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Research Article

Conformal Geometry of Hypersurfaces in Lorentz Space Forms

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Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface without umbilical points in the Lorentz space form $M_1^{n+1}(c)$. We define the conformal metric and the conformal second fundamental form on the hypersurface, which determines the hypersurface up to conformal transformation of $M_1^{n+1}(c)$. We calculate the Euler-Lagrange equations of the volume functional of the hypersurface with respect to the conformal metric, whose critical point is called a Willmore hypersurface, and we give a conformal characteristic of the hypersurfaces with constant mean curvature and constant scalar curvature. Finally, we prove that if the hypersurface x with constant mean curvature and constant scalar curvature is Willmore, then x is a hypersurface in $H_1^{n+1}(-1)$.

1. Introduction

Let $x : M^n \to S^{n+p}$ be an immersed submanifold in sphere S^{n+k}. In [1], on the submanifold the Wang has constructed a complete invariant system of the Möbius transformation group of S^{n+p}. Especially for the hypersurface, the Möbius invariants, the Möbius metric, and the Möbius second fundamental form determine the hypersurface up to Möbius transformations provided the dimension of hypersurface $n \geq 3$ (also see [2]). After that, the study of the Möbius geometry has been a topic of increasing interest (see [3–6]).

In this paper we study space-like hypersurfaces in the Lorentz space form $M_1^{n+1}(c)$ under the conformal transformation group. We follow Wang's idea and construct conformal invariants of space-like hypersurfaces which determine hypersurfaces up to a conformal transformation.

For the Lorentz space form, there exists a united conformal compactification $\mathbb{Q}_1^{n+1}$, which is the projectivized light cone in $\mathbb{R}P^{n+2}$ induced from R_2^{n+3} (see [7, 8]). Using conformal compactification $\mathbb{Q}_1^{n+1}$, we define the conformal metric g and the conformal second fundamental form on a hypersurface in the Lorentz space form, which determines a hypersurface up to a conformal transformation. Clearly, the volume functional with respect to the conformal metric is a conformal invariant. We call a critical hypersurface of the volume functional Willmore hypersurface. There are many studies about the Willmore hypersurface in the Lorentz space form (see, [7, 9, 10]).

Our main goal is to calculate the Euler-Lagrange equation for the volume functional by conformal invariants and to find some special Willmore hypersurfaces. We find that maximal hypersurfaces in Lorentz space form are not Willmore in general if the dimension $n \geq 3$. We give a conformal characteristic of the hypersurfaces with constant mean curvature and constant scalar curvature. By the conformal characteristic, we prove that if the hypersurfaces are Willmore, then the hypersurfaces must be in $H_1^{n+1}(-1)$. Thus, isoparametric hypersurfaces in $S_1^{n+1}(1)$ and R_1^{n+1} are not Willmore.

We organize the paper as follows. In Section 2, we define the conformal invariants and give conformal congruent theorem of hypersurfaces in the Lorentz space form. In Section 3, we calculate the Euler Lagrange equation for the volume functional. In Section 4, we give a conformal characteristic of space-like hypersurfaces with constant mean curvature and constant scalar curvature. In Section 5, we give some examples of the Willmore hypersurface and prove that some special hypersurfaces are not Willmore in general.

2. Conformal Invariants of Hypersurfaces in Lorentz Space

In this section, we define some conformal invariants of hypersurface and give a congruent theorem of hypersurfaces under the conformal group of $M_1^{n+1}(c)$.

Let R_t^{n+2} be the real vector space R^{n+2} with the Lorenz inner product $\langle\, ,\, \rangle$ given by

$$\langle X, Y\rangle := -\sum_{i=1}^{t} x_i y_i + \sum_{j=t+1}^{n+2} x_j y_j. \tag{1}$$

Let $M_1^{n+1}(c)$ be a Lorentz space form. When $c = 0$, $M_1^{n+1}(c) = R_1^{n+1}$; when $c = 1$, $M_1^{n+1}(c) = S_1^{n+1}(1)$, and when $c = -1$, $M_1^{n+1}(c) = H_1^{n+1}(-1)$, where

$$\begin{aligned} S_1^{n+1}(1) &= \left\{x \in R_1^{n+2} \mid \langle x, x\rangle = 1\right\}, \\ H_1^{n+1}(-1) &= \left\{x \in R_2^{n+2} \mid \langle x, x\rangle = -1\right\}. \end{aligned} \tag{2}$$

