Research Article

Semiclassical Strings in $(2 + 1)$ -Dimensional Backgrounds

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This study analyzes the geometrical relationship between a classical string and its semiclassical quantum model. From an arbitrary $(2 + 1)$ -dimensional geometry, a specific ansatz for a classical string is used to generate a semi-classical quantum model. In this framework, examples of quantum oscillations and quantum free particles are presented that uniquely determine a classical string and the space-time geometry where its motion takes place.

1. Introduction

Quantization schemes in string theory are characterized by their background dependency. From this standpoint, space-time is something more fundamental than the strings and thus cannot be framed in terms of them. This conceptual impediment seems to prevent string theory from being a quantum model of gravity. However, even if we disregard philosophical questions, the technical difficulties in string theory are also great and a general quantization procedure for string theory is as yet unknown.

On the other hand, quantization of string theory is possible in specific cases such as the semiclassical method developed for the pulsating string in $AdS_5 \times S^5$ [1, 2], a method that has been applied to various backgrounds [3–8]. The procedure is linked to a particular geometry where classical string motion takes place, and in this paper a generalization of the method that enables a description of a wider class of classical strings in various $(2+1)$ -dimensional spaces is presented. Strings can be understood to move in an effective space, which is a subspace of the ten-dimensional space-time where string theory is consistently defined.

The basis for this generalization is the observation that metric tensor elements determine the potential of the classical Hamiltonian which is used to build a quantum model. From this simple idea, it is possible to vary the potential and to establish correspondence between the classical model and the quantum model based on the geometry of the effective

$(2 + 1)$ -dimensional space-time where the string moves. The correspondence between the quantum model and the classical model does not mean equivalence or duality, as in the AdS/CFT correspondence, but an association between a classical motion and a quantum model with a common space-time geometry. It is not clear if there are any other classical strings that can generate an identical quantum model, but this question is not posed here, because the aim is to demonstrate the existence of correspondence only. The potential is the central element relating the quantum model to the classical model, and a specific potential requires a particular space-time. The results show that quantum oscillation and quantum free particles occur at different space-time topologies. A space-time that could be used to construct a quantum oscillation and a quantum free particle for the same string has not been found. There is an important question about the consistency of the proposed geometries as supergravity backgrounds. This point has not been developed, and the results show that it is worthwhile to study it in the future, particularly because there is no reason for preventing the proposed geometries from being string backgrounds.

The paper is organized as follows. Section 2 describes dynamics of a string in a $(2 + 1)$ -dimensional and the embedding of the string world-sheet in a 3-dimensional plane space. In Section 3, string motion and space-time geometry are determined from specific potentials of the quantum model, and the quantum spectrum is also studied. Section 4 contains the author's conclusions.

2. The String and Space

(2 + 1)-dimensional space-time where the string moves is described by the line element

$$ds^2 = -dt^2 + dx^2 + f^2 d\varphi^2, \quad (1)$$

where x is a general coordinate, which can be a distance or an angle, φ is an angular coordinate, and f is an arbitrary function of the coordinates. In this space, a classical string performs an arbitrary motion. Depending on the string, the equations of motion generate the conditions that must be fulfilled by f , and the first task is to choose a string.

2.1. Classical and Quantum Dynamics. The string of interest is described by the ansatz

$$t = \kappa\tau, \quad x = x(\tau), \quad \varphi = m\sigma, \quad (2)$$

where κ is a constant and the string is wrapped m times along φ and executes some motion along x . If $f = f(x)$, the Nambu-Goto action for this string, namely,

$$\mathcal{A} = -m\sqrt{\lambda} \int d\tau f \sqrt{\kappa^2 - \dot{x}^2}, \quad (3)$$

determines its equation of motion

$$\frac{f'}{f} = -\frac{\ddot{x}}{\kappa^2 - \dot{x}^2}, \quad (4)$$

where the prime denotes a derivative with respect to x and the dot denotes a derivative with respect to τ . The above equation is nonlinear and difficult to use; thus, it will be substituted by the Virasoro constraint

$$\dot{x}^2 = \kappa^2 - m^2 f^2. \quad (5)$$

From (3) and (5), the canonical momentum and energy are

$$\Pi = \frac{m\sqrt{\lambda}}{\sqrt{\kappa^2 - \dot{x}^2}} f \dot{x}, \quad \mathcal{E} = \sqrt{\lambda} \kappa \quad (6)$$

whose canonical Hamiltonian is

$$\mathcal{H} = \kappa \sqrt{\Pi^2 + m^2 \lambda f^2}. \quad (7)$$

This classical formalism can be used to semiclassically quantize the string using the square of the Hamiltonian (7) to express the Schrödinger equation as $\mathcal{H}^2 \Psi = (\mathcal{E}^2 / \kappa^2) \Psi$, where $\mathcal{E}^2$ is the squared quantum energy. From this analysis, it follows that the coefficient f of the line element (1) determines the geometry of the space as well as the classical potential and the quantum wave function.

