Research Article

Uniqueness of Meromorphic Functions Sharing Fixed Point

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We study the uniqueness of meromorphic functions concerning differential polynomials sharing fixed point and obtain some significant results, which improve the results due to Lin and Yi (2004).

1. Introduction and Main Results

Let \( f(z) \) be a nonconstant meromorphic function in the whole complex plane \( \mathbb{C} \). We will use the following standard notations of value distribution theory: \( T(r,f), m(r,f), N(r,f), \ldots \) (see [1, 2]). We denote by \( S(r,f) \) any function satisfying

\[
S(r,f) = o\{T(r,f)\}, \quad \text{as } r \to +\infty, \quad (1)
\]

possibly outside of a set with finite measure.

Let \( a \) be a finite complex number and \( k \) a positive integer. We denote by \( N_k(r,1/(f-a)) \) the counting function for the zeros of \( f(z) - a \) in \( |z| \leq r \) with multiplicity \( \leq k \) and by \( \overline{N}_k(r,1/(f-a)) \) the corresponding one for which multiplicity is not counted. Let \( N_k(r,1/(f-a)) \) be the counting function for the zeros of \( f(z) - a \) in \( |z| \leq r \) with multiplicity \( \geq k \) and \( \overline{N}_k(r,1/(f-a)) \) the corresponding one for which multiplicity is not counted. Set

\[
N_k \left( r, \frac{1}{f-a} \right) = \overline{N}_k \left( r, \frac{1}{f-a} \right) + N_{k-1} \left( r, \frac{1}{f-a} \right) + \cdots + N_1 \left( r, \frac{1}{f-a} \right). \quad (2)
\]

We say that \( f \) and \( g \) share a CM (counting multiplicity) if \( f-a \) and \( g-a \) have the same zeros with the same multiplicities. Similarly, we say that \( f \) and \( g \) share a IM (ignoring multiplicity) if \( f-a \) and \( g-a \) have the same zeros with ignoring multiplicities.

In 2004, Lin and Yi [3] obtained the following results.

Theorem A. Let \( f \) and \( g \) be two transcendental meromorphic functions, \( n \geq 12 \) an integer. If \( f^n(f-1)f' \) and \( g^n(g-1)g' \) share \( z \) CM, then either \( f(z) \equiv g(z) \) or

\[
g = \frac{(n+2)\left(1 - h^{n+1}\right)}{(n+1)(1 - h^{n+2})}, \quad f = \frac{(n+2)h\left(1 - h^{n+1}\right)}{(n+1)(1 - h^{n+2})}, \quad (3)
\]

where \( h \) is a nonconstant meromorphic function.

Theorem B. Let \( f \) and \( g \) be two transcendental meromorphic functions, \( n \geq 13 \) an integer. If \( f^n(f-1)^2f' \) and \( g^n(g-1)^2g' \) share \( z \) CM, then \( f(z) \equiv g(z) \).

In this paper, we study the uniqueness problems of entire or meromorphic functions concerning differential polynomials sharing fixed point, which improves Theorems A and B.

1.1. Main Results

Theorem 1. Let \( f \) and \( g \) be two nonconstant meromorphic functions, \( n \geq 11 \) a positive integer. If \( f^n(f-1)f' \) and
\[ g^n(1/g - 1)g' \text{ share } z \text{ CM, } f \text{ and } g \text{ share } \infty \text{ IM, then either } f(z) \equiv g(z) \text{ or } \]

\[ g = \frac{(n + 2) (1 - h^{1/(n+1)})}{(n + 1) (1 - h^{1/(n+2)})}, \quad f = \frac{(n + 2) h (1 - h^{1/(n+1)})}{(n + 1) (1 - h^{1/(n+2)})}, \quad (4) \]

where \( h \) is a nonconstant meromorphic function.

**Theorem 2.** Let \( f \) and \( g \) be two nonconstant meromorphic functions, \( n \geq 12 \) a positive integer. If \( f^n(f - 1)f' \) and \( g^n(1/g - 1)g' \) share \( z \) CM, \( f \) and \( g \) share \( \infty \) IM, then \( f(z) \equiv g(z) \).

**Theorem 3.** Let \( f \) and \( g \) be two nonconstant entire functions, \( n \geq 7 \) an integer. If \( f^n(f - 1)f' \) and \( g^n(1/g - 1)g' \) share \( z \) CM, then \( f(z) \equiv g(z) \).

2. Some Lemmas

**Lemma 4** (see [4]). Let \( f_1, f_2, \) and \( f_3 \) be nonconstant meromorphic functions such that \( f_1 + f_2 + f_3 = 1 \). If \( f_1, f_2, \) and \( f_3 \) are linearly independent, then

\[ T(r, f_1) < 3 \sum_{i=1}^{3} N_i \left( r, \frac{1}{f_i} \right) + 3 \sum_{i=1}^{3} N_i \left( r, f_i \right) + o(T(r)), \quad (5) \]

where \( T(r) = \max_{1 \leq i \leq 3} \{ T(r, f_i) \} \) and \( r \notin E \).

**Lemma 5** (see [1]). Let \( f_1 \) and \( f_2 \) be two nonconstant meromorphic functions. If \( c_1 f_1 + c_2 f_2 = c_3 \), where \( c_1, c_2, \) and \( c_3 \) are non-zero constants, then

\[ T(r, f_1) \leq N(r, f_1) + N \left( r, \frac{1}{f_1} \right) + N \left( r, \frac{1}{f_2} \right) + S(r, f_1). \quad (6) \]

Lemmas 4 and 5 play a very important role in proving our theorems.

