Research Article

A Global Convergence of LS-CD Hybrid Conjugate Gradient Method

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Conjugate gradient method is one of the most effective algorithms for solving unconstrained optimization problem. In this paper, a modified conjugate gradient method is presented and analyzed which is a hybridization of known LS and CD conjugate gradient algorithms. Under some mild conditions, the Wolfe-type line search can guarantee the global convergence of the LS-CD method. The numerical results show that the algorithm is efficient.

1. Introduction

Consider the following nonlinear programs:

$$\min_{x \in R^n} f(x), \quad (1)$$

where R^n denotes an n -dimensional Euclidean space and $f(x)$ is continuously differentiable function.

As you know, conjugate gradient method is a line search method that takes the following form:

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \dots, \quad (2)$$

where d_k is a descent direction of $f(x)$ at x_k and is a stepsize obtained by some one-dimensional line search. If x_k is the current iterate, we denote $f(x_k) \triangleq f_k$, $\nabla f(x_k) \triangleq g_k$, and $\nabla^2 f(x_k) \triangleq G_k$, respectively. If G_k is available and inverse, then $d_k = -G_k^{-1} g_k$ leads to the Newton method and $d_k = -g_k$ results in the steepest descent method [1]. The search direction d_k is generally required to satisfy $g_k^T d_k \leq 0$, which guarantees that d_k is a descent direction of $f(x)$ at x_k [2]. In order to guarantee the global convergence, we sometimes require d_k to satisfy a sufficient descent condition as follows:

$$g_k^T d_k \leq -c \|g_k\|^2, \quad (3)$$

where $c > 0$ is a constant and $\|\cdot\|$ is the Euclidean norm. In line search methods, the well-known conjugate gradient method has the following form:

$$d_k = \begin{cases} -g_k, & \text{if } k = 0 \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1. \end{cases} \quad (4)$$

Different conjugate gradient algorithms correspond to different choices for the parameter β_k , where β_k can be defined by

$$\begin{aligned} \beta_k^{\text{FR}} &= \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, & \beta_k^{\text{PRP}} &= \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2}, \\ \beta_k^{\text{DY}} &= \frac{\|g_k\|^2}{d_{k-1}^T (g_{k-1} - g_k)}, & \beta_k^{\text{LS}} &= -\frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}, \\ \beta_k^{\text{CD}} &= -\frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}, & \beta_k^{\text{HS}} &= -\frac{g_k^T (g_{k-1} - g_k)}{d_{k-1}^T (g_{k-1} - g_k)}, \end{aligned} \quad (5)$$

or by other formulae. The corresponding methods are called FR (Fletcher-Reeves) [3], PRP (Polak-Ribière-Polyak) [4, 5], DY (Dai-Yuan) [6], CD (conjugate descent [7]), LS (Liu-Storey [8]), and HS (Hestenes-Stiefel [9]) conjugate gradient method, respectively.

Although the above mentioned conjugate gradient algorithms are equivalent to each other for minimizing strong convex quadratic functions under exact line search, they

have different performance when using them to minimize nonquadratic functions or when using inexact line searches. For general objective function, the FR, DY, and CD methods have strong convergence properties, but they may have modest practical performance due to jamming. On the other hand, the methods of PRP, LS, and HS in general may not be convergent, but they often have better computational performance.

Touati-Ahmed and Storey [10] have given the first hybrid conjugate algorithm; the method is combinations of different conjugate gradient algorithms; mainly it is being proposed to avoid the jamming phenomenon. Recently, some kinds of new hybrid conjugate gradient methods are given in [11–17]. Based on the new method, we focus on hybrid conjugate gradient methods and analyze the global convergence of the methods with Wolfe-type line search.

The rest of this paper is organized as follows. The algorithm is presented in Section 2. In Section 3 the global convergence is analyzed. We give the numerical experiments in Section 4.

2. Description of Algorithm

Algorithm 1.

Step 0. Initialization:

given a starting point $x_0 \in R^n$, choose parameters

$$0 < \varepsilon \ll 1, \quad 0 < \delta < \frac{1}{2}, \quad \delta < \sigma < 1, \quad d_0 = -g_0. \quad (6)$$

Set $k = 0$.

Step 1. If $\|g_k\| < \varepsilon$, stop; else go to Step 2.