We denote by C^{n+2} the cone in $\mathbb{R}_2^{n+3}$ and by $\mathbb{Q}_1^{n+1}$ the conformal compactification space in $\mathbb{R}P^{n+3}$:

$$\begin{aligned} C^{n+2} &:= \left\{X \in \mathbb{R}_2^{n+3} \mid \langle X, X\rangle = 0,\ X \neq 0\right\}, \\ \mathbb{Q}_1^{n+1} &:= \left\{[X] \in \mathbb{R}P^{n+2} \mid \langle X, X\rangle = 0\right\}. \end{aligned} \tag{3}$$

Let $O(n+3, 2)$ be the Lorentz group of R_2^{n+3} keeping the Lorentz inner product $\langle\, ,\, \rangle$ invariant. Then, $O(n+3, 2)$ is a transformation group on $\mathbb{Q}_1^{n+1}$ defined by

$$T([X]) := [XT], \quad X \in C^{n+2},\ T \in O(n+3, 2). \tag{4}$$

Topologically, $\mathbb{Q}_1^{n+1}$ is identified with the compact space $S^n \times S^1/S^0$, which is endowed by a standard Lorentz metric $h = g_{S^n} \oplus (-g_{S^1})$. $\mathbb{Q}_1^{n+1}$ has conformal metric:

$$[h] := \left\{e^{\tau} h \mid \tau \in C^{\infty}\left(\mathbb{Q}_1^{n+1}\right)\right\}, \tag{5}$$

and $[O(n+3, 2)]$ is the conformal transformation group of $\mathbb{Q}_1^{n+1}$.

We define the following mappings (without ambiguity all denote by σ):

$$\begin{aligned} \sigma : R_1^{n+1} &\longrightarrow \mathbb{Q}_1^{n+1}, \qquad u \longmapsto \left[\left(\frac{\langle u, u\rangle + 1}{2}, u, \frac{\langle u, u\rangle - 1}{2}\right)\right], \\ \sigma : S_1^{n+1}(1) &\longrightarrow \mathbb{Q}_1^{n+1}, \qquad u \longmapsto [(1, u)], \\ \sigma : H_1^{n+1}(-1) &\longrightarrow \mathbb{Q}_1^{n+1}, \qquad u \longmapsto [(u, 1)]. \end{aligned} \tag{6}$$

Using σ, we can regard hypersurfaces in $M_1^{n+1}(c)$ as submanifolds in $\mathbb{Q}_1^{n+1}$. A classical theorem states the following.

Theorem 1. *Two hypersurfaces $x, \overline{x} : M^n \to M_1^{n+1}(c)$ are conformal equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\sigma \circ x = T(\sigma \circ \overline{x}) : M^n \to \mathbb{Q}_1^{n+1}$.*

Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface without umbilical points; then, $(\sigma \circ x) * (TM^n)$ is a positive definite subbundle of $T\mathbb{Q}_1^{n+1}$. For any local lift Z of the standard projection $\pi : C^{n+2} \to \mathbb{Q}_1^{n+1}$, we get a local lift $y := Z \circ \sigma \circ x : U \to C^{n+1}$ of $\sigma \circ x : M \to \mathbb{Q}_1^{n+1}$, which is defined in an open subset U of M^n. Thus, $\langle dy, dy\rangle = \lambda^2 dx \cdot dx$ is a local metric, which is conformal to the induced metric $dx \cdot dx$. We denote by Δ and κ the Laplacian operator and the normalized scalar curvature with respect to the local positive definite metric $\langle dy, dy\rangle$, respectively. Then, we have the following.

