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*Research Article*

# Uniqueness of Meromorphic Functions Sharing Fixed Point

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We study the uniqueness of meromorphic functions concerning differential polynomials sharing fixed point and obtain some significant results , which improve the results due to Lin and Yi (2004).

## 1. Introduction and Main Results

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. We will use the following standard notations of value distribution theory: $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots$ (see [1, 2]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\}, \quad \text{as } r \longrightarrow +\infty, \tag{1}$$

possibly outside of a set with finite measure.

Let $a$ be a finite complex number and $k$ a positive integer. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for the zeros of $f(z)-a$ in $|z| \le r$ with multiplicity $\le k$ and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for the zeros of $f(z)-a$ in $|z| \le r$ with multiplicity $\ge k$ and $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right). \tag{2}$$

Let $g(z)$ be a nonconstant meromorphic function. We denote by $\overline{N}_L(r, 1/(f-a))$ the counting function for $a$-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, where multiplicity is not counted. Similarly, we have notation $\overline{N}_L(r, 1/(g-a))$.

We say that $f$ and $g$ share $a$ CM (counting multiplicity) if $f-a$ and $g-a$ have same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM (ignoring multiplicity) if $f-a$ and $g-a$ have same zeros with ignoring multiplicities.

In 2004, Lin and Yi [3] obtained the following results.

**Theorem A.** *Let $f$ and $g$ be two transcendental meromorphic functions, $n \ge 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, then either $f(z) \equiv g(z)$ or*

$$g = \frac{(n+2)\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \qquad f = \frac{(n+2)\,h\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \tag{3}$$

*where $h$ is a nonconstant meromorphic function.*

**Theorem B.** *Let $f$ and $g$ be two transcendental meromorphic functions, $n \ge 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $z$ CM, then $f(z) \equiv g(z)$.*

In this paper, we study the uniqueness problems of entire or meromorphic functions concerning differential polynomials sharing fixed point, which improves Theorems A and B.

### 1.1. Main Results

**Theorem 1.** *Let $f$ and $g$ be two nonconstant meromorphic functions, $n \ge 11$ a positive integer. If $f^n(f-1)f'$ and*

*$g^n(g-1)g'$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) \equiv g(z)$ or*

$$g = \frac{(n+2)\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \qquad f = \frac{(n+2)\,h\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \quad (4)$$

*where $h$ is a nonconstant meromorphic function.*

**Theorem 2.** *Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then $f(z) \equiv g(z)$.*

**Theorem 3.** *Let $f$ and $g$ be two nonconstant entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, then $f(z) \equiv g(z)$.*

## 2. Some Lemmas

**Lemma 4** (see [4]). *Let $f_1$, $f_2$, and $f_3$ be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 = 1$. If $f_1$, $f_2$, and $f_3$ are linearly independent, then*

$$T(r, f_1) < \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r, f_i) + o(T(r)), \quad (5)$$

*where $T(r) = \max_{1\leq i\leq 3}\{T(r, f_i)\}$ and $r \notin E$.*

**Lemma 5** (see [1]). *Let $f_1$ and $f_2$ be two nonconstant meromorphic functions. If $c_1 f_1 + c_2 f_2 = c_3$, where $c_1$, $c_2$, and $c_3$ are non-zero constants, then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + S(r, f_1). \quad (6)$$

*Lemmas 4 and 5 play a very important role in proving our theorems.*

**Lemma 6** (see [1]). *Let $f$ be a nonconstant meromorphic function and let $k$ be a nonnegative integer, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \quad (7)$$

*The following lemmas play a cardinal role in proving our results.*

**Lemma 7.** *Let $f$ and $g$ be nonconstant meromorphic functions. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM and $n > 6$, then*

$$T(r, g) \leq \left(\frac{n+3}{n-6}\right) T(r, f) + \log r + S(r, g). \quad (8)$$

*Proof.* Applying Nevanlinna's second fundamental theorem to $g^n(g-1)g'$, we have

$$\begin{aligned} T\left(r, g^n(g-1)g'\right) &\leq \overline{N}\left(r, g^n(g-1)g'\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^n(g-1)g'-z}\right) + S(r, g) \\ &\leq \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f^n(f-1)f'-z}\right) + S(r, g). \end{aligned} \quad (9)$$

By the first fundamental theorem and (9), we have

$$\begin{aligned} (n+1)T(r, g) &\leq T(r, g^n(g-1)) + S(r, g) \\ &\leq T\left(r, g^n(g-1)g'\right) + T\left(r, \frac{1}{g'}\right) + S(r, g) \\ &\leq \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f^n(f-1)f'-z}\right) \\ &\quad + T\left(r, g'\right) + S(r, g). \end{aligned} \quad (10)$$

We know that

$$\begin{aligned} \overline{N}\left(r, \frac{1}{f^n(f-1)f'-z}\right) &\leq T\left(r, \frac{1}{f^n(f-1)f'-z}\right) \\ &= T\left(r, f^n(f-1)f'-z\right) + O(1) \\ &\leq T\left(r, f^n(f-1)f'\right) \\ &\quad + T(r, z) + O(1) \\ &\leq nT(r, f) + T(r, f-1) \\ &\quad + T\left(r, f'\right) + \log r + O(1) \\ &\leq (n+3)T(r, f) + \log r + O(1). \end{aligned} \quad (11)$$