**Lemma 6** (see [1]). Let \( f \) be a nonconstant meromorphic function and let \( k \) be a nonnegative integer, then

\[ N \left( r, \frac{1}{f^k} \right) \leq N \left( r, \frac{1}{f} \right) + kN \left( r, f \right) + S(r, f). \quad (7) \]

The following lemmas play a cardinal role in proving our results.

**Lemma 7.** Let \( f \) and \( g \) be nonconstant meromorphic functions. If \( f^n(f - 1)f' \) and \( g^n(1/g - 1)g' \) share \( z \) CM and \( n > 6 \), then

\[ T(r, g) \leq \frac{n + 3}{n - 6} T(r, f) + \log r + S(r, g). \quad (8) \]

Proof. Applying Nevanlinna's second fundamental theorem to \( g^n(1/g - 1)g' \), we have

\[ T \left( r, g^n(1/g - 1)g' \right) \leq N \left( r, g^n(1/g - 1)g' \right) + N \left( r, \frac{1}{g^n(1/g - 1)g'} \right) + S(r, g) \]

\[ \leq N(r, g) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{g'} \right) \]

\[ + N \left( r, \frac{1}{g - 1} \right) + N \left( r, \frac{1}{g'} \right) + T(r, g) \cdot (9) \]

By the first fundamental theorem and (9), we have

\[ (n + 1) T(r, g) \leq T \left( r, g^n(1/g - 1)g' \right) + T(r, g) \]

\[ \leq T \left( r, g^n(1/g - 1)g' \right) + T \left( r, \frac{1}{g'} \right) + S(r, g) \]

\[ \leq N(r, g) + N \left( r, \frac{1}{g} \right) + N \left( r, \frac{1}{g'} \right) \]

\[ + T \left( r, \frac{1}{g - 1} \right) + T \left( r, \frac{1}{g'} \right) + T \left( r, g' \right) + S(r, g) \cdot (10) \]

We know that

\[ N \left( r, \frac{1}{f^n(1/f - 1)f'} \right) \leq T \left( r, \frac{1}{f^n(1/f - 1)f'} \right) \]

\[ = T \left( r, f^n(1/f - 1)f' \right) + O(1) \]

\[ \leq T \left( r, f^n(1/f - 1)f' \right) + T(r, f - 1) \]

\[ + T \left( r, f' \right) + \log r + O(1) \]

\[ \leq (n + 3) T \left( r, f \right) + \log r + O(1) \cdot (11) \]
Therefore using Lemma 6, (10) becomes

\[(n + 1)T(r, g) \leq \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + (n + 3)T(r, f) + T\left(r, \frac{1}{g'}\right) + \log r + S(r, g)\]

\[\leq 7T(r, g) + (n + 3)T(r, f) + \log r + S(r, g)\]

(12)

\[\Rightarrow (n - 6)T(r, g) \leq (n + 3)T(r, f) + \log r + S(r, g);\]

(13)

since \(n > 6\), we have

\[T(r, g) \leq \left(\frac{n + 3}{n - 6}\right)T(r, f) + \log r + S(r, g)\]

(14)

This completes the proof of Lemma 7.

**Lemma 8.** Let \(f\) and \(g\) be nonconstant entire functions. If \(f^n(f - 1)f'\) and \(g^n(g - 1)g'\) share \(z\) CM and \(n > 3\), then

\[T(r, g) \leq \left(\frac{n + 2}{n - 3}\right)T(r, f) + \log r + S(r, g)\]

(15)

**Proof.** Applying Nevanlinna's second fundamental theorem to \(g^n(g - 1)g'\), we have

\[T\left(r, \frac{1}{f^n(f - 1)f' - z}\right) \leq \overline{N}\left(r, \frac{1}{g^n(g - 1)g'}\right) + \overline{N}\left(r, \frac{1}{g^n(g - 1)g' - z}\right) + S(r, g)\]

\[\leq \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - 1}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f' - z}\right) + S(r, g)\]

(16)

Since \(g\) is an entire function, we have \(\overline{N}(r, g) = 0\) and the above equation becomes

\[T\left(r, \frac{1}{f^n(f - 1)f' - z}\right) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - 1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f'}\right) + S(r, g)\]

(17)

By the first fundamental theorem and (17), we have

\[(n + 1)T(r, g) \leq T\left(r, \frac{1}{f^n(f - 1)f'}\right) + S(r, g)\]

\[\leq T\left(r, \frac{1}{g^n(g - 1)g'}\right) + T\left(r, \frac{1}{g'}\right) + S(r, g)\]

\[\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - 1}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f'}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f' - z}\right) + S(r, g)\]

\[\leq 4T(r, g) + (n + 2)T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + S(r, g)\]

(18)

We know that

\[\overline{N}\left(r, \frac{1}{f^n(f - 1)f' - z}\right) \leq T\left(r, \frac{1}{f^n(f - 1)f'}\right) + O(1)\]

\[\leq T\left(r, \frac{1}{f^n(f - 1)f'}\right) + O(1)\]

\[\leq nT(r, f) + T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + O(1)\]

\[\leq (n + 2)T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + O(1)\]

(19)