Theorem 2. *Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface; then the 2-form $g := -(\langle \Delta y, \Delta y\rangle - n^2\kappa)\langle dy, dy\rangle$ is a globally defined conformal invariant. Moreover, g is positive definite at any nonumbilical point of M^n.*

Proof. First, we prove that g is well defined. Suppose that $y : U \to C^{n+1}$, $\widetilde{y} : \widetilde{U} \to C^{n+2}$ are different lifts defined in open subsets U and $\widetilde{U}$ of M. For the local positive definite metrics $\langle\, ,\, \rangle_y = \langle dy, dy\rangle$, we denote by Δ the Laplace operator, by ∇f the gradient of a function f and by κ the normalized scalar curvatures with respect to $\langle\, ,\, \rangle_y$, respectively. Analogously for $\langle d\widetilde{y}, d\widetilde{y}\rangle$, we denote by $\widetilde{\Delta}$ the Laplace operator and by $\widetilde{\kappa}$ the normalized scalar curvatures. On $U \cap \widetilde{U}$, we have $\widetilde{y} = e^{\tau} y$, where $\tau \in C^{\infty}(U \cap \widetilde{U})$. Therefore, $\langle d\widetilde{y}, d\widetilde{y}\rangle = e^{2\tau}\langle dy, dy\rangle$. By some computations, we have

$$\begin{aligned} e^{2\tau}\widetilde{\Delta} f &= \Delta f + (n-2)\left\langle \nabla\tau, \nabla f\right\rangle_y, \\ e^{2\tau}\widetilde{\kappa} &= \kappa - \frac{2}{m}\Delta\tau - \frac{n-2}{n}\langle \nabla\tau, \nabla\tau\rangle_y. \end{aligned} \tag{7}$$

Using these formula, it follows that

$$\left(\langle \Delta y, \Delta y\rangle - n^2\kappa\right)\langle dy, dy\rangle = \left(\left\langle \widetilde{\Delta}\widetilde{y}, \widetilde{\Delta}\widetilde{y}\right\rangle - n^2\widetilde{\kappa}\right)\langle d\widetilde{y}, d\widetilde{y}\rangle. \tag{8}$$

Next, we prove that g is invariant under conformal transformations of $M_1^{n+1}(c)$. Let ϕ be a conformal transformation of $M_1^{n+1}(c)$, and we denote $\overline{x} = \phi(x)$; then, there is a $T \in O(n+3, 2)$ acting on $\mathbb{Q}_1^{n+1}$, and $y : U \to C^{n+2}$ is a lift of $\sigma \circ x : M \to \mathbb{Q}_1^{n+1}$ defined in open subsets U; then, the submanifold $\overline{x} = \phi(x)$ must have a local lift like $\overline{y} = e^{\tau} yT$. Since T preserves the Lorentz inner product and the dilatation of the local lift y will not impact the term $(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle dy, dy\rangle$, therefore the 2-form g is invariant under conformal transformations.

Now, let $M_1^{n+1}(c) = S_1^{n+1}(1)$ and take local lift $y = (1, x)$; then

$$-\left(\langle \Delta y, \Delta y\rangle - n^2\kappa\right) = \frac{n}{n-1}\left(|II|^2 - nH^2\right), \tag{9}$$

where II and H are the second fundamental form and the mean curvature of x, respectively. Thus, g is positive definite at any nonumbilical point of M^n; analogously for hypersurfaces in R_1^{n+1} and H_1^{n+1}. Thus; we complete the proof of Theorem 2. □

Now, we assume that space-like hypersurface $x : M^n \to M_1^{n+1}(c)$ is umbilical-free; thus, the 2-form g is a positive definite. We call g the conformal metric of hypersurface x. There exists a unique lift:

$$Y : M \longrightarrow C^{n+2} \tag{10}$$

such that $g = \langle dY, dY\rangle$. We call Y the conformal position vector of x. Theorem 2 implies the following.