Therefore using Lemma 6, (10) becomes

$$\begin{aligned}(n+1)T(r,g) &\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\quad+(n+3)T(r,f)+T\left(r,g'\right)\\ &\quad+\log r+S(r,g)\\ &\le 7T(r,g)+(n+3)T(r,f)+\log r+S(r,g)\end{aligned} \tag{12}$$

$$\Longrightarrow (n-6)T(r,g)\le (n+3)T(r,f)+\log r+S(r,g); \tag{13}$$

since $n>6$, we have

$$T(r,g)\le\left(\frac{n+3}{n-6}\right)T(r,f)+\log r+S(r,g) \tag{14}$$

This completes the proof of Lemma 7. □

**Lemma 8.** *Let $f$ and $g$ be nonconstant entire functions. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM and $n>3$, then*

$$T(r,g)\le\left(\frac{n+2}{n-3}\right)T(r,f)+\log r+S(r,g). \tag{15}$$

*Proof.* Applying Nevanlinna's second fundamental theorem to $g^n(g-1)g'$, we have

$$\begin{aligned}T\left(r,g^n(g-1)g'\right) &\le \overline{N}\left(r,g^n(g-1)g'\right)\\ &\quad+\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{g^n(g-1)g'-z}\right)+S(r,g)\\ &\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{f^n(f-1)f'-z}\right)+S(r,g).\end{aligned} \tag{16}$$

Since $g$ is an entire function, we have $\overline{N}(r,g)=0$ and the above equation becomes

$$\begin{aligned}&T\left(r,g^n(g-1)g'\right)\\ &\quad\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{f^n(f-1)f'-z}\right)+S(r,g).\end{aligned} \tag{17}$$

By the first fundamental theorem and (17), we have

$$\begin{aligned}(n+1)T(r,g) &\le T\left(r,g^n(g-1)\right)+S(r,g)\\ &\le T\left(r,g^n(g-1)g'\right)+T\left(r,\frac{1}{g'}\right)+S(r,g)\\ &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{f^n(f-1)f'-z}\right)\\ &\quad+T\left(r,g'\right)+S(r,g).\end{aligned} \tag{18}$$

We know that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^n(f-1)f'-z}\right) &\le T\left(r,\frac{1}{f^n(f-1)f'-z}\right)\\ &= T\left(r,f^n(f-1)f'-z\right)+O(1)\\ &\le T\left(r,f^n(f-1)f'\right)\\ &\quad+T(r,z)+O(1)\\ &\le nT(r,f)+T(r,f-1)\\ &\quad+T\left(r,f'\right)+\log r+O(1)\\ &\le (n+2)T(r,f)+\log r+O(1).\end{aligned} \tag{19}$$

Therefore using Lemma 6, (18) becomes

$$\begin{aligned}(n+1)T(r,g) &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+\overline{N}\left(r,\frac{1}{g'}\right)\\ &\quad+(n+2)T(r,f)+T\left(r,g'\right)\\ &\quad+\log r+S(r,g)\\ &\le 4T(r,g)+(n+2)T(r,f)+\log r+S(r,g)\end{aligned} \tag{20}$$

or

$$(n-3)T(r,g)\le (n+2)T(r,f)+\log r+S(r,g); \tag{21}$$

since $n>3$, we have

$$T(r,g)\le\left(\frac{n+2}{n-3}\right)T(r,f)+\log r+S(r,g). \tag{22}$$

This completes the proof of Lemma 8. □

**Lemma 9** (see [5]). *Suppose that $f(z)$ is a meromorphic function in the complex plane and $P(f)=a_0f^n+a_1f^{n-1}+\cdots+a_n$, where $a_0(\not\equiv 0)$, $a_1,\ldots,a_n$ are small meromorphic functions of $f(z)$. Then*

$$T(r,P(f))=nT(r,f)+S(r,f). \tag{23}$$

**Lemma 10** (see [6]). *Let $f_1, f_2$, and $f_3$ be three meromorphic functions satisfying $\sum_{j=1}^{3} f_j = 1$, let $g_1 = -f_3/f_2$, $g_2 = 1/f_2$, and $g_3 = -f_1/f_2$. If $f_1, f_2$, and $f_3$ are linearly independent, then $g_1, g_2$, and $g_3$ are linearly independent.*

## 3. Proof of Theorems

*Proof of Theorem 1.* By assumption, $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, and $f$ and $g$ share $\infty$ IM. Let

$$H = \frac{f^n(f-1)f' - z}{g^n(g-1)g' - z}. \tag{24}$$

Then, $H$ is a meromorphic function satisfying

$$\begin{aligned} T(r,H) &= T\left(r, \frac{f^n(f-1)f' - z}{g^n(g-1)g' - z}\right) \\ &\le T\left(r, f^n(f-1)f' - z\right) \\ &\quad + T\left(r, g^n(g-1)g' - z\right) + O(1) \\ &\le (n+3)\left(T(r,f) + T(r,g)\right) + O(\log r). \end{aligned} \tag{25}$$

Therefore,

$$T(r,H) = O\left(T(r,f) + T(r,g)\right). \tag{26}$$

From (24), we easily see that the zeros and poles of $H$ are multiple and satisfy

$$\overline{N}(r,H) \le \overline{N}_L(r,f), \qquad \overline{N}\left(r, \frac{1}{H}\right) \le \overline{N}_L(r,g). \tag{27}$$

Let

$$f_1 = \frac{f^n(f-1)f'}{z}, \qquad f_2 = H, \qquad f_3 = \frac{-Hg^n(g-1)g'}{z}. \tag{28}$$

Then, $f_1 + f_2 + f_3 = 1$ and $T(r)$ denote the maximum of $T(r, f_j)$, $j = 1, 2, 3$.

We have

$$\begin{aligned} T(r,f_1) &= O\left(T(r,f)\right), \\ T(r,f_2) &= O\left(T(r,f) + T(r,g)\right), \\ T(r,f_3) &= O\left(T(r,f) + T(r,g)\right). \end{aligned} \tag{29}$$

Therefore, $T(r) = O(T(r,f) + T(r,g))$, and thus

$$S(r,f) + S(r,g) = o\left(T(r)\right). \tag{30}$$

Now, we discuss the following three cases.