2.2. Space Geometry. The spatial motion of the string is constrained by the metric (1), which defines the world-sheet of the string. Consequently, the world-sheet of a string moving through the whole of space is identical to space, and thus the embedding of the 2-dimensional surface into a 3-dimensional plane space enables both the geometry of the space and

the motion of the string to be visualized. Expressing (1) as $ds^2 = -dt^2 + d\mathcal{J}^2$, the two-dimensional subspace generated by x and φ must be embedded into a three-dimensional plane space with the metric

$$dS^2 = g_{ij} dx^i dx^j = (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \quad (8)$$

It is assumed that the coordinates of the plane space have a cylindrical symmetry, so that

$$x^1 = \rho \sin \phi, \quad x^2 = \rho \cos \phi, \quad x^3 = z, \quad (9)$$

where $\rho = \rho(x)$, $z = z(x)$, and $\phi = \phi(\varphi)$. The embedding is obtained through the identity

$$d\mathcal{J}^2 = h_{ab} d\xi^a d\xi^b = g_{ij} \frac{\partial x^i}{\partial \xi^a} \frac{\partial x^j}{\partial \xi^b} d\xi^a d\xi^b, \quad (10)$$

$$\text{where } (x, \vartheta) = (\xi^1, \xi^2).$$

Using (9) in the metric tensor h_{ab} , we obtained the system of differential equations

$$\rho_x^2 + z_x^2 = 1, \quad \rho^2 \phi_\varphi^2 = f^2, \quad (11)$$

where indices x and φ represent derivatives with respect to these coordinates. It can immediately be seen that $\phi = C\varphi$, and as both of the coordinates have the same range, $C = 1$ and $\rho = f$. As f is known from the beginning, the embedding must be obtained from the integration

$$z = \pm \int dx \sqrt{1 - f'^2}. \quad (12)$$

3. Moving Strings

In the preceding section the general formalism which associates a moving string in a (2 + 1)-dimensional space to a quantum model has been presented. In order to relate the geometry and the topology of the space to the classical motion of the string, f and x are specialized, and then we expect some quantum-geometry relation to arise. Maintaining a rotational symmetry for φ , x may be either a radial or an angular coordinate. For a radial $x = r > 0$, the motion of the string is constrained to an open surface, and for an angular coordinate, the surface can be closed. However, this division does not exhaust the possibilities, because these two categories can be subdivided. Below, the radial coordinate case has been divided into the pulsating string case and the falling string case.

3.1. Radial Coordinate Pulsating String. In this case, the ansatz $f = \ell r^{n/2}$ will be used, where n is a positive integer and ℓ is a dimensional constant responsible for f having length dimension. By choosing $y = (m\sqrt{\lambda} f / \kappa)^2$ and $\kappa\tau = rg(y)$, we obtain the equation of motion from (5)

$$g + nyg_y = \frac{1}{\sqrt{1-y}}. \quad (13)$$

FIGURE 1: Time for the pulsating string. $n = 2$ solid line; $n = 5$ dotted line; $n = 50$ dashed line.

Using the hypergeometric function relation $F(a, b; b; z) = (1 - z)^{-a}$ and contiguous hypergeometric function relations, it is possible to ascertain that the general solution for (13) is

$$\kappa\tau = rF\left(\frac{1}{2}, \frac{1}{n}; 1 + \frac{1}{n}; \frac{m^2\lambda\ell}{\kappa^2}r^n\right). \quad (14)$$

For $n = 1$, $\kappa\tau \propto \sqrt{1 - y}$, which implies that the time is limited; hence, this can be discarded as a physical solution. The inverse of (14) gives $r = r(\tau)$ for each n . Although this inverse function is unknown, some considerations can be stated by observing Figure 1.