Therefore using Lemma 6, (18) becomes

\[(n + 1)T(r, g) \leq T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + S(r, g)\]

\[\leq T\left(r, \frac{1}{g^n(g - 1)g'}\right) + T\left(r, \frac{1}{g'}\right) + \log r + S(r, g)\]

\[\leq \overline{N}\left(r, \frac{1}{g^n(g - 1)g'}\right) + \overline{N}\left(r, \frac{1}{g'}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f'}\right) + \overline{N}\left(r, \frac{1}{f^n(f - 1)f' - z}\right) + \overline{N}\left(r, \frac{1}{g^n(g - 1)g' - z}\right) + \overline{N}\left(r, \frac{1}{g^n(g - 1)g' - z}\right) + S(r, g)\]

\[\leq 4T(r, g) + (n + 2)T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + S(r, g)\]

(20)

or

\[(n - 3)T(r, g) \leq (n + 2)T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + S(r, g);\]

(21)

since \(n > 3\), we have

\[T(r, g) \leq \left(\frac{n + 2}{n - 3}\right)T\left(r, \frac{1}{f^n(f - 1)f'}\right) + \log r + S(r, g)\]

(22)

This completes the proof of Lemma 8.

**Lemma 9** (see [5]). Suppose that \(f(z)\) is a meromorphic function in the complex plane and \(P(f) = a_0f^n + a_1f^{n-1} + \cdots + a_n\), where \(a_0(\neq 0), a_1, \ldots, a_n\) are small meromorphic functions of \(f(z)\). Then

\[T(r, P(f)) = nT\left(r, \frac{1}{f^n(f - 1)f'}\right) + S(r, f)\]

(23)
Lemma 10 (see [6]). Let \( f_1, f_2, \) and \( f_3 \) be three meromorphic functions satisfying \( \sum_{j=1}^{3} f_j = 1 \), let \( g_1 = -f_3/f_2, \) \( g_2 = 1/f_2, \) and \( g_3 = -f_1/f_2. \) If \( f_1, f_2, \) and \( f_3 \) are linearly independent, then \( g_1, g_2, \) and \( g_3 \) are linearly independent.

3. Proof of Theorems

Proof of Theorem 1. By assumption, \( f^n(f - 1)f' \) and \( g^n(g - 1)g' \) share \( z \) CM, and \( f \) and \( g \) share \( \infty \) IM. Let

\[
H = \frac{f^n(f - 1)f' - z}{g^n(g - 1)g' - z}.
\]  

Then, \( H \) is a meromorphic function satisfying

\[
T(r, H) = T \left( r, \frac{f^n(f - 1)f' - z}{g^n(g - 1)g' - z} \right)
\leq T \left( r, f^n(f - 1)f' - z \right)
+ T \left( r, g^n(g - 1)g' - z \right) + O(1)
\leq (n + 3) \left( T(r, f) + T(r, g) \right) + O(\log r).
\]

Therefore,

\[
T(r, H) = O \left( T(r, f) + T(r, g) \right).
\]  

From (24), we easily see that the zeros and poles of \( H \) are multiple and satisfy

\[
N(r, H) \leq N_L(r, f), \quad N \left( r, \frac{1}{H} \right) \leq N_L(r, g).
\]  

Let

\[
f_1 = \frac{f^n(f - 1)f'}{z}, \quad f_2 = H,
\]

\[
f_3 = \frac{-Hg^n(g - 1)g'}{z}.
\]

Then, \( f_1 + f_2 + f_3 = 1 \) and \( T(r) \) denote the maximum of \( T(r, f_j), j = 1, 2, 3. \)

We have

\[
T(r, f_1) = O \left( T(r, f) \right),
\]

\[
T(r, f_2) = O \left( T(r, f) + T(r, g) \right),
\]

\[
T(r, f_3) = O \left( T(r, f) + T(r, g) \right).
\]

Therefore, \( T(r) = O(\max(T(r, f) + T(r, g))) \), and thus

\[
S(r, f) + S(r, g) = o(T(r)).
\]

Now, we discuss the following three cases.

Case 1. Suppose that neither \( f_2 \) nor \( f_3 \) is a constant.

If \( f_1, f_2, \) and \( f_3 \) are linearly independent, then by Lemma 4 and (28), we have

\[
T(r, f_1) < \sum_{i=1}^{3} N_2 \left( r, \frac{1}{f_i} \right) + \sum_{i=1}^{3} N \left( r, f_i \right) + o(T(r))
\]

\[
\leq N_2 \left( r, \frac{1}{f_1} \right) + N_2 \left( r, \frac{1}{f_2} \right) + N_2 \left( r, \frac{1}{f_3} \right)
\]

\[
+ N \left( r, f_1 \right) + N \left( r, f_2 \right) + N \left( r, f_3 \right) + o(T(r))
\]

\[
\leq N_2 \left( r, \frac{z}{f^n(f - 1)f'} \right) + N_2 \left( r, \frac{1}{H} \right)
\]

\[
+ N_2 \left( r, \frac{Hg^n(g - 1)g'}{z} \right)
\]

\[
+ N \left( r, f^n(f - 1)f' \right)
\]

\[
+ N \left( r, H \right) + N \left( r, Hg^n(g - 1)g' \right) + 2 \log r + o(T(r)).
\]

Using (27), we note that

\[
N_2 \left( r, \frac{1}{Hg^n(g - 1)g'} \right)
\]

\[
\leq N_2 \left( r, \frac{1}{H} \right) + N_2 \left( r, \frac{1}{g^n(g - 1)g'} \right)
\]

\[
\leq 2N \left( r, \frac{1}{H} \right) + N_2 \left( r, \frac{1}{g^n(g - 1)g'} \right)
\]

\[
\leq 2N_L \left( r, g \right) + N_2 \left( r, \frac{1}{g^n(g - 1)g'} \right).
\]