*Case 1.* Suppose that neither $f_2$ nor $f_3$ is a constant.

If $f_1$, $f_2$, and $f_3$ are linearly independent, then by Lemma 4 and (28), we have

$$\begin{aligned} T(r,f_1) &< \sum_{i=1}^{3} N_2\left(r, \frac{1}{f_i}\right) + \sum_{i=1}^{3} \overline{N}(r,f_i) + o\left(T(r)\right) \\ &\le N_2\left(r, \frac{1}{f_1}\right) + N_2\left(r, \frac{1}{f_2}\right) + N_2\left(r, \frac{1}{f_3}\right) \\ &\quad + \overline{N}(r,f_1) + \overline{N}(r,f_2) + \overline{N}(r,f_3) + o\left(T(r)\right) \\ &\le N_2\left(r, \frac{z}{f^n(f-1)f'}\right) + N_2\left(r, \frac{1}{H}\right) \\ &\quad + N_2\left(r, \frac{z}{Hg^n(g-1)g'}\right) \\ &\quad + \overline{N}\left(r, \frac{f^n(f-1)f'}{z}\right) \\ &\quad + \overline{N}(r,H) + \overline{N}\left(r, \frac{Hg^n(g-1)g'}{z}\right) + o\left(T(r)\right) \\ &= N_2\left(r, \frac{1}{f^n(f-1)f'}\right) + N_2\left(r, \frac{1}{H}\right) \\ &\quad + N_2\left(r, \frac{1}{Hg^n(g-1)g'}\right) \\ &\quad + \overline{N}\left(r, f^n(f-1)f'\right) \\ &\quad + \overline{N}(r,H) + \overline{N}\left(r, Hg^n(g-1)g'\right) \\ &\quad + 2\log r + o\left(T(r)\right). \end{aligned} \tag{31}$$

Using (27), we note that

$$\begin{aligned} &N_2\left(r, \frac{1}{Hg^n(g-1)g'}\right) \\ &\quad \le N_2\left(r, \frac{1}{H}\right) + N_2\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &\quad \le 2\overline{N}\left(r, \frac{1}{H}\right) + N_2\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &\quad \le 2\overline{N}_L(r,g) + N_2\left(r, \frac{1}{g^n(g-1)g'}\right). \end{aligned} \tag{32}$$

Since $\overline{N}_L(r,g) = 0$, we obtain that

$$N_2\left(r, \frac{1}{Hg^n(g-1)g'}\right) \le N_2\left(r, \frac{1}{g^n(g-1)g'}\right), \tag{33}$$

$$\begin{aligned} \overline{N}\left(r, Hg^n(g-1)g'\right) &\le \overline{N}(r,H) + \overline{N}\left(r, g^n(g-1)g'\right) \\ &\le \overline{N}_L(r,f) + \overline{N}(r,g). \end{aligned} \tag{34}$$

But $\overline{N}_L(r,f)=0$, so we get

$$\overline{N}\left(r, Hg^n(g-1)g'\right) \le \overline{N}(r,g). \tag{35}$$

Using (33) and (35) in (31), we get

$$\begin{aligned} T(r,f_1) \le{}& N_2\left(r, \frac{1}{f^n(f-1)f'}\right) + N_2\left(r, \frac{1}{H}\right) \\ &+ N_2\left(r, \frac{1}{g^n(g-1)g'}\right) + \overline{N}(r,f) \\ &+ \overline{N}(r,H) + \overline{N}(r,g) + 2\log r + o(T(r)). \end{aligned} \tag{36}$$

Since $f$ and $g$ share $\infty$ IM, we have $\overline{N}(r,f)=\overline{N}(r,g)$.

Using this with (27), we get

$$\begin{aligned} T(r,f_1) \le{}& N_2\left(r, \frac{1}{f^n(f-1)f'}\right) + 2\overline{N}_L(r,g) \\ &+ N_2\left(r, \frac{1}{g^n(g-1)g'}\right) + \overline{N}_L(r,f) \\ &+ 2\overline{N}(r,f) + 2\log r + o(T(r)) \\ \le{}& N\left(r, \frac{1}{f^n(f-1)f'}\right) \\ &- \left[N_{(3}\left(r, \frac{1}{f^n(f-1)f'}\right)\right. \\ &\left. -2\overline{N}_{(3}\left(r, \frac{1}{f^n(f-1)f'}\right)\right] \\ &+ N\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &- \left[N_{(3}\left(r, \frac{1}{g^n(g-1)g'}\right)\right. \\ &\left. -2\overline{N}_{(3}\left(r, \frac{1}{g^n(g-1)g'}\right)\right] \\ &+ 2\overline{N}_L(r,g) + \overline{N}_L(r,f) + 2\overline{N}(r,f) \\ &+ 2\log r + o(T(r)). \end{aligned} \tag{37}$$

If $z_0$ is a zero of $f$ with multiplicity $p$, then $z_0$ is a zero of $f^n(f-1)f'$ with multiplicity $np+p-1\ge 3$; we have

$$\begin{aligned} &\left[N_{(3}\left(r, \frac{1}{f^n(f-1)f'}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{f^n(f-1)f'}\right)\right] \\ &\quad \ge (n-2)N\left(r, \frac{1}{f}\right). \end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned} &\left[N_{(3}\left(r, \frac{1}{g^n(g-1)g'}\right) - 2\overline{N}_{(3}\left(r, \frac{1}{g^n(g-1)g'}\right)\right] \\ &\quad \ge (n-2)N\left(r, \frac{1}{g}\right). \end{aligned} \tag{39}$$

Let

$$f_1^* = \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}. \tag{40}$$

By Lemma 9, we have $T(r,f_1^*)=(n+2)T(r,f)+S(r,f)$.