For $n = 2$, $t = \arcsin(y)$, and thus periodic oscillatory behavior for r is warranted. For other values of n , the graph shows that the maximum value of y is one and that the maximum of y is reached at a value of t , which approaches one, the greater n . For an even number n , t is an odd function where negative values are allowed for the argument, indicating a range of y in the interval $[-1, 1]$ for the function. This fact may be observed in Figure 2 by formally inverting the first terms of the infinite series generated by (14). The same fact cannot be stated for odd n , where t does not have a definite parity for the negative values of the argument. On the other hand, the inversion of the first terms confirms the existence of the maximum value of r to be one in Figure 3. However, as the negative values are not actually allowed in both of the cases in this particular problem, the behavior of $r(\tau)$ given by the positive $y(t)$ can be described as oscillatory. Of course, at $y = 0$, the string changes its direction and the derivative of $y(t)$ is not defined there. Only when negative values of y are allowed for even values of n , the derivative of $y(t)$ is well defined at these points. The conclusion is that the classical motion of $r(\tau)$ is possibly oscillatory for any $n \geq 2$.

Another aspect of the problem to be considered is the geometry of the two-dimensional surface where the string

FIGURE 2: Formal inversion of $t(y)$. $n = 2$ solid line; $n = 6$ dashed line; $n = 8$ dotted line.

FIGURE 3: Formal inversion of $t(y)$; $n = 3$ dotted line; $n = 5$ dashed line; $n = 7$ solid line.

moves. Defining the variable $w = f_r^2 = (n^2\ell^2/4)r^{n-2}$ and the coordinate $z = rh(w)$, from (12), we obtain

$$h + (n - 2)wh_w = \sqrt{1 - w}, \quad (15)$$

and consequently

$$z = rF\left(-\frac{1}{2}, \frac{1}{n-2}; 1 + \frac{1}{n-2}; \frac{n^2\ell^2}{4}r^{n-2}\right). \quad (16)$$

Equation (16) is valid for $n > 2$. If $n = 1$, $w \propto 1/r$, so that $r > 1$ and, as the string vibrates for $r < 1$, there is no physical solution for (15). For $n = 2$, z is constant and the whole plane is allowed for vibration of the string. This is an expected result coherent with the known solutions [1, 3]. The graph of (16) in Figure 4 gives an idea of the surface where the string is allowed to move.

Of course, for each particular n , the pervious graph has an identical reflected line in the negative z direction. Besides this, the surface is cylindrically symmetric, so that the whole surface is similar to a cone. As n increases, the graph approaches a straight line and the whole surface approaches a rectangular cone. It is also interesting to note that the space is finite. Only the $n = 2$ case generates an infinite plane surface.

FIGURE 4: Embedded space for the pulsating string. $n = 3$ dotted line; $n = 10$ solid line; $n = 25$ dashed line.

The description of the space and the motion of the string concludes the analysis of the classical behavior of the string. The next goal is to semiclassically study the quantum features of this system. For each n , a specific quantum model can be obtained, and it is described by the Schrödinger equation

$$-\Psi'' - \frac{n}{2r}\Psi' + m^2\lambda\ell^2 r^n \Psi = \frac{\mathcal{E}^2}{\kappa^2}\Psi. \quad (17)$$

Except for the nonphysical $n = 1$ case, the exactly solvable solution of (17) only occurs when $n = 2$. For this particular situation, the wave function and the energy spectrum are

$$\Psi_N = \mathcal{N} e^{-(m\sqrt{\lambda}\ell/2)r^2} L_N(m\sqrt{\lambda}\ell r^2), \quad (18)$$

$$\frac{\mathcal{E}_N^2}{\kappa^2 m \sqrt{\lambda} \ell} = 4N + 2,$$

where $\mathcal{N}$ is the normalization constant, N is a positive integer, and L_N are the Laguerre polynomials. In this case, ℓ is dimensionless and can be set to one.