Since \( N_L(r, g) = 0 \), we obtain that

\[
N_2 \left( r, \frac{1}{Hg^n(g - 1)g'} \right) \leq N_2 \left( r, \frac{1}{g^n(g - 1)g'} \right),
\]

\[
N \left( r, Hg^n(g - 1)g' \right) \leq N \left( r, H \right) + N \left( r, g^n(g - 1)g' \right)
\]

\[
\leq N_L \left( r, f \right) + N \left( r, g \right).
\]
But $\overline{N}(r, f) = 0$, so we get
\begin{equation}
\overline{N} (r, H g^n (g - 1) g') \leq \overline{N} (r, g).
\end{equation}
(35)

Using (33) and (35) in (31), we get
\begin{equation}
T(r, f) \leq N_2 \left( r, \frac{1}{f^n (f - 1) f'} \right) + N_2 \left( r, \frac{1}{H} \right) + N_2 \left( r, \frac{1}{g^n (g - 1) g'} \right) + \overline{N} (r, f) + \overline{N} (r, H) + \overline{N} (r, g) + 2 \log r + o (T(r)).
\end{equation}
(36)

Similarly,
\begin{equation}
\left[ N_3 \left( r, \frac{1}{g^n (g - 1) g'} \right) - 2 \overline{N}_3 \left( r, \frac{1}{g^n (g - 1) g'} \right) \right] \geq (n - 2) N \left( r, \frac{1}{g} \right).
\end{equation}
(39)

Let
\begin{equation}
f_1^* = \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}.
\end{equation}
(40)

By Lemma 9, we have $T(r, f_1^*) = (n + 2)T(r, f) + S(r, f)$.

Since $(f_1^*)' = zf_1$, we have
\begin{equation}
m \left( r, \frac{1}{f_1^*} \right) \leq m \left( r, \frac{1}{zf_1} \right) + m \left( r, \frac{(f_1^*)'}{f_1^*} \right) \leq m \left( r, \frac{1}{f_1} \right) + \log r + S(r, f).
\end{equation}
(41)

By the first fundamental theorem, we have
\begin{equation}
T(r, f_1^*) \leq T(r, f) + N \left( r, \frac{1}{f_1^*} \right) - N \left( r, \frac{1}{f_1} \right) + \log r + S(r, f);
\end{equation}
(42)

we have
\begin{equation}
N \left( r, \frac{1}{f_1^*} \right) = (n + 1) N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f - (n + 2) / (n + 1)} \right).
\end{equation}
(43)

From (37)–(43), we get
\begin{equation}
T(r, f_1^*) \leq N \left( r, \frac{1}{f^n (f - 1) f'} \right) - (n - 2) N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g^n (g - 1) g'} \right) - (n - 2) N \left( r, \frac{1}{g} \right) + 2 \overline{N}_L (r, g) + 2 \overline{N}_L (r, f) + 2 \overline{N} (r, f) + 2 \log r + o (T(r)) + \overline{N}_L (r, f) + 2 \overline{N} (r, f) + (n + 1) N \left( r, \frac{1}{f - (n + 2) / (n + 1)} \right) + N \left( r, \frac{1}{f - (n + 2) / (n + 1)} \right) + N \left( r, \frac{1}{f - (n + 2) / (n + 1)} \right) + 3 \log r + o (T(r)).
\end{equation}
(44)
Using Lemma 6, we get

\[ T(r, f_1^*) \leq 3N\left( r, \frac{1}{f} \right) + 3N\left( r, \frac{1}{g} \right) \]

\[ + 2N(r, f) + N(r, g) + 2N_L(r, f) \]

\[ + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{g-1} \right) \]

\[ + N\left( r, \frac{1}{f-(n+2)/(n+1)} \right) \]

\[ + 3 \log r + o(T(r)) \]

Adding (45) and (47) gives

\[ (n+2)T(r, f) + (n+2)T(r, g) \]

\[ \leq 6 \left( N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{g} \right) \right) \]

\[ + \left( N(r, f) + N(r, g) \right) \]

\[ + \left( N\left( r, \frac{1}{f-1} \right) + N\left( r, \frac{1}{g-1} \right) \right) \]

\[ + N\left( r, \frac{1}{g-(n+2)/(n+1)} \right) \]

\[ + 6 \log r + o(T(r)) \]

\[ (n-6)T(r, f) + (n-6)T(r, g) \]

\[ \leq 3 \left( N(r, f) + N(r, g) \right) \]

\[ + 3 \left( N_L(r, f) + N_L(r, g) \right) + 6 \log r + o(T(r)). \]

Using (48), we get

\[ (n-6)(T(r, f) + T(r, g)) \]

\[ \leq 3N(r, f) + 6N(r, g) + 6 \log r + o(T(r)) \]

or

\[ (n-6)(T(r, f) + T(r, g)) \]

\[ \leq 6N(r, f) + 3N(r, g) + 6 \log r + o(T(r)). \]

Combining (50) and (51), we get

\[ (n-6)(T(r, f) + T(r, g)) \leq \frac{9}{2} \left( N(r, f) + N(r, g) \right) \]

\[ + 6 \log r + o(T(r)) \]

\[ \left( n - \frac{21}{2} \right) (T(r, f) + T(r, g)) \leq 6 \log r + o(T(r)). \]

By \( n \geq 11 \) and (30), we get a contradiction. Thus \( f_1, f_2, \) and \( f_3 \) are linearly dependent. Then, there exists three constants \((c_1, c_2, c_3) \neq (0, 0, 0)\) such that

\[ c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \]

If \( c_1 = 0 \), from (53) \( c_2 \neq 0, c_3 \neq 0 \), and

\[ f_3 = -\frac{c_2}{c_3} f_2; \]

\[ \implies g''(g-1)g' = \frac{c_2}{c_3} z. \]
On integrating, we get
\[ \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} = c_2 \frac{z^2}{2} + k, \quad k \text{ is a constant}, \]
\[ T \left( r, \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} \right) \leq T \left( r, z^2 \right) + O(1), \]
\[(n + 2) T \left( r, g \right) \leq 2 \log r + O(1); \tag{56}\]
since \( n \geq 11 \), we get a contradiction.