Step 2. Compute step size α_k , such that

$$\begin{aligned} f(x_k) - f(x_k + \alpha_k d_k) &\geq -\delta \alpha_k g_k^T d_k, \\ \sigma g_k^T d_k &\leq g(x_k + \alpha_k d_k)^T d_k \leq 0. \end{aligned} \quad (7)$$

Step 3. Let $x_{k+1} = x_k + \alpha_k d_k$; if $\|g_{k+1}\| < \varepsilon$, stop; otherwise, go to Step 4.

Step 4. Compute the search direction

$$d_{k+1} = -g_{k+1} + \beta_{k+1}^{\text{LS-CD}} d_k, \quad (8)$$

where $\beta_k^{\text{LS-CD}} = \max\{0, \min\{\beta_k^{\text{LS}}, \beta_k^{\text{CD}}\}\}$.

Step 5. Let $k := k + 1$, and go to Step 2.

Throughout this paper, the following basic assumptions on the objective function are assumed, which have been widely used in the literature to analyze the global convergence of the conjugate gradient methods.

(H2.1) The objective function $f(x)$ is continuously differentiable and has a lower bound on the level set $L_0 = \{x \in R^n \mid f(x) \leq f(x_0)\}$, where x_0 is the starting point.

(H2.2) The gradient $g(x)$ of $f(x)$ is Lipschitz continuous in some neighborhood U of L_0 ; that is, there exists a constant $L > 0$, such that

$$\|g(x) - g(y)\| \leq L \|x - y\|, \quad \forall x, y \in U. \quad (9)$$

Since $\{f(x_k)\}$ is decreasing, it is clear that the sequence $\{x_k\}$ generated by Algorithm 1 is contained in L_0 .

3. Global Convergence of Algorithm

Now we analyze the global convergence of Algorithm 1.

Lemma 2. Suppose that assumptions (H2.1) and (H2.2) hold; the sequences $\{g_k\}$ and $\{d_k\}$ are to be generated by Algorithm 1, if $g_k \neq 0$ for all $k \geq 0$; then

$$g_k^T d_k < 0. \quad (10)$$

Proof. If $k = 0$, then $d_0 = -g_0$, and we get

$$g_0^T d_0 = -\|g_0\|^2 < 0. \quad (11)$$

When $k \geq 1$, multiplying g_k^T by

$$d_k = -g_k + \beta_k^{\text{LS-CD}} d_{k-1}, \quad (12)$$

we obtain

$$g_k^T d_k = -\|g_k\|^2 + \beta_k^{\text{LS-CD}} g_k^T d_{k-1}. \quad (13)$$

It follows from $\beta_k^{\text{LS-CD}} \geq 0$ and $g_k^T d_{k-1} \leq 0$ that

$$g_k^T d_k \leq -\|g_k\|^2 < 0. \quad (14)$$

Therefore, the result is true. $\square$

Lemma 3. Suppose that assumptions (H2.1) and (H2.2) hold, and consider any iteration of the form (2), where d_k is a descent direction and α_k satisfies the Wolfe conditions (7). Then, the Zoutendijk condition

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < +\infty \quad (15)$$

holds.

Proof. From (7), we have

$$(\sigma - 1) g_k^T d_k \leq (g_{k+1} - g_k)^T d^k. \quad (16)$$

In addition, the assumption (H2.2) gives

$$L \alpha_k \|d_k\|^2 \geq (g_{k+1} - g_k)^T d^k. \quad (17)$$

Combing these two relations, we have

$$\alpha_k \geq \frac{(\sigma - 1) \|g_k^T d_k\|}{L \|d_k\|^2}, \quad (18)$$

TABLE 1: Test results for CD algorithm.

Prob	x_0	x_k	f_*	NI
1	(4, 5, 10.1)	(NaN, NaN, NaN)	NaN	5000
2	(6.5, 23)	(14.92256489843370, 14.92256511253481)	-0.99999999999993	18
3	(7, 11)	(0.00000007501213, -0.49999972373559)	-0.24999999999992	16
4	(2.5, 11.9)	(5.9999999984147, 15.99997571516633)	-0.99999999998863	160
5	(7, 9.8)	(NaN, NaN)	NaN	5000
6	(-3, -1, -3, -1)	(NaN, NaN, NaN, NaN)	NaN	5000

x_0 : the initial point; x_k : the final point; f_* : the final value of the objective function; NI: the number of times of iteration for each problem.

TABLE 2: Test results for LS algorithm.

