) \right\} \\ &= \sup \left\{ |u_\rho(z)| : |z - \rho y| < 5(1 - |z|) \right\}, \end{aligned} \quad (27)$$

where $\rho = (3 + r)/4$, $z = w/\rho$, $|(w/\rho) - y| < 5|1 - (|w|/\rho)|$, and $|w - \rho| < 5(\rho - |w|)$.

Hence, by Theorem 1,

$$\begin{aligned} &\int_{\mathbf{B}} (u^*(z))^p (1 - |z|^2)^\alpha dv(z) \\ &\leq n \int_0^1 (1 - r^2)^\alpha r^{n-1} dr \int_{\mathbf{S}} (u^*(ry))^p d\sigma(y) \\ &\leq n \int_0^1 (1 - r^2)^\alpha r^{n-1} dr \int_{\mathbf{S}} (u_\rho^*(y))^p d\sigma(y) \\ &\leq n \int_0^1 (1 - r^2)^\alpha r^{n-1} dr \int_{\mathbf{S}} (u_\rho^+(y))^p d\sigma(y) \\ &\leq C \int_{\mathbf{B}} u(z)^p (1 - |z|^2)^\alpha dv(z). \end{aligned} \quad (28)$$

This completes the proof of the theorem. $\square$

References

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