Thus \( c_1 \neq 0 \), and by (53) we have
\[ c_1 f_1 = -c_2 f_2 - c_3 f_3 \]
\[ \Rightarrow f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{57}\]
Substituting this in \( f_1 + f_2 + f_3 = 1 \), we get
\[ \frac{-c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3 + f_2 + f_3 = 1; \tag{58}\]
that is,
\[ \left( 1 - \frac{c_2}{c_1} \right) f_2 + \left( 1 - \frac{c_3}{c_1} \right) f_3 = 1, \quad \text{where } c_1 \neq c_3, \ c_2 \neq c_3. \tag{59}\]

From (28), we obtain
\[ \left( 1 - \frac{c_2}{c_1} \right) H + \left( 1 - \frac{c_3}{c_1} \right) \left( \frac{-H g^n (g - 1) g'}{z} \right) = 1 \]
\[ \Rightarrow \left( 1 - \frac{c_2}{c_1} \right) - \left( 1 - \frac{c_3}{c_1} \right) \left( \frac{g^n (g - 1) g'}{z} \right) = 1 \]
\[ \Rightarrow \left( 1 - \frac{c_2}{c_1} \right) \left( \frac{g^n (g - 1) g'}{z} \right) + \frac{1}{H} = \left( 1 - \frac{c_3}{c_1} \right). \tag{60}\]
Applying Lemma 5 to the above equation, we get
\[ T \left( r, \frac{g^n (g - 1) g'}{z} \right) \leq \overline{N} \left( r, \frac{g^n (g - 1) g'}{z} \right) \]
\[ + \overline{N} \left( r, \frac{g^n (g - 1) g'}{z} \right) + \overline{N} \left( r, H \right) + S \left( r, g \right). \tag{61}\]

Using (61), we get
\[ T \left( r, g^n (g - 1) g' \right) \leq \overline{N} \left( r, g^n (g - 1) g' \right) + \overline{N} \left( r, \frac{1}{g^n (g - 1) g'} \right) + 2 \log r + S \left( r, g \right), \]
\[ \leq \overline{N} \left( r, \frac{1}{g^n (g - 1) g'} \right) + 2 \overline{N} \left( r, g \right) + 2 \log r + S \left( r, g \right). \tag{63}\]

By Lemmas 9 and 6 and (63), we have
\[(n + 1) T \left( r, g \right) = T \left( r, g^n (g - 1) \right) + S \left( r, g \right)
\leq T \left( r, g^n (g - 1) g' \right) + T \left( r, \frac{1}{g} \right) + S \left( r, g \right)
\leq \overline{N} \left( r, \frac{1}{g^n (g - 1) g'} \right) + 2 \overline{N} \left( r, g \right) + 2 \log r + S \left( r, g \right)
\leq 8 T \left( r, g \right) + 2 \log r + S \left( r, g \right)
\Rightarrow (n - 7) T \left( r, g \right) \leq 2 \log r + S \left( r, g \right); \tag{64}\]
we obtain \( n \leq 7 \), which contradicts \( n \geq 11 \).

Case 2. Suppose that \( f_2 = c \ (\neq 0) \), where \( c \) is a constant. If \( c \neq 1 \), then we have
\[ f_1 + f_2 + f_3 = 1 \]
\[ \Rightarrow f_1 = \frac{f_1 \left( g^n (f - 1) f' \right)}{z} - \frac{c \left( g^n (g - 1) g' \right)}{z} = 1; \tag{65}\]
\[ \Rightarrow f_1 = \frac{f_1 \left( g^n (f - 1) f' \right)}{z} - \frac{c \left( g^n (g - 1) g' \right)}{z} = 1 - c. \tag{66}\]
Applying Lemma 5 to the above equation, we have
\[ T \left( r, \frac{f^n (f - 1) f'}{z} \right) \leq \overline{N} \left( r, \frac{f^n (f - 1) f'}{z} \right) \]
\[ + \overline{N} \left( r, \frac{z}{g^n (g - 1) g'} \right) + \overline{N} \left( r, \frac{z}{f^n (f - 1) f'} \right) + S \left( r, f \right). \]
\[
\leq N(r, f) + N\left( r, \frac{1}{g^n(g-1)g'} \right) + N\left( r, \frac{1}{g'} \right) + N\left( r, \frac{1}{g-1} \right) + 2T(r, f) + 2 \log r + S(r, f).
\]

Note that

\[
T\left( r, f^n(f-1)f' \right) \leq T\left( r, \frac{f^n(f-1)f'}{z} \right) + \log r. \tag{68}
\]

Therefore,

\[
T\left( r, f^n(f-1)f' \right) \leq N(r, f) + N\left( r, \frac{1}{g^n(g-1)g'} \right) + N\left( r, \frac{1}{f^n(f-1)f'} \right) + 2 \log r + S(r, f). \tag{69}
\]