Prob	x_0	x_k	f_*	NI
1	(4, 5, 10.1)	(-1.744444446043301, -1.74444445793385, 6.97777777040829)	4.09877659651031	27
2	(6.5, 23)	(14.92256524764775, 14.92256496282052)	-0.99999999999992	26
3	(7, 11)	(-0.00000003058419, -0.49999989218958)	-0.24999999999999	34
4	(2.5, 11.9)	(5.99999999851030, 15.99997486954042)	-0.99999999998783	323
5	(7, 9.8)	(-0.43256275978261, 0.10814076312008)	0.78917703640310	19
6	(-3, -1, -3, -1)	(2.50000015783668, 2.50000010044152, 5.24999984443432, -3.50000007174394)	-79.87499999999992	29

x_0 : the initial point; x_k : the final point; f_* : the final value of the objective function; NI: the number of times of iteration for each problem.

which with $f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k$ implies that

$$f_k - f_{k+1} \geq \delta \frac{(1 - \sigma) (g_k^T d_k)^2}{L \|d_k\|^2}. \quad (19)$$

Thus,

$$\sum_{k=0}^{\infty} f_k - f_{k+1} \geq \sum_{k=0}^{\infty} \delta \frac{(1 - \sigma) (g_k^T d_k)^2}{L \|d_k\|^2}. \quad (20)$$

Noting that f is bounded below, (15) holds.

Furthermore, from Lemma 2 and (3), we can easily obtain the following condition:

$$\sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < +\infty. \quad (21)$$

□

Theorem 4. Suppose that x_0 is a starting point for which assumptions (H2.1) and (H2.2) hold. Consider Algorithm 1; then, one has either $g_k = 0$ for some finite k or

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0. \quad (22)$$

Proof. The first statement is easy to show, since the only stopping point is in Step 3. Thus, assume that the algorithm generates an infinite sequence $\{g_k\}$; if the statement is false, there exists a constant $\varepsilon > 0$, such that

$$\|g_k\| \geq \varepsilon, \quad \forall k \geq 0. \quad (23)$$

From (8), we have

$$d_k = -g_k + \beta_k^{\text{LS-CD}} d_{k-1}. \quad (24)$$

Squaring both sides of the above equation, we get

$$\|d_k\|^2 + 2g_k^T d_k + \|g_k\|^2 = (\beta_k^{\text{LS-CD}})^2 \|d_{k-1}\|^2, \quad (25)$$

that is,

$$\|d_k\|^2 = (\beta_k^{\text{LS-CD}})^2 \|d_{k-1}\|^2 - 2g_k^T d_k - \|g_k\|^2. \quad (26)$$

From the definitions of β_k^{LS} , and β_k^{CD} , we have

$$0 \leq \beta_k^{\text{LS-CD}} \leq \beta_k^{\text{CD}}. \quad (27)$$

Thus, we can get

$$\|d_k\|^2 \leq (\beta_k^{\text{CD}})^2 \|d_{k-1}\|^2 - 2g_k^T d_k - \|g_k\|^2. \quad (28)$$

On the other hand, multiplying g_k^T by

$$d_k = -g_k + \beta_k^{\text{LS-CD}} d_{k-1}, \quad (29)$$

we obtain

$$g_k^T d_k = -\|g_k\|^2 + \beta_k^{\text{LS-CD}} g_k^T d_{k-1}. \quad (30)$$

Considering that $\beta_k^{\text{LS-CD}} \geq 0$ and $g_k^T d_{k-1} \leq 0$, we have

$$g_k^T d_k \leq -\|g_k\|^2 < 0, \quad (31)$$

which indicates that

$$(g_k^T d_k)^2 \geq \|g_k\|^4. \quad (32)$$

TABLE 3: Test results for LS-CD algorithm.

Prob	x_0	x_k	f_*	NI
1	(4, 5, 10.1)	(-1.744444443639771, -1.74444443412601, 6.97777778236907)	1.92102149037102	25
2	(6.5, 23)	(14.92256501285285, 14.92256521587144)	-0.999999999999996	15
3	(7, 11)	(0.00000000238593, -0.49999999997767)	-0.250000000000000	23
4	(2.5, 11.9)	(5.99999999856486, 15.99997486532800)	-0.99999999998782	311
5	(7, 9.8)	(-0.43256260505806, 0.10814059874208)	0.78917703640311	23
6	(-3, -1, -3, -1)	(2.49999974693426, 2.49999983895816, 5.25000008609708, -3.49999988497012)	-79.87499999999987	16

x_0 : the initial point; x_k : the final point; f_* : the final value of the objective function; NI: the number of times of iteration for each problem.