Theorem 3. *Two space-like hypersurfaces $x, \overline{x} : M^n \to M_1^{n+1}(c)$ are conformal equivalent if and only if there exists $T \in O(n+3, 2)$ such that $\overline{Y} = YT$, where $Y, \overline{Y}$ are the conformal position vectors of $x, \overline{x}$, respectively.*

Let $\{E_1, \ldots, E_n\}$ be a local orthonormal basis of M^n with respect to g with dual basis $\{\omega_1, \ldots, \omega_n\}$. Denote $Y_i = E_i(Y)$. We define

$$N := -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y, \tag{11}$$

where Δ is the Laplace operator of g; then, we have

$$\begin{gathered}\langle N, Y\rangle = 1, \qquad \langle N, N\rangle = 0, \qquad \langle N, Y_k\rangle = 0, \\ \left\langle Y_i, Y_j\right\rangle = \delta_{ij}, \quad 1 \le k \le n.\end{gathered} \tag{12}$$

We may decompose R_2^{n+3} such that

$$R_2^{n+3} = \operatorname{span}\{Y, N\} \oplus \operatorname{span}\{Y_1, \ldots, Y_n\} \oplus \mathbb{V}, \tag{13}$$

where $\mathbb{V} \perp \operatorname{span}\{Y, N, Y_1, \ldots, Y_n\}$. We call $\mathbb{V}$ the conformal normal bundle of x, which is linear bundle. Let ξ be a local section of $\mathbb{V}$ and $\langle \xi, \xi\rangle = -1$; then, $\{Y, N, Y_1, \ldots, Y_n, \xi\}$ forms a moving frame in R_2^{n+3} along M^n. We may write the structure equations as follows:

$$\begin{aligned} dY &= \sum_i \omega_i Y_i, \\ dN &= \sum_i A_i Y_i + C\xi, \\ dY_i &= -A_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + B_i \xi, \\ d\xi &= CY + \sum_i B_i Y_i, \end{aligned} \tag{14}$$

where $\{A_i, C, B_i, \omega_{ij}\}$ are 1-forms on M^n with $\omega_{ij} = -\omega_{ij}$. It is clear that $A := \sum_i A_i \otimes \omega_i$, $B := \sum_i B_i \otimes \omega_i$, C are globally defined conformal invariants. We call B the conformal second fundamental form, A the conformal 2-tensor, and C conformal 1-form, respectively. If we write

$$A_i = \sum_j A_{ij}\omega_j, \qquad B_i = \sum_j B_{ij}\omega_j, \qquad C = \sum_i C_i \omega_i, \tag{15}$$

then we can define the covariant derivatives of these tensors and curvature tensor with respect to conformal metric g:

$$\begin{aligned} \sum_j C_{i,j}\omega_j &= dC_i + \sum_k C_k \omega_{kj}, \\ \sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \\ \sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}. \end{aligned} \tag{16}$$

By exterior differentiation of structure equations (14) and the definition of the covariant derivative of conformal invariants, we can get the integrable conditions of the structure equations:

$$A_{ij} = A_{ji}, \qquad B_{ij} = B_{ji}, \tag{17}$$

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{18}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{19}$$

$$C_{i,j} - C_{j,i} = \sum_k \left(B_{ik}A_{kj} - B_{jk}A_{ki}\right), \tag{20}$$

$$R_{ijkl} = B_{il}B_{jk} - B_{ik}B_{jl} + A_{ik}\delta_{jl} + A_{jl}\delta_{ik} - A_{il}\delta_{jk} - A_{jk}\delta_{il}. \tag{21}$$

Since $g = \langle dY, dY\rangle$, we get

$$\langle \Delta Y, \Delta Y\rangle = n^2\kappa - 1. \tag{22}$$

From structure equation, we have

$$\Delta Y = -\operatorname{tr}(A) - nN + \operatorname{tr}(B)\,\xi. \tag{23}$$

Furthermore, we have

$$\begin{gathered} \operatorname{tr}(A) = \frac{1}{2n}\left(n^2\kappa - 1\right), \\ R_{ij} = \operatorname{tr}(A)\,\delta_{ij} + (n-2)\,A_{ij} + \sum_k B_{ik}B_{kj}, \\ (1-n)\,C_i = \sum_j B_{ij,j}, \qquad \sum_{ij} B_{ij}^2 = \frac{n-1}{n}, \qquad \sum_i B_{ii} = 0, \end{gathered} \tag{24}$$

where κ is the normalized scalar curvature of g. From (24), we see that when $n \ge 3$, all coefficients in the structure equations are determined by the conformal metric g and the conformal second fundamental form B; thus, we get the following conformal congruent theorem.