Using Lemmas 9 and 6 and (69), we have

\[
(n+1)T(r, f) = T(r, f^n(f-1)) + S(r, f)
\]

\[
\leq T\left( r, f^n(f-1)f' \right) + T\left( r, \frac{1}{f^n} \right) + S(r, f)
\]

\[
\leq N(r, f) + N\left( r, \frac{1}{f^n(f-1)f'} \right) + N\left( r, \frac{1}{g^n(g-1)g'} \right) + T(r, f') + 2 \log r + S(r, f)
\]

\[
\leq N(r, f) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f-1} \right) + N\left( r, \frac{1}{f'} \right) + 2T(r, f) + 2 \log r + S(r, f).
\]

Using Lemma 7, we get

\[
(n-6)T(r, f) \leq 4 \left( \frac{n+3}{n-6} \right) T(r, f) + 2 \log r + S(r, f); \tag{71}
\]

since \( n \geq 11 \), we get a contradiction.

Therefore \( c = 1 \), and by (27) and (24) we have

\[
f^n(f-1)f' = g^n(g-1)g'. \tag{72}
\]

On integrating, we get

\[
\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} + k \tag{73}
\]

\[F^* = G^* + k, \text{ where } k \text{ is a constant.}\]

We claim that \( k = 0 \). Suppose that \( k \neq 0 \), then

\[
\Theta(0, F^*) + \Theta(k, F^*) + \Theta(\infty, F^*) = \Theta(0, F^*) + \Theta(0, G^*) + \Theta(\infty, F^*). \tag{74}
\]

We have

\[
N\left( r, \frac{1}{F^*} \right)
\]

\[
= N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f - (n+2)/(n+1)} \right) \leq 2T(r, f). \tag{75}
\]

Similarly,

\[
N\left( r, \frac{1}{G^*} \right) \leq 2T(r, g), \tag{76}
\]

\[N(r, F^*) = N(r, f) \leq T(r, f). \]

Using Lemma 9, we have

\[
T(r, F^*) = (n+2)T(r, f) + S(r, f), \tag{77}
\]

\[T(r, G^*) = (n+2)T(r, g) + S(r, g). \]

Thus,

\[
\Theta(0, F^*) = 1 - \lim_{r \to \infty} \frac{N(r, 1/F^*)}{T(r, F^*)} \geq 1 - \frac{2}{n+2}. \tag{78}
\]
Similarly,
\[
\Theta(0, G^*) \geq 1 - \frac{2}{n+2},
\]
\[
\Theta(\infty, F^*) = 1 - \lim_{r \to \infty} \frac{N(r, F^*)}{T(r, F^*)} \geq 1 - \frac{1}{n+2}.
\]
Therefore, (74) becomes
\[
\Theta(0, F^*) + \Theta(k, F^*) + \Theta(\infty, F^*)
\]
\[
\geq 2 \left(1 - \frac{2}{n+2}\right) + 1 - \frac{1}{n+2} = \frac{3n+1}{n+2} \geq 2 \text{ for } n \geq 11,
\]
which contradicts \(\sum_{a \in \mathbb{T}} \Theta(a, f) \leq 2\). Thus, we have
\[
\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} = g
\]
(81).

Let \(h = \frac{f}{g}\). If \(h \neq 1\), then we easily obtain that
\[
g = \frac{(n+2)\left(h^{n+1} - 1\right)}{(n+1)\left(h^{n+2} - 1\right)}, \quad f = \frac{(n+2)h\left(h^{n+1} - 1\right)}{(n+1)\left(h^{n+2} - 1\right)},
\]
(82).

If \(h \equiv 1\), that is, \(f \equiv g\).

Case 3. Suppose that \(f_3 = c\) (\(\neq 0\)), where \(c\) is a constant.

If \(c \neq 1\), then we have
\[
f_1 + f_2 + f_3 = 1
\]
\[
f_1 + f_2 = 1 - c
\]
(83).

Applying Lemma 5 to the above equation, we have
\[
T\left(r, \frac{f^n(f-1)f'}{z}\right) \leq N\left(r, \frac{f^n(f-1)f'}{z}\right)
\]
\[
+ N\left(r, \frac{z}{f^n(f-1)f'}\right)
\]
\[
+ N\left(r, \frac{g^n(g-1)g'}{z}\right) + S(r, f)
\]
\[
\leq N(r, f) + N\left(r, \frac{1}{f^n(f-1)f'}\right)
\]
\[
+ N(r, g) + 2 \log r + S(r, f).
\]
(84).

Therefore using (84), we have
\[
T\left(r, \frac{f^n(f-1)f'}{z}\right) \leq N\left(r, \frac{1}{f^n(f-1)f'}\right)
\]
\[
+ N(r, g) + 3 \log r + S(r, f).
\]
(86).

Using Lemmas 9 and 6 and (86), we have
\[
(n+1)T(r, f) = T(r, f^n(f-1)) + S(r, f)
\]
\[
\leq T\left(r, \frac{1}{f^n}f'\right)
\]
\[
+ T\left(r, \frac{1}{f^{n+1}}\right) + S(r, f)
\]
\[
\leq N(r, f) + N\left(r, \frac{1}{f^n(f-1)f'}\right)
\]
\[
+ N\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^{n+1}}\right)
\]
\[
+ N\left(r, \frac{1}{f^n}\right) + N\left(r, \frac{1}{f^{n+1}}\right) + 2T(r, f)
\]
\[
+ 3 \log r + S(r, f)
\]
\[
\leq 7T(r, f) + T(r, g) + 3 \log r + S(r, f)
\]
\[
\Rightarrow (n-6)T(r, f) \leq T(r, g) + 3 \log r + S(r, f).
\]
(87).