Since $(f_1^*)' = zf_1$, we have

$$\begin{aligned} m\left(r, \frac{1}{f_1^*}\right) &\le m\left(r, \frac{1}{zf_1}\right) + m\left(r, \frac{(f_1^*)'}{f_1^*}\right) \\ &\le m\left(r, \frac{1}{f_1}\right) + \log r + S(r,f). \end{aligned} \tag{41}$$

By the first fundamental theorem, we have

$$\begin{aligned} T(r,f_1^*) \le{}& T(r,f_1) + N\left(r, \frac{1}{f_1^*}\right) - N\left(r, \frac{1}{f_1}\right) \\ &+ \log r + S(r,f); \end{aligned} \tag{42}$$

we have

$$\begin{aligned} &N\left(r, \frac{1}{f_1^*}\right) \\ &\quad = (n+1)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-(n+2)/(n+1)}\right). \end{aligned} \tag{43}$$

From (37)–(43), we get

$$\begin{aligned} T(r,f_1^*) \le{}& N\left(r, \frac{1}{f^n(f-1)f'}\right) - (n-2)N\left(r, \frac{1}{f}\right) \\ &+ N\left(r, \frac{1}{g^n(g-1)g'}\right) - (n-2)N\left(r, \frac{1}{g}\right) \\ &+ 2\overline{N}_L(r,g) \\ &+ \overline{N}_L(r,f) + 2\overline{N}(r,f) + (n+1)N\left(r, \frac{1}{f}\right) \\ &+ N\left(r, \frac{1}{f-(n+2)/(n+1)}\right) \\ &- N\left(r, \frac{1}{f^n(f-1)f'}\right) + 3\log r + o(T(r)). \end{aligned} \tag{44}$$

Using Lemma 6, we get

$$\begin{aligned}
T(r,f_1^*) \le{}& 3N\left(r,\frac{1}{f}\right)+3N\left(r,\frac{1}{g}\right)\\
&+2\overline{N}(r,f)+\overline{N}(r,g)+2\overline{N}_L(r,g)\\
&+\overline{N}_L(r,f)+N\left(r,\frac{1}{g-1}\right)\\
&+N\left(r,\frac{1}{f-(n+2)/(n+1)}\right)\\
&+3\log r+o(T(r))\\
(n+2)T(r,f) \le{}& 3N\left(r,\frac{1}{f}\right)+3N\left(r,\frac{1}{g}\right)\\
&+2\overline{N}(r,f)+\overline{N}(r,g)\\
&+2\overline{N}_L(r,g)+\overline{N}_L(r,f)+N\left(r,\frac{1}{g-1}\right)\\
&+N\left(r,\frac{1}{f-(n+2)/(n+1)}\right)\\
&+3\log r+o(T(r)).
\end{aligned} \tag{45}$$

Let

$$\begin{aligned}
g_1 &= -\frac{f_3}{f_2}=\frac{g^n(g-1)g'}{z},\\
g_2 &= \frac{1}{f_2}=\frac{1}{H}, \qquad g_3=-\frac{f_1}{f_2}=\frac{f^n(f-1)f'}{zH}.
\end{aligned} \tag{46}$$

Then $g_1+g_2+g_3=1$. By Lemma 10, $g_1$, $g_2$, and $g_3$ are linearly independent. In the same manner as above, we get

$$\begin{aligned}
(n+2)T(r,g) \le{}& 3N\left(r,\frac{1}{g}\right)+3N\left(r,\frac{1}{f}\right)\\
&+2\overline{N}(r,g)+\overline{N}(r,f)+2\overline{N}_L(r,f)\\
&+\overline{N}_L(r,g)+N\left(r,\frac{1}{f-1}\right)\\
&+N\left(r,\frac{1}{g-(n+2)/(n+1)}\right)\\
&+3\log r+o(T(r)).
\end{aligned} \tag{47}$$

Note that

$$\overline{N}_L(r,f)+\overline{N}_L(r,g)\le\overline{N}(r,f)=\overline{N}(r,g). \tag{48}$$

Adding (45) and (47) gives

$$\begin{aligned}
&(n+2)(T(r,f)+T(r,g))\\
&\quad\le 6\left(N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right)+3\left(\overline{N}(r,f)+\overline{N}(r,g)\right)\\
&\qquad+\left(\overline{N}_L(r,f)+\overline{N}_L(r,g)\right)\\
&\qquad+\left(N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{g-1}\right)\right)\\
&\qquad+N\left(r,\frac{1}{f-(n+2)/(n+1)}\right)\\
&\qquad+N\left(r,\frac{1}{g-(n+2)/(n+1)}\right)+6\log r+o(T(r))\\
&(n-6)(T(r,f)+T(r,g))\\
&\quad\le 3\left(\overline{N}(r,f)+\overline{N}(r,g)\right)\\
&\qquad+3\left(\overline{N}_L(r,f)+\overline{N}_L(r,g)\right)+6\log r+o(T(r)).
\end{aligned} \tag{49}$$

Using (48), we get

$$\begin{aligned}
&(n-6)(T(r,f)+T(r,g))\\
&\quad\le 3\overline{N}(r,f)+6\overline{N}(r,g)+6\log r+o(T(r))
\end{aligned} \tag{50}$$

or

$$\begin{aligned}
&(n-6)(T(r,f)+T(r,g))\\
&\quad\le 6\overline{N}(r,f)+3\overline{N}(r,g)+6\log r+o(T(r)).
\end{aligned} \tag{51}$$

Combining (50) and (51), we get

$$\begin{aligned}
(n-6)(T(r,f)+T(r,g)) &\le \frac{9}{2}\left(\overline{N}(r,f)+\overline{N}(r,g)\right)\\
&\quad+6\log r+o(T(r))\\
\left(n-\frac{21}{2}\right)(T(r,f)+T(r,g)) &\le 6\log r+o(T(r)).
\end{aligned} \tag{52}$$