Theorem 4. *Two space-like hypersurfaces $x : M^n \to M_1^{n+1}(c)$ and $\overline{x} : M^n \to M_1^{n+1}(c)$ $(n \ge 3)$ are conformal equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ which preserves the conformal metric and the conformal second fundamental form.*

Remark 5. By using the same method as in [1], we can define conformal invariants of space-like submanifold

$x : M^m \to M_1^{n+1}(c)$. But for time-like submanifolds, the situation is very different. In fact, for space-like submanifolds, the globally defined conformal invariant $g := -(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle dy, dy\rangle$ is positive definite or semipositive definite, and for time-like submanifolds, the globally defined conformal invariant $g := -(\langle \Delta y, \Delta y\rangle - m^2\kappa)\langle dy, dy\rangle$ is non-definite.

Next, we give the relations between the conformal invariants and isometric invariants of $x : M^n \to M_1^{n+1}(c)$.

First, we consider space-like hypersurface in R_1^{n+1}. Let $x : M^n \to R_1^{n+1}$ be a space-like hypersurface without umbilical points. Let $\{e_1, \ldots, e_n\}$ be an orthonormal local basis for the induced metric $I = \langle dx, dx\rangle$ with dual basis $\{\theta_1, \ldots, \theta_n\}$. Let e_{n+1} be a normal vector field of x, and $\langle e_{n+1}, e_{n+1}\rangle = -1$. Then, we have the first and second fundamental forms I, II and the mean curvature H. We may write $I = \sum_i \theta_i \otimes \theta_i$; $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$; $H = (1/n)\sum_i h_{ii}$. Denote by Δ_M the Laplacian and κ_M the normalized scalar curvature for I. By structure equation and Gauss equation of x we get that

$$\Delta_M x = nHe_{n+1}, \qquad \kappa_M = \frac{-1}{n(n-1)}\left(n^2|H|^2 - |II|^2\right). \tag{25}$$

For $x : M^n \to R_1^{n+1}$, there is a lift:

$$y : M^n \longrightarrow C^{n+2}, \qquad y = \left(\frac{\langle x, x\rangle + 1}{2}, x, \frac{\langle x, x\rangle - 1}{2}\right). \tag{26}$$

So, we get

$$\langle dy, dy\rangle = \langle dx, dx\rangle = I, \qquad \Delta = \Delta_M, \qquad \kappa = \kappa_M. \tag{27}$$

It follows from (25) that

$$\langle \Delta Y, \Delta Y\rangle - n^2\kappa = \frac{n}{n-1}\left(-|II|^2 + n|H|^2\right). \tag{28}$$

Therefore, the conformal metric and conformal position vector of x are as follows:

$$g = \frac{n}{n-1}\left(|II|^2 - n|H|^2\right)\langle dx, dx\rangle := e^{2\tau} I,$$
$$Y = \sqrt{\frac{n}{n-1}\left(|II|^2 - n|H|^2\right)}\left(\frac{\langle x, x\rangle + 1}{2}, x, \frac{\langle x, x\rangle - 1}{2}\right). \tag{29}$$

Let $E_i = e^{-\tau}e_i$; then $\{E_i \mid 1 \le i \le n\}$ are the local orthonormal basis for g, and with the dual basis $\omega_i = e^{\tau}\theta_i$. Let

$$\begin{aligned} y_i &= \left(\langle x, e_i\rangle, e_i, \langle x, e_i\rangle\right), \\ y_{n+1} &= \left(\langle x, e_{n+1}\rangle, e_{n+1}, \langle x, e_{n+1}\rangle\right). \end{aligned} \tag{30}$$

By some calculations, we can obtain that

$$\begin{gathered} Y = e^{\tau} y, \qquad Y_i = e^{\tau}\left(\tau_i y + y_i\right), \qquad \xi = -Hy + y_{n+1}, \\ -e^{\tau}N = \frac{1}{2}\left(|\nabla\tau|^2 - |H|^2\right)y + \sum_i \tau_i y_i + Hy_{n+1} + \left(1, \vec{0}, 1\right), \end{gathered} \tag{31}$$

where $\tau_i = e_i(\tau)$ and $|\nabla\tau|^2 = \sum_i \tau_i^2$.