The $n > 2$ solutions must be studied perturbatively, and the nonperturbed case is calculated by excluding the potential term of (17). The solutions are

$$\Psi = \frac{1}{r^{(n-2)/4}} \left[A J_{(n-2)/4} \left(\frac{\mathcal{E}}{\kappa} r \right) + B Y_{(n-2)/4} \left(\frac{\mathcal{E}}{\kappa} r \right) \right], \quad (19)$$

where A and B are integration constants. The space is finite and the range of the radial coordinate can be assumed to be $[0, R]$. Solution (19) describes quantum free particles; however, the permitted energies can be either continuous or quantized. Wave functions where $\Psi(r = R) \neq 0$ have continuous energies and wave functions where $\Psi(r = R) = 0$

have a quantized energy spectrum given according to the zeros of the Bessel functions,

$$\frac{\mathcal{E}_N}{\kappa} = \frac{R^{(N)}}{R}, \quad \text{so that } N \in \mathbb{N}, \quad \mathcal{Z} = \{J, Y\}. \quad (20)$$

The index $\mathcal{Z}$ is due to the fact that the zeros of the Bessel functions J and Y are not common and then there are two independent wave functions and two energy spectra. Both of the wave functions are normalizable, except only the Y wave function for $n = 6, 10, 14$, and so forth.

The perturbative calculations for energy need a wave function given by an orthogonal set, and thus only the quantized energy wave functions can be used. The orthogonal set can be obtained from the Bessel function J , which obeys the condition

$$\int_0^1 dx x J_\mu(\alpha x) J_\mu(\beta x) = \frac{\beta J_\mu(\alpha) J_{\mu-1}(\beta) - \alpha J_{\mu-1}(\alpha) J_\mu(\beta)}{\alpha^2 - \beta^2}, \quad (21)$$

which is zero if α and β are different Bessel function zeros. From (21), we also get the normalization condition

$$\int_0^1 dx x J_\mu^2(R^{(N)} x) = \frac{J_{\mu+1}^2(R^{(N)})}{2R_j^{(N)2}}. \quad (22)$$

Thus, the energy in the first order of perturbation for the potential $f^2 = \ell^2 r^n$ is

$$\begin{aligned} \frac{\delta \mathcal{E}_N^2}{\kappa^2 m^2 \lambda \ell^2} &= R^{(N)n+1} \int_0^1 dx x^{n+1} J_{(n-2)/2}^2(R^{(N)} x) \\ &= \frac{\lambda R^{(N)2n+2}}{n \Gamma(n/2) 2^{n-2} J_{(n-2)/2}^2(R^{(N)}) 2^F_3} \\ &\quad \times \left(\left[n, \frac{n-1}{2} \right], \left[n-1, \frac{n}{2}, n+1 \right], -R^{(N)2} \right), \end{aligned} \quad (23)$$

where ${}_2F_3$ is a generalized hypergeometric function. As the series defined by this object converges for every finite argument, (23) is expected to be a well-behaved value that does not diverge for any zero of J .

The result rounds off the analysis, which comprises the geometrical correspondence between the classical dynamics and quantum dynamics of a string. Of course, there is no correspondence in the terms of gauge/gravity duality, as the classical string does not have a quantized spectrum and so the models are not identical in this sense. However, the example does show that a classical pulsating string and a quantum oscillation are connected through a specific geometry, which determines the string motion and the quantum energy spectrum. Another example of this correspondence is provided in the next section.

3.2. Free Falling String. This model is constructed using $f^2 = \ell^2 r^{-n}$, with $n \in \mathbb{N}$, and the analysis follows the manner

FIGURE 5: Time for the free falling string. $n = 2$ dashed line; $n = 5$ dotted line; $n = 50$ solid line.

developed for the pulsating string, comprising of the classical string motion, the geometry of the space, and semiclassical quantization.

Choosing $y = (m\sqrt{\lambda\ell}f/\kappa)^2$ and $\kappa\tau = rg(y)$, we ascertain from (5) that the classical motion obeys

$$g - nyg_y = \frac{1}{\sqrt{1-y}}. \quad (24)$$

From the hypergeometric function relation $F(a, b; z) = (1-z)^{-a}$ and the contiguous hypergeometric function relations, it is possible to ascertain that the general solutions for (24) are

$$\kappa\tau = rF\left(\frac{1}{2}, -\frac{1}{n}; 1 - \frac{1}{n}; \frac{n^2\lambda\ell^2}{\kappa^2}r^{-n}\right). \quad (25)$$

This solution holds for $n > 1$, because $n = 1$ allows negative values for τ , thus comprising an unphysical solution. These solutions are different from the former pulsating case, because the radial coordinate and the time coordinate continuously increase, as can be seen in Figure 5.

The greater the n value, the more the solutions approach the straight line $\kappa\tau = r$. This string goes continuously to infinity, asymptotically approaching a constant velocity of a free particle; hence, it can be described as a string in free fall.