By $n\ge 11$ and (30), we get a contradiction. Thus $f_1$, $f_2$, and $f_3$ are linearly dependent. Then, there exists three constants $(c_1,c_2,c_3)\ne(0,0,0)$ such that

$$c_1f_1+c_2f_2+c_3f_3=0. \tag{53}$$

If $c_1=0$, from (53) $c_2\ne 0$, $c_3\ne 0$, and

$$f_3=-\frac{c_2}{c_3}f_2; \tag{54}$$

$$\Longrightarrow g^n(g-1)g'=\frac{c_2}{c_3}z. \tag{55}$$

On integrating, we get

$$\begin{gathered}\frac{g^{n+2}}{n+2}-\frac{g^{n+1}}{n+1}=\frac{c_2}{c_3}\frac{z^2}{2}+k, \quad k \text{ is a constant,}\\ T\left(r,\frac{g^{n+2}}{n+2}-\frac{g^{n+1}}{n+1}\right)\le T\left(r,z^2\right)+O(1),\\ (n+2)T(r,g)\le 2\log r+O(1);\end{gathered} \tag{56}$$

since $n \ge 11$, we get a contradiction.

Thus $c_1 \ne 0$, and by (53) we have

$$\begin{gathered}c_1 f_1=-c_2 f_2-c_3 f_3\\ \Longrightarrow f_1=-\frac{c_2}{c_1}f_2-\frac{c_3}{c_1}f_3.\end{gathered} \tag{57}$$

Substituting this in $f_1+f_2+f_3=1$, we get

$$-\frac{c_2}{c_1}f_2-\frac{c_3}{c_1}f_3+f_2+f_3=1; \tag{58}$$

that is,

$$\left(1-\frac{c_2}{c_1}\right)f_2+\left(1-\frac{c_3}{c_1}\right)f_3=1, \quad \text{where } c_1\ne c_3,\ c_2\ne c_3. \tag{59}$$

From (28), we obtain

$$\begin{gathered}\left(1-\frac{c_2}{c_1}\right)H+\left(1-\frac{c_3}{c_1}\right)\left(\frac{-Hg^n(g-1)g'}{z}\right)=1\\ \Longrightarrow \left(1-\frac{c_2}{c_1}\right)-\left(1-\frac{c_3}{c_1}\right)\left(\frac{g^n(g-1)g'}{z}\right)=1\\ \Longrightarrow \left(1-\frac{c_3}{c_1}\right)\left(\frac{g^n(g-1)g'}{z}\right)+\frac{1}{H}=\left(1-\frac{c_2}{c_1}\right).\end{gathered} \tag{60}$$

Applying Lemma 5 to the above equation, we get

$$\begin{aligned}T\left(r,\frac{g^n(g-1)g'}{z}\right)&\le \overline{N}\left(r,\frac{g^n(g-1)g'}{z}\right)\\ &\quad+\overline{N}\left(r,\frac{z}{g^n(g-1)g'}\right)\\ &\quad+\overline{N}(r,H)+S(r,g).\end{aligned} \tag{61}$$

Note that

$$\begin{aligned}T\left(r,g^n(g-1)g'\right)&\le T\left(r,\frac{g^n(g-1)g'}{z}\right)+T(r,z)\\ &\le T\left(r,\frac{g^n(g-1)g'}{z}\right)+\log r.\end{aligned} \tag{62}$$

Using (61), we get

$$\begin{aligned}T\left(r,g^n(g-1)g'\right)&\le \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)\\ &\quad+\overline{N}(r,g)+2\log r+S(r,g)\\ &=\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)+2\overline{N}(r,g)\\ &\quad+2\log r+S(r,g).\end{aligned} \tag{63}$$

By Lemmas 9 and 6 and (63), we have

$$\begin{aligned}(n+1)T(r,g)&=T\left(r,g^n(g-1)\right)+S(r,g)\\ &\le T\left(r,g^n(g-1)g'\right)+T\left(r,\frac{1}{g'}\right)+S(r,g)\\ &\le \overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)+2\overline{N}(r,g)\\ &\quad+2\log r+2T(r,g)+S(r,g)\\ &\le \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)\\ &\quad+2\overline{N}(r,g)+2T(r,g)+2\log r+S(r,g)\\ &\le 8T(r,g)+2\log r+S(r,g)\\ \Longrightarrow (n-7)T(r,g)&\le 2\log r+S(r,g);\end{aligned} \tag{64}$$

we obtain $n \le 7$, which contradicts $n \ge 11$.

*Case 2.* Suppose that $f_2=c\,(\ne 0)$, where $c$ is a constant.

If $c \ne 1$, then we have

$$\begin{gathered}f_1+f_2+f_3=1\\ \Longrightarrow \frac{f^n(f-1)f'}{z}+c-\frac{cg^n(g-1)g'}{z}=1;\end{gathered} \tag{65}$$

$$\Longrightarrow \frac{f^n(f-1)f'}{z}-\frac{cg^n(g-1)g'}{z}=1-c. \tag{66}$$

Applying Lemma 5 to the above equation, we have

$$\begin{aligned}T\left(r,\frac{f^n(f-1)f'}{z}\right)&\le \overline{N}\left(r,\frac{f^n(f-1)f'}{z}\right)\\ &\quad+\overline{N}\left(r,\frac{z}{g^n(g-1)g'}\right)\\ &\quad+\overline{N}\left(r,\frac{z}{f^n(f-1)f'}\right)+S(r,f)\end{aligned}$$

$$\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)+\overline{N}\left(r,\frac{1}{f^n(f-1)f'}\right)+\log r+S(r,f). \tag{67}$$