By a direct calculation, we get the following expression of the conformal invariants A, B, and C:

$$\begin{gathered} A_{ij} = e^{-2\tau}\left[\tau_i\tau_j - h_{ij}H - \tau_{i,j} + \frac{1}{2}\left(-|\nabla\tau|^2 + |H|^2\right)\delta_{ij}\right], \\ B_{ij} = e^{-\tau}\left(h_{ij} - H\delta_{ij}\right), \\ C_i = e^{-2\tau}\left(H\tau_i - H_i - \sum_j h_{ij}\tau_j\right), \end{gathered} \tag{32}$$

where $\tau_{i,j}$ is the Hessian of τ for I and $H_i = e_i(H)$.

Using the same methods, we can obtain relations between the conformal invariants and isometric invariants of $x : M^n \to S_1^{n+1}(1)$ and $x : M^n \to H_1^{n+1}(-1)$. We have the following unitied expression of the conformal invariants A, B, and C:

$$\begin{gathered} A_{ij} = e^{-2\tau}\left[\tau_i\tau_j - \tau_{i,j} - h_{ij}H + \frac{1}{2}\left(-|\nabla\tau|^2 + |H|^2 + \epsilon\right)\delta_{ij}\right], \\ B_{ij} = e^{-\tau}\left(h_{ij} - H\delta_{ij}\right), \\ C_i = e^{-2\tau}\left(H\tau_i - H_i - \sum_j h_{ij}\tau_j\right), \end{gathered} \tag{33}$$

where $\epsilon = 1$ for $x : M^n \to S_1^{n+1}(1)$ and $\epsilon = -1$ for $x : M^n \to H_1^{n+1}(-1)$.

3. The First Variation of the Conformal Volume Functional

Let $x_0 : M^n \to M_1^{n+1}(c)$ be a compact space-like hypersurface with boundary ∂M^n. We define the generalized Willmore functional $W(M^n)$ (as the volume functional of the conformal metric g):

$$\begin{aligned} &\mathrm{Vol}_g\left(M^n\right) \\ &\quad = \int_M dM_g = \left(\frac{n}{n-1}\right)^{n/2}\int_{M^n}\left(|II|^2 - nH^2\right)^{n/2} dM_I. \end{aligned} \tag{34}$$

A critical hypersurface of the conformal volume functional is called a Willmore hypersurface.

Let $x : M^n \times R \to M_1^{n+1}(c)$ be an admissible variation of x_0 such that

$$x(\cdot, t)|_{\partial M^n} = x_0|_{\partial M^n}, \tag{35}$$

$$dx_t\left(T_pM^n\right)\big|_{\partial M^n} = dx_0\left(T_pM^n\right)\big|_{\partial M^n}, \tag{36}$$

for each small t. For each t, x_t has the conformal metric g_t. As in Section 2, we have a moving frame $\{Y, N, Y_i, \xi\}$ in R_2^{n+3} and the Willmore functional $W(x_t)$. Let ξ be a local basis for

the conformal normal bundle $\mathbb{V}_t$ of x_t. Denote by d and d_M the differential operators on $M^n \times R$ and M^n, respectively. Then, we have

$$d = d_M + dt \wedge \frac{\partial}{\partial t}. \tag{37}$$

By (31), we can find functions $w, v_i, v : M^n \times R \to R$ such that

$$\frac{\partial Y}{\partial t} = wY + \sum_i v_i Y_i + v\xi. \tag{38}$$

Since $\{Y, N, Y_i, \xi\}$ is a moving frame along $M^n \times R$, it follows from (37) and (38) that

$$\begin{aligned} dY &= wdtY + \sum_i \Omega_i Y_i + V\xi, \\ dN &= -wdtN + \sum_i \Psi_i Y_i + \Phi\xi, \\ dY_i &= -\Psi_i Y - \Omega_i N + \sum_j \Omega_{ij} Y_j + F_i \xi, \\ d\xi &= -\Phi Y - VN - \sum_i F_i Y_i, \end{aligned} \tag{39}$$

where $\Omega_{ij} = -\Omega_{ji}$, $\Omega_i = \omega_i + v_i dt$, $V = vdt$. By exterior differentiation of (39), we get