The other aspect of the classical picture, the geometry of the space, is obtained from (12) in the same manner that it was obtained for the pulsating string case, and it is described by

$$z = \pm rF\left(-\frac{1}{2}, -\frac{1}{n+2}; 1 - \frac{1}{n+2}; \frac{n^2\ell^2}{4}r^{-n-2}\right), \quad (26)$$

whose positive part can be seen in Figure 6.

Of course, each surface has a cylindrical symmetry and it consists of two infinite sheets with a hole in the center. The existence of the hole is naturally predictable, as the metric is not defined at $r = 0$, and then the puncture in space is expected. The existence of the two sheets where the free

FIGURE 6: Embedded space for the free falling string. $n = 2$ dotted line; $n = 4$ solid line; $n = 50$ dashed line.

fall of the string can occur seems somewhat surprising. However, this kind of situation has already been observed in a sphere [1, 3], where the string independently pulsates in each hemisphere.

After the classical description, the quantum fluctuations are studied through the Schrödinger equation

$$-\Psi'' + \frac{n}{2r}\Psi' + \frac{m^2\lambda\ell^2}{r^n}\Psi = \frac{\mathcal{E}^2}{\kappa^2}\Psi. \quad (27)$$

The $n = 1$ has already been observed not to have a classical physical solution, and then the analysis comprises $n \geq 2$. There is an exact solution for the $n = 2$ case, namely,

$$\Psi = r \left[AJ_a\left(\frac{\mathcal{E}}{\kappa}r\right) + BY_a\left(\frac{\mathcal{E}}{\kappa}r\right) \right], \quad (28)$$

where $a = \sqrt{1 + m^2\lambda\ell^2}$ and A and B are integration constants. For $n > 2$, the exact solutions are unknown and the free particle solutions are very similar to the aforementioned exact solution, given by

$$\Psi = r^{(n+2)/4} \left[AJ_{(n+2)/4}\left(\frac{\mathcal{E}}{\kappa}r\right) + BY_{(n+2)/4}\left(\frac{\mathcal{E}}{\kappa}r\right) \right]. \quad (29)$$

Although the intensity of the wave function increases with r and diverges at infinity, the solutions are indeed free particles. One manner of visualizing this is to see that it comes from the fact that the nonnormalizable free particle solutions define a Dirac delta function and then obey a localization condition [9],

$$\int_{-\infty}^{\infty} \Psi\Psi^* \sqrt{-g}dx = \delta(x). \quad (30)$$

The Dirac delta function in terms of the Bessel functions in a $(d + 1)$ -dimensional space is

$$\delta^{d+1}(\epsilon^2 - \eta^2) = \int_0^\infty dr r J_\mu(\epsilon r) J_\mu(\eta r), \quad (31)$$

and it fits perfectly with (30). Thus, even the exact solution for $n = 2$ is a free particle, and for other values of n , the same interpretation holds.

As for the previous case, there are continuous and quantized energies. The quantized energies obey the condition that the wave function is zero at the edge of space, and then (20) is valid in this case also. The energies of the $n > 2$ cases are calculated using perturbation theory, so that

$$\frac{\delta \mathcal{E}_N^2}{\kappa^2 m^2 \lambda \ell^2} = R^{(N)n+1} \int_0^1 dx x^{n+1} J_{(n-2)/2}^2(R^{(N)} x). \quad (32)$$

The exact expression of the above integral, given in terms of generalized ${}_2F_3$ functions, is complicated and not really pertinent to this study. However, as in the former case, the series that represents the function is convergent for any value of the argument, and this is enough to assure a finite correction to the energy.

4. Conclusion

In this paper, examples of classical strings were presented that can be semiclassically quantized through a well-known prescription. The examples demonstrate that the classical string and its quantum fluctuations are connected through the space where the motion takes place. The geometry and the topology of the space determine both the classical string and the quantum Hamiltonian.

Although the results extend the range of quantum models that can be obtained from a string motion, from the point of view of the author of this paper, it is somewhat frustrating that the potential that goes with the inverse of the distance is not permitted in the models presented. The string motion that could model the relevant physical phenomena described by this potential, namely, gravity and electromagnetism, remains unknown. On the other hand, the results are evidence that the link between quantum theory and general relativity through geometry seems not to be merely a myth.

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