Note that

$$T\left(r,f^n(f-1)f'\right)\leq T\left(r,\frac{f^n(f-1)f'}{z}\right)+\log r. \tag{68}$$

Therefore,

$$T\left(r,f^n(f-1)f'\right)\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)+\overline{N}\left(r,\frac{1}{f^n(f-1)f'}\right)+2\log r+S(r,f). \tag{69}$$

Using Lemmas 9 and 6 and (69), we have

$$\begin{aligned}(n+1)T(r,f)&=T\left(r,f^n(f-1)\right)+S(r,f)\\&\leq T\left(r,f^n(f-1)f'\right)+T\left(r,\frac{1}{f'}\right)+S(r,f)\\&\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^n(f-1)f'}\right)+\overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right)+T\left(r,f'\right)+2\log r+S(r,f)\\&\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+2T(r,f)+2\log r+S(r,f)\\&\leq 7T(r,f)+4T(r,g)+2\log r+S(r,f)\\&\Longrightarrow (n-6)T(r,f)\leq 4T(r,g)+2\log r+S(r,f).\end{aligned} \tag{70}$$

Using Lemma 7, we get

$$(n-6)T(r,f)\leq 4\left(\frac{n+3}{n-6}\right)T(r,f)+2\log r+S(r,f); \tag{71}$$

since $n\geq 11$, we get a contradiction.

Therefore $c=1$, and by (27) and (24) we have

$$f^n(f-1)f'=g^n(g-1)g'. \tag{72}$$

On integrating, we get

$$\begin{gathered}\frac{f^{n+2}}{n+2}-\frac{f^{n+1}}{n+1}=\frac{g^{n+2}}{n+2}-\frac{g^{n+1}}{n+1}+k\\ F^*=G^*+k,\quad \text{where } k \text{ is a constant.}\end{gathered} \tag{73}$$

We claim that $k=0$. Suppose that $k\neq 0$, then

$$\begin{aligned}&\Theta(0,F^*)+\Theta(k,F^*)+\Theta(\infty,F^*)\\&\quad=\Theta(0,F^*)+\Theta(0,G^*)+\Theta(\infty,F^*).\end{aligned} \tag{74}$$

We have

$$\overline{N}\left(r,\frac{1}{F^*}\right)=\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-(n+2)/(n+1)}\right)\leq 2T(r,f). \tag{75}$$

Similarly,

$$\begin{gathered}\overline{N}\left(r,\frac{1}{G^*}\right)\leq 2T(r,g),\\ \overline{N}(r,F^*)=\overline{N}(r,f)\leq T(r,f).\end{gathered} \tag{76}$$

Using Lemma 9, we have

$$\begin{gathered}T(r,F^*)=(n+2)T(r,f)+S(r,f),\\ T(r,G^*)=(n+2)T(r,g)+S(r,g).\end{gathered} \tag{77}$$

Thus,

$$\Theta(0,F^*)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/F^*)}{T(r,F^*)}\geq 1-\frac{2}{n+2}. \tag{78}$$

Similarly,

$$\Theta(0, G^*) \geq 1 - \frac{2}{n+2},$$
$$\Theta(\infty, F^*) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, F^*)}{T(r, F^*)} \geq 1 - \frac{1}{n+2}. \tag{79}$$

Therefore, (74) becomes

$$\begin{aligned}\Theta(0, F^*) + \Theta(k, F^*) + \Theta(\infty, F^*) &\geq 2\left(1 - \frac{2}{n+2}\right) + 1 - \frac{1}{n+2} \\ &= \frac{3n+1}{n+2} > 2 \quad \text{for } n \geq 11,\end{aligned} \tag{80}$$

which contradicts $\sum_{a\in\overline{\mathscr{E}}} \Theta(a, f) \leq 2$. Thus, we have

$$\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}. \tag{81}$$

Let $h = f/g$. If $h \not\equiv 1$, then we easily obtain that

$$g = \frac{(n+2)\left(h^{n+1} - 1\right)}{(n+1)\left(h^{n+2} - 1\right)}, \qquad f = \frac{(n+2)\,h\left(h^{n+1} - 1\right)}{(n+1)\left(h^{n+2} - 1\right)}. \tag{82}$$

If $h \equiv 1$, that is, $f \equiv g$.

*Case 3*. Suppose that $f_3 = c\,(\neq 0)$, where $c$ is a constant.

If $c \neq 1$, then we have

$$\begin{aligned} f_1 + f_2 + f_3 &= 1 \\ f_1 + f_2 &= 1 - c \\ \frac{f^n(f-1)f'}{z} - \frac{cz}{g^n(g-1)g'} &= 1 - c. \end{aligned} \tag{83}$$

Applying Lemma 5 to the above equation, we have

$$\begin{aligned} T\left(r, \frac{f^n(f-1)f'}{z}\right) &\leq \overline{N}\left(r, \frac{f^n(f-1)f'}{z}\right) \\ &\quad + \overline{N}\left(r, \frac{z}{f^n(f-1)f'}\right) \\ &\quad + \overline{N}\left(r, \frac{g^n(g-1)g'}{z}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^n(f-1)f'}\right) \\ &\quad + \overline{N}(r, g) + 2\log r + S(r, f). \end{aligned} \tag{84}$$

Note that

$$\begin{aligned} T\left(r, f^n(f-1)f'\right) &\leq T\left(r, \frac{f^n(f-1)f'}{z}\right) + T(r, z) \\ &\leq T\left(r, \frac{f^n(f-1)f'}{z}\right) + \log r. \end{aligned} \tag{85}$$

Therefore using (84), we have

$$\begin{aligned} T\left(r, f^n(f-1)f'\right) &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^n(f-1)f'}\right) \\ &\quad + \overline{N}(r, g) + 3\log r + S(r, f). \end{aligned} \tag{86}$$