$$\begin{aligned} d\Omega_i &= \sum_j \Omega_{ij} \wedge \Omega_j + wdt \wedge \Omega_i + vdt \wedge F_i, \\ dv \wedge dt &= \sum_i \Omega_i \wedge F_i, \\ dF_i &= \sum_j F_j \wedge \Omega_{ji} - \Omega_i \wedge \Phi - \Psi_i \wedge V, \\ d\Omega_{ij} &= \sum_k \Omega_{ik} \wedge \Omega_{kj} - \Omega_i \wedge \Psi_j - \Psi_i \wedge \Omega_j + F_i \wedge F_j. \end{aligned} \tag{40}$$

Since $T^*(M^n \times R) = T^*M^n \oplus T^*R$, we have the following decomposition:

$$\begin{aligned} \Omega_{ij} &= \omega_{ij} + L_{ij} dt, \qquad \Psi_i = A_i + u_i dt, \\ \Phi &= C + udt, \qquad F_i = B_i + b_i dt, \end{aligned} \tag{41}$$

where $\{u_i, u, L_{ij}, b_i\}$ are local functions on $M \times \mathbb{R}$. Using (40) and comparing the terms in $T^*M \wedge dt$, we get

$$\frac{\partial \omega_i}{\partial t} = \sum_j \left(v_{i,j} + L_{ij} + B_{ij} v\right) \omega_j + w\omega_i, \tag{42}$$

where $\{v_{i,j}\}$ is the covariant derivative of $\sum v_i E_i$ with respect to g_t. Here, we have used the notations of conformal invariants $\{A_{ij}, B_{ij}, C_i\}$ for x_t. By the same way, we get from (40) that

$$\begin{aligned} b_i &= e_i(v) + \sum_j B_{ij} v_j, \\ \frac{\partial F_i}{\partial t} &= \sum_j \left(b_{i,j} + \sum_k L_{ik} B_{kj} + A_{ij} v - v_i C_j\right) \omega_j + u\omega_i, \end{aligned} \tag{43}$$

where $\{b_{i,j}\}$ are covariant derivatives of $\sum_i b_i \omega_i$. Using (42) and (43), we get

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} + wB_{ij} &= b_{i,j} + \sum_k \left(L_{ik} B_{ik} - B_{ik} L_{kj}\right) - v_i C_j \\ &\quad + A_{ij} v + u\delta_{ij} - \sum_k B_{ik} \left(v_{k,j} + B_{kj} v\right), \\ b_{i,j} &= v_{,ij} + \sum_k \left(B_{ik,j} v_k + B_{ik} v_{k,j}\right), \\ v_{,ij} &= d_M e_i(v) + \sum_j e_j(v) \omega_{ji}. \end{aligned} \tag{44}$$

Therefore, we have

$$\begin{aligned} \frac{\partial B_{ij}}{\partial t} + wB_{ij} &= v_{,ij} + \sum_k \left(L_{ik} B_{ik} - B_{ik} L_{kj} + B_{ij,k} v_k\right) \\ &\quad + \left(A_{ij} - \sum_k B_{ik} B_{kj}\right) + \left(u - \sum_k v_k C_k\right) \delta_{ij}, \end{aligned} \tag{45}$$

$$\frac{n-1}{n} w = \sum_{ij} B_{ij} \left[v_{,ij} + \left(A_{ij} - \sum_k B_{ik} B_{kj}\right) v\right]. \tag{46}$$

Now, we calculate the first variation of the following conformal volume functional:

$$W(t) = \operatorname{vol}(g_t) = \int_{M^n} \omega_1 \wedge \cdots \wedge \omega_n = \int_{M^n} dM_t, \tag{47}$$

where dM is the volume for g_t. From (42) and (46) we get

$$\begin{aligned} &W'(t) \\ &= \sum_i \int_{M^n} \omega_1 \wedge \cdots \wedge \frac{\partial \omega_i}{\partial t} \wedge \cdots \wedge \omega_n = n \int_{M^n} w dM_t \\ &= \frac{n^2}{n-1} \int_{M^n} \left\{\sum_{ij} B_{ij} \left[v_{,ij} + \left(A_{ij} - \sum_k B_{ik} B_{kj}\right) v\right]\right\} dM_t. \end{aligned} \tag{48}$$