Dividing the above inequality (28) by $(g_k^T d_k)^2$, we obtain

$$\begin{aligned}
\frac{\|d_k\|^2}{(g_k^T d_k)^2} &\leq \frac{(\beta_k^{\text{CD}})^2 \|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2g_k^T d_k}{(g_k^T d_k)^2} - \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\
&= \frac{(-\|g_k\|^2/d_{k-1}^T g_{k-1})^2 \|d_{k-1}\|^2}{(g_k^T d_k)^2} \\
&\quad - \frac{2g_k^T d_k}{(g_k^T d_k)^2} - \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\
&\leq \frac{(-\|g_k\|^2/d_{k-1}^T g_{k-1})^2 \|d_{k-1}\|^2}{\|g_k\|^4} \\
&\quad - \frac{2g_k^T d_k}{\|g_k\|^4} - \frac{\|g_k\|^2}{\|g_k\|^4} \\
&\leq \frac{\|d_{k-1}\|^2}{(d_{k-1}^T g_{k-1})^2} + \frac{2}{\|g_k\|^2} - \frac{1}{\|g_k\|^2} \\
&= \frac{\|d_{k-1}\|^2}{(d_{k-1}^T g_{k-1})^2} + \frac{1}{\|g_k\|^2}.
\end{aligned} \tag{33}$$

Using the above inequality recursively and noting that

$$\|d_0\|^2 = -g_0^T d_0 = \|g_0\|^2, \tag{34}$$

we have

$$\frac{\|d_k\|^2}{(g_k^T d_k)^2} \leq \sum_{l=0}^{k-1} \frac{1}{\|g_l\|^2}. \tag{35}$$

Then, from (23) and (35), it holds that

$$\frac{(g_k^T d_k)^2}{\|d_k\|^2} \geq \frac{\varepsilon^2}{k}. \tag{36}$$

Thus, it is easy to obtain

$$\sum_{k=0}^{\infty} \frac{(g_k^T d_k)^2}{\|d_k\|^2} = +\infty. \tag{37}$$

This contradicts the Zoutendijk condition (15). Therefore, the conclusion holds. $\square$

TABLE 4: The performance of the LS method, CD method, and LS-CD method.

Prob	Dim	LS	CD	LS-CD
		NI/NF/NG	NI/NF/NG	NI/NF/NG
PEN 1	100	51/142/92	62/223/182	51/168/125
	1000	33/125/83	52/181/165	33/164/117
	10000	21/118/72	31/157/121	21/132/102
TRIG	100	305/399/398	NaN	305/399/398
	500	343/424/423	NaN	343/424/423
ROSEX	500	52/112/107	92/267/238	50/186/157
	1000	70/149/145	98/287/255	70/246/183

Prob: the test problem name from [18]; Dim: the problem dimension; NI: the iterations number; NF: the function evaluations number; NG: the gradient evaluations number.

4. Numerical Experiments

In this section, we give the numerical results of Algorithm 1 to show that the method is efficient for unconstrained optimization problems. We set the parameters $\delta = 0.3$, and $\delta = 0.7$ and use MATLAB 7.0 to test the chosen problems on a PC with 2.10 GHz CPU processor, 1.0 GB RAM memory, and Linux operation system. We also use the condition $\|g_k\| \leq 10^{-6}$ or It-max > 5000 as the stopping criterion (It-max denotes the maximal number of iterations). When the limit of 5000 function evaluations was exceeded, the run was stopped, which is indicated by "NaN." The problems that we tested are from [17, 19].

$$\text{Prob 1 } f(x) = (x_1 + 3x_2 + x_3)^2 + 4(x_1 - x_2)^2,$$

$$\text{Prob 2 } f(x) = \sin(x_1 + x_2) + (x_1 - x_2)^2,$$

$$\text{Prob 3 } f(x) = \ln(1 + x_1^2) + x_2^2 + x_2,$$

$$\text{Prob 4 } f(x) = \sin(\pi x_1/12) \cos(\pi x_2/16),$$

$$\text{Prob 5 } f(x) = e^{x_1} + x_1^2 + 2x_1x_2 + 4x_2^2,$$

$$\text{Prob 6 } f(x) = x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5(x_1 + x_2) - 21x_3 + 7x_4.$$

Tables 1, 2, and 3 show the computation results.

Because conjugate gradient algorithms are devised for solving large-scale unconstrained optimization problems, we chose some large-scale problems from [18] and compared the performance of the hybrid LS-CD method (Algorithm 1 in Section 2) with the LS method and CD method.

From Tables 1, 2, 3, and 4, we see that the performance of Algorithm 1 is better than that of the CD and the LS methods

for some problems. Therefore, our numerical experiments show that the algorithm is efficient.

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