Using Lemmas 9 and 6 and (86), we have

$$\begin{aligned} (n+1)T(r, f) &= T(r, f^n(f-1)) + S(r, f) \\ &\leq T\left(r, f^n(f-1)f'\right) \\ &\quad + T\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^n(f-1)f'}\right) \\ &\quad + \overline{N}(r, g) + T\left(r, f'\right) + 3\log r + S(r, f) \\ &\leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) \\ &\quad + N\left(r, \frac{1}{f'}\right) + \overline{N}(r, g) + 2T(r, f) \\ &\quad + 3\log r + S(r, f) \\ &\leq 7T(r, f) + T(r, g) + 3\log r + S(r, f) \\ \Longrightarrow (n-6)T(r, f) &\leq T(r, g) + 3\log r + S(r, f). \end{aligned} \tag{87}$$

Using Lemma 7, we get

$$(n-6)T(r, f) \leq \frac{(n+3)}{(n-6)}T(r, f) + 3\log r + S(r, f); \tag{88}$$

since $n \geq 11$, we get a contradiction. Thus, $c = 1$. Hence,

$$\begin{aligned} \frac{f^n(f-1)f'}{z} - \frac{z}{g^n(g-1)g'} &= 0 \\ \Longrightarrow f^n(f-1)f'g^n(g-1)g' &= z^2. \end{aligned} \tag{89}$$

Let $z_0$ be a zero of $f$ of order $p$. From (89), we know that $z_0$ is a pole of $g$. Suppose that $z_0$ is a pole of $g$ of order $q$. From (89), we obtain

$$\begin{aligned} np + p - 1 &= nq + 2q + 1 \\ \Longrightarrow (n+1)(p-q) &= q + 2, \end{aligned} \tag{90}$$

which implies that $p \geq q + 1$ and $q + 2 \geq n + 1$. Hence,

$$p \geq n. \tag{91}$$

Let $z_1$ be a zero of $(f-1)$ of order $p_1$, then from (89) $z_1$ is a pole of $g$ (say order $q_1$). By (89), we get

$$\begin{aligned} p_1 + p_1 - 1 &= nq_1 + 2q_1 + 1 \\ 2p_1 - 1 &\geq n + 3 \\ 2p_1 \geq n+4 &\Longrightarrow p_1 \geq \frac{n+4}{2}. \end{aligned} \tag{92}$$

Let $z_2$ be a zero of $f'$ of order $p_2$ that is not zero of $f(f-1)$, then from (89), $z_2$ is a pole of $g$ of order $q_2$. Again by (89), we get

$$\begin{aligned} p_2 &= nq_2 + 2q_2 + 1 \\ &\Longrightarrow p_2 \geq n+3. \end{aligned} \tag{93}$$

In the same manner as above, we have similar results for the zeros of $g^n(g-1)g'$. From (89)–(93), we have

$$\overline{N}\left(r, f^n(f-1)f'\right) = \overline{N}\left(r, \frac{z^2}{g^n(g-1)g'}\right); \tag{94}$$

that is,

$$\begin{aligned} \overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + N\left(r,\frac{1}{g'}\right) \\ &\leq \frac{1}{11}N\left(r,\frac{1}{g}\right) + \frac{2}{15}N\left(r,\frac{1}{g-1}\right) + \frac{1}{14}N\left(r,\frac{1}{g'}\right) \\ &\leq \frac{1}{11}T(r,g) + \frac{2}{15}T(r,g) + \frac{2}{14}T(r,g) + s(r,g) \\ &= \left(\frac{1}{11}+\frac{2}{15}+\frac{2}{7}\right)T(r,g) + S(r,g) \\ &\Longrightarrow \overline{N}(r,f) < \frac{2}{3}T(r,g) + S(r,g). \end{aligned} \tag{95}$$

By Nevanlinna's second fundamental theorem, we have from (91), (92), and (95) that

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}(r,f) + S(r,f) \\ &< \frac{1}{11}T(r,f) + \frac{2}{15}T(r,f) + \frac{2}{3}T(r,g) \\ &\quad + S(r,f) + S(r,g) \\ &\Longrightarrow T(r,f) \\ &\quad \leq \frac{37}{165}T(r,f) + \frac{2}{3}T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{96}$$

Similarly,

$$T(r,g) \leq \frac{37}{165}T(r,g) + \frac{2}{3}T(r,f) + S(r,f) + S(r,g). \tag{97}$$

From (96) and (97), we get

$$\begin{aligned} &T(r,f) + T(r,g) \\ &\quad \leq \frac{37}{165}(T(r,f) + T(r,g)) \\ &\qquad + \frac{2}{3}(T(r,f) + T(r,g)) + S(r,f) + S(r,g) \\ &\frac{18}{165}(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g); \end{aligned} \tag{98}$$

since $n \geq 11$, we get a contradiction.

This completes the proof of Theorem 1. □

Using the same argument as in the proof of Theorem 1, we can prove Theorem 2.

*Proof of Theorem 3.* By the assumption of the theorem, we know that either both $f$ and $g$ are two transcendental entire functions or both $f$ and $g$ are polynomials.

If $f$ and $g$ are transcendental entire functions, putting $\overline{N}(r,f) = 0$, $\overline{N}(r,g) = 0$ and using similar arguments as in the proof of Theorem 1, we easily obtain Theorem 3.