Using Lemma 7, we get
\[
(n-6)T(r, f) \leq \frac{(n+3)}{(n-6)}T(r, f) + 3 \log r + S(r, f); \quad (88)
\]
since \(n \geq 11\), we get a contradiction. Thus, \(c = 1\). Hence,
\[
\frac{f^n(f-1)f'}{z} - \frac{z}{g^n(g-1)g'} = 0
\]
(89).

Let \(z_0\) be a zero of \(f\) of order \(p\). From (89), we know that \(z_0\) is a pole of \(g\). Suppose that \(z_0\) is a pole of \(g\) of order \(q\). From (89), we obtain
\[
np + p - 1 = nq + 2q + 1
\]
\[
\Rightarrow (n+1)(p - q) = q + 2,
\]
which implies that \(p \geq q + 1\) and \(q + 2 \geq n + 1\). Hence,
\[
p \geq n.
\]
(91).
Let $z_1$ be a zero of $(f-1)$ of order $p_1$, then from (89) $z_1$ is a pole of $g$ (say order $q_1$). By (89), we get
\[
\begin{align*}
p_1 + p_1 - 1 &= mq_1 + 2q_1 + 1 \\
2p_1 - 1 &\geq n + 3 \\
2p_1 &\geq n + 4 \implies p_1 \geq \frac{n + 4}{2}.
\end{align*}
\] (92)

Let $z_2$ be a zero of $f'$ of order $p_2$ that is not zero of $f(f-1)$, then from (89), $z_2$ is a pole of $g$ of order $q_2$. Again by (89), we get
\[
p_2 = mq_2 + 2q_2 + 1 \implies p_2 \geq n + 3.
\] (93)

In the same manner as above, we have similar results for the zeros of $g^n(g-1)g'$. From (89)--(93), we have
\[
N(r, f^n(f-1)f') = N\left(r, z, \frac{z^2}{g^n(g-1)g'}\right);
\] (94)

that is,
\[
\begin{align*}
N(r, f) &\leq N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-1}\right) + N\left(r, \frac{1}{g'}\right) \\
&\leq \frac{1}{11} N\left(r, \frac{1}{g}\right) + \frac{2}{15} N\left(r, \frac{1}{g-1}\right) + \frac{1}{14} N\left(r, \frac{1}{g'}\right) \\
&\leq \frac{1}{11} T(r, g) + \frac{2}{15} T(r, g) + \frac{2}{14} T(r, g) + S(r, g) \\
&= \left(\frac{1}{11} + \frac{2}{15} + \frac{2}{14}\right) T(r, g) + S(r, g) \\
&\implies N(r, f) < \frac{2}{3} T(r, g) + S(r, g).
\end{align*}
\] (95)

By Nevanlinna’s second fundamental theorem, we have from (91), (92), and (95) that
\[
\begin{align*}
T(r, f) &\leq N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N(r, f) + S(r, f) \\
&< \frac{1}{11} T(r, f) + \frac{2}{15} T(r, f) + \frac{2}{3} T(r, g) \\
&\quad + S(r, f) + S(r, g) \\
&\quad + T(r, f) \\
&\leq \frac{37}{165} T(r, f) + \frac{2}{3} T(r, g) + S(r, f) + S(r, g).
\end{align*}
\] (96)

Similarly,
\[
\begin{align*}
T(r, g) &\leq \frac{37}{165} T(r, g) + \frac{2}{3} T(r, f) + S(r, f) + S(r, g).
\end{align*}
\] (97)

From (96) and (97), we get
\[
\begin{align*}
T(r, f) + T(r, g) \\
&\leq \frac{37}{165} \left(T(r, f) + T(r, g)\right) + \frac{2}{3} \left(T(r, f) + T(r, g)\right) + S(r, f) + S(r, g)
\end{align*}
\] (98)

\[
\begin{align*}
&+ \frac{18}{165} \left(T(r, f) + T(r, g)\right) \leq S(r, f) + S(r, g);
\end{align*}
\]

since $n \geq 11$, we get a contradiction.

This completes the proof of Theorem 1.

Using the same argument as in the proof of Theorem 1, we can prove Theorem 2.

**Proof of Theorem 3.** By the assumption of the theorem, we know that either both $f$ and $g$ are two transcendental entire functions or both $f$ and $g$ are polynomials.

If $f$ and $g$ are transcendental entire functions, putting $N(r, f) = 0, N(r, g) = 0$ and using similar arguments as in the proof of Theorem 1, we easily obtain Theorem 3.