If $f$ and $g$ are polynomials, $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, we get

$$f^n(f-1)f' - z = k\left(g^n(g-1)g' - z\right), \tag{99}$$

where $k$ is a nonzero constant. Suppose that $k \neq 1$, (99) can be written as

$$\frac{f^n(f-1)f'}{z} - k\frac{g^n(g-1)g'}{z} = 1-k. \tag{100}$$

Applying Lemma 5 to the above equation, we have

$$\begin{aligned} T\left(r,\frac{f^n(f-1)f'}{z}\right) &\leq \overline{N}\left(r,\frac{f^n(f-1)f'}{z}\right) \\ &\quad + \overline{N}\left(r,\frac{z}{g^n(g-1)g'}\right) \\ &\quad + \overline{N}\left(r,\frac{z}{f^n(f-1)f'}\right) + S(r,f). \end{aligned} \tag{101}$$

Since $f$ is a polynomial, so it does not have any poles. Thus, we have,

$$\begin{aligned} &T\left(r,\frac{f^n(f-1)f'}{z}\right) \\ &\quad \leq \overline{N}\left(r,\frac{1}{g^n(g-1)g'}\right) \\ &\qquad + \overline{N}\left(r,\frac{1}{f^n(f-1)f'}\right) + 2\log r + S(r,f). \end{aligned} \tag{102}$$

Note that

$$T\left(r, f^n(f-1)f'\right) \leq T\left(r,\frac{f^n(f-1)f'}{z}\right) + \log r. \tag{103}$$

Therefore,

$$\begin{aligned} T\left(r, f^n(f-1)f'\right) &\le \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f^n(f-1)f'}\right) + 3\log r + S(r,f). \end{aligned} \tag{104}$$

Using Lemmas 9 and 6 and (104), we have

$$\begin{aligned} (n+1)T(r,f) &= T\left(r, f^n(f-1)\right) + S(r,f) \\ &\le T\left(r, f^n(f-1)f'\right) \\ &\quad + T\left(r, \frac{1}{f'}\right) + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{f^n(f-1)f'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right) \\ &\quad + T\left(r, f'\right) + 3\log r + S(r,f) \\ &\le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) \\ &\quad + N\left(r, \frac{1}{g'}\right) + T(r,f) \\ &\quad + 3\log r + S(r,f) \\ &\le 4T(r,f) + 3T(r,g) + 3\log r + S(r,f) \\ \Longrightarrow (n-3)T(r,f) &\le 3T(r,g) + 3\log r + S(r,f). \end{aligned} \tag{105}$$

Using Lemma 8, we get

$$(n-3)T(r,f) \le 3\left(\frac{n+2}{n-3}\right)T(r,f) + 3\log r + S(r,f); \tag{106}$$

since $n \ge 7$, we get a contradiction.

Therefore, $k = 1$; so (99) becomes

$$f^n(f-1)f' = g^n(g-1)g'. \tag{107}$$

On integrating, we get

$$\begin{aligned} \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} &= \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} + c \\ F^* &= G^* + c, \quad \text{where } c \text{ is a constant.} \end{aligned} \tag{108}$$

We claim that $c = 0$. Suppose that $c \ne 0$, then

$$\Theta(0, F^*) + \Theta(c, F^*) = \Theta(0, F^*) + \Theta(0, G^*). \tag{109}$$

We have

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F^*}\right) \\ &\quad = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-(n+2)/(n+1)}\right) \le 2T(r,f). \end{aligned} \tag{110}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{G^*}\right) \le 2T(r,g). \tag{111}$$

Using Lemma 9, we have

$$\begin{aligned} T(r, F^*) &= (n+2)T(r,f) + S(r,f), \\ T(r, G^*) &= (n+2)T(r,g) + S(r,g). \end{aligned} \tag{112}$$

Thus,

$$\Theta(0, F^*) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/F^*)}{T(r, F^*)} \ge 1 - \frac{2}{n+2}. \tag{113}$$

Similarly,

$$\Theta(0, G^*) \ge 1 - \frac{2}{n+2}. \tag{114}$$

Therefore, (109) becomes

$$\begin{aligned} \Theta(0, F^*) + \Theta(c, F^*) &\ge 2\left(1 - \frac{2}{n+2}\right) \\ &= \frac{2n}{n+2} \ge \frac{14}{9} > 1 \quad \text{for } n \ge 7, \end{aligned} \tag{115}$$

which contradicts $\sum_{a\in\mathscr{C}} \Theta(a, f) \le 1$. Thus, we have

$$\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}. \tag{116}$$

Let $h = f/g$. If $h \not\equiv 1$, then by (116) we have

$$g = \frac{(n+2)\left(1 + h + h^2 + \cdots + h^n\right)}{(n+1)\left(1 + h + h^2 + \cdots + h^{n+1}\right)}. \tag{117}$$

By Picard's theorem, $h(z)$ is a constant. Hence, $g$ is a constant, which is a contradiction. Therefore, $h(z) \equiv 1$, that is, $f(z) \equiv g(z)$. □

## 4. Remarks

If the condition "$f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM" is replaced by the condition "$f^n(f-1)f'$ and $g^n(g-1)g'$ share $\alpha(z)$ CM," where $\alpha$ is a meromorphic function such that $\alpha \not\equiv 0, \infty$ and $T(r,\alpha) = o\{T(r,f), T(r,g)\}$; the conclusion of Theorems 1, 2, and 3 still holds. We, thus, obtain the following results.

**Theorem 11.** *Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 11$ a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $\alpha(z)$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) \equiv g(z)$ or*

$$g = \frac{(n+2)\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \qquad f = \frac{(n+2)\,h\left(1-h^{n+1}\right)}{(n+1)\left(1-h^{n+2}\right)}, \tag{118}$$

*where $h$ is a nonconstant meromorphic function.*

**Theorem 12.** *Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $\alpha(z)$ CM, $f$ and $g$ share $\infty$ IM, then $f(z) \equiv g(z)$.*

**Theorem 13.** *Let $f$ and $g$ be two nonconstant entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

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