If $f$ and $g$ are polynomials, $f^n(f-1)f'$ and $g^n(g-1)g'$ share $z$ CM, we get
\[
f^n(f-1)f' - z = k(g^n(g-1)g' - z),
\] (99)

where $k$ is a nonzero constant. Suppose that $k \neq 1, (99)$ can be written as
\[
f^n(f-1)f' - \frac{k g^n(g-1)g'}{z} = 1 - k.
\] (100)

Applying Lemma 5 to the above equation, we have
\[
\begin{align*}
T\left(r, \frac{f^n(f-1)f'}{z}\right) &\leq N\left(r, \frac{f^n(f-1)f'}{z}\right) \\
&\quad + N\left(r, \frac{z}{g^n(g-1)g'}\right) \\
&\quad + N\left(r, \frac{z}{f^n(f-1)f'}\right) + S(r, f).
\end{align*}
\] (101)

Since $f$ is a polynomial, so it does not have any poles. Thus, we have,
\[
\begin{align*}
T\left(r, \frac{f^n(f-1)f'}{z}\right) &\leq N\left(r, \frac{1}{g^n(g-1)g'}\right) \\
&\quad + \log r + S(r, f).
\end{align*}
\] (102)

Note that
\[
T\left(r, f^n(f-1)f'\right) \leq T\left(r, \frac{f^n(f-1)f'}{z}\right) + \log r.
\] (103)
Therefore,
\[ T(r, f^n (f - 1) f') \leq N\left( r, \frac{1}{g^n (g - 1) g'} \right) + N\left( r, f^{n+1} (f - 1) f' \right) + 3 \log r + S(r, f). \]  
(104)

Using Lemmas 9 and 6 and (104), we have
\[ (n + 1) T(r, f) = T(r, f^n (f - 1)) + S(r, f) \leq T\left( r, f^n (f - 1) f' \right) + T\left( r, f^{n+1} (f - 1) f' \right) + 3 \log r + S(r, f) \leq N\left( r, \frac{1}{g^n (g - 1) g'} \right) + N\left( r, 1 \right) + 3 \log r + S(r, f). \]  
(105)

Using Lemma 8, we get
\[ (n - 3) T(r, f) \leq 3 T(r, g) + 3 \log r + S(r, f). \]  
(106)

since \( n \geq 7 \), we get a contradiction.

Therefore, \( k = 1 \); so (99) becomes
\[ f^n (f - 1) f' = g^n (g - 1) g'. \]  
(107)

On integrating, we get
\[ \frac{f^{m+2}}{n+2} - \frac{f^{m+1}}{n+1} = \frac{g^{m+2}}{n+2} - \frac{g^{m+1}}{n+1} + c \]  
(108)
\[ F^* = G^* + c, \]  
where \( c \) is a constant.

We claim that \( c = 0 \). Suppose that \( c \neq 0 \), then
\[ \Theta(0, F^*) + \Theta(c, F^*) = \Theta(0, F^*) + \Theta(0, G^*). \]  
(109)

We have
\[ N\left( r, \frac{1}{F^*} \right) = N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{f - (n + 2) / (n + 1)} \right) \leq 2T(r, f). \]  
(110)

Similarly,
\[ N\left( r, \frac{1}{G^*} \right) \leq 2T(r, g). \]  
(111)

Using Lemma 9, we have
\[ T(r, F^*) = (n + 2) T(r, f) + S(r, f), \]  
(112)
\[ T(r, G^*) = (n + 2) T(r, g) + S(r, g). \]

Thus,
\[ \Theta(0, F^*) = 1 - \lim_{r \to \infty} \frac{N(r, 1/F^*)}{T(r, F^*)} \geq 1 - \frac{2}{n + 2}. \]  
(113)

Similarly,
\[ \Theta(0, G^*) \geq 1 - \frac{2}{n + 2}. \]  
(114)

Therefore, (109) becomes
\[ \Theta(0, F^*) + \Theta(c, F^*) \geq 2 \left( 1 - \frac{2}{n + 2} \right) = \frac{2n}{n + 2} \geq \frac{14}{9} > 1 \text{ for } n \geq 7, \]  
(115)

which contradicts \( \sum_{a \in C} \Theta(a, f) \leq 1 \). Thus, we have
\[ \frac{f^{m+2}}{n+2} - \frac{f^{m+1}}{n+1} = \frac{g^{m+2}}{n+2} - \frac{g^{m+1}}{n+1} + c. \]  
(116)

Let \( h = f / g \). If \( h \equiv 1 \), then by (116) we have
\[ g = \frac{(n + 2) (1 + h + h^2 + \cdots + h^m)}{(n + 1) (1 + h + h^2 + \cdots + h^{m+1})}. \]  
(117)

By Picard’s theorem, \( h(z) \) is a constant. Hence, \( g \) is a constant, which is a contradiction. Therefore, \( h(z) \equiv 1 \), that is, \( f(z) \equiv g(z) \).

\[ \square \]

4. Remarks

If the condition “\( f^n (f - 1) f' \) and \( g^n (g - 1) g' \) share \( z \) CM” is replaced by the condition “\( f^n (f - 1) f' \) and \( g^n (g - 1) g' \) share \( \alpha(z) \) CM,” where \( \alpha \) is a meromorphic function such that \( \alpha \neq 0, \infty \) and \( T(r, \alpha) = o(T(r, f), T(r, g)) \); the conclusion of Theorems 1, 2, and 3 still holds. We, thus, obtain the following results.
Theorem 11. Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 11$ a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $\alpha(z)$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) \equiv g(z)$ or
\[
g = \frac{(n+2)\left(1 - \frac{1}{h^{n+1}}\right)}{(n+1)\left(1 - \frac{1}{h^{n+2}}\right)}, \quad f = \frac{(n+2)h\left(1 - \frac{1}{h^{n+1}}\right)}{(n+1)\left(1 - \frac{1}{h^{n+2}}\right)}, \tag{118}
\]
where $h$ is a nonconstant meromorphic function.

Theorem 12. Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 12$ a positive integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $\alpha(z)$ CM, $f$ and $g$ share $\infty$ IM, then $f(z) \equiv g(z)$.

Theorem 13. Let $f$ and $g$ be two nonconstant entire functions, $n \geq 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.

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