From the fact that the variation is admissible, we know that $v_i = 0$, $v = 0$ and $e_i(v) = 0$ on ∂M^n. It follows from (48) and Green's formula that

$$\begin{aligned} W'(t) &= \frac{n^2}{n-1} \\ &\quad \times \int_{M^n} \left\{\sum_{ij} B_{ij,ij} + \sum_{ij} A_{ij} B_{ij} - \sum_{ijk} B_{ik} B_{kj} B_{ij}\right\} \\ &\quad \times v dM_t. \end{aligned} \tag{49}$$

Thus, we have the following theorem.

Theorem 6. *A space-like hypersurface $x : M^n \to M_1^{n+1}(c)$ is a Willmore hypersurface (i.e., a critical hypersurface to the conformal volume functional) if and only if*

$$\sum_{ij} B_{ij,ij} + \sum_{ij} A_{ij}B_{ij} - \sum_{ijk} B_{ik}B_{kj}B_{ij} = 0. \tag{50}$$

Using (24), we can write the Euler-Lagrange equations as

$$\sum_i C_{i,i} + \sum_{ij}\left(\frac{1}{n-1}R_{ij} - A_{ij}\right)B_{ij} = 0. \tag{51}$$

Theorem 7. *Any maximal (or zero mean curvature) space-like surface in Lorentz space forms is a Willmore surface.*

Proof. Let $x : M^2 \to R_1^3$ be a space-like surface. Let $\{e_1, e_2\}$ be a local orthonormal basis of $\langle dx, dx\rangle$ and e_3 a local normal vector field. From (32), we get that

$$\sum_{ij} A_{ij}B_{ij} = \sum_{ij} e^{-2\tau}B_{ij}\left(\tau_i\tau_j - \tau_{i,j}\right) + \sum_{ij} e^{-\tau}B_{ij}B_{ij}H. \tag{52}$$

Since we have the following relations of connections:

$$\omega_{ij} = \theta_{ij} + \tau_i\omega_j - \tau_j\omega_i, \tag{53}$$

a direct calculation implies that

$$\sum_i C_{i,i} = \sum_{ij} e^{-2\tau}B_{ij}\left(\tau_i\tau_j - \tau_{i,j}\right) - e^{-3\tau}\Delta_M H. \tag{54}$$

From (51), we have

$$\sum_i C_{i,i} - \sum_{ij} A_{ij}B_{ij} = -e^{-3\tau}\left(\Delta_M H + e^{2\tau}\sum_{ij} B_{ij}B_{ij}H\right). \tag{55}$$

If x is a maximal space-like surface, that is, $H = 0$ and $R_{ij} = \kappa\delta_{ij}$, thus x is Willmore.

Similarly, we can verify that maximal space-like surfaces in $S_1^3(1)$ and $H_1^3(-1)$ are also Willmore. Thus, we complete the proof of Theorem 7. □

4. A Characteristic of CMC Hypersurfaces and κ_M = Constant

In this section, we consider space-like hypersurfaces with constant mean curvature and constant scalar curvature.

Proposition 8. *Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface without umbilical points. If the mean curvature and scalar curvature of x are constant, then conformal invariants of x satisfy*

$$(1),\ C = 0, \qquad (2),\ A = \mu B + \lambda g, \tag{56}$$

where μ, λ are constant.

Proof. First, we consider the space-like hypersurface $x : M^n \to R_1^{n+1}$. Since H and κ are constant, then by the Gauss equation we have that

$$e^{2\tau} = \frac{n}{n-1}\left(|II|^2 - nH^2\right) = \text{constant}. \tag{57}$$

From (32), we get that

$$C_i = 0, \qquad B_{ij} = e^{-\tau}\left(h_{ij} - H\delta_{ij}\right),$$
$$A_{ij} = e^{-2\tau}\left[\frac{1}{2}H^2 - Hh_{ij}\right]. \tag{58}$$

From (58), Let $\mu = -e^{-\tau}H$ and $\lambda = -(1/2)e^{-2\tau}H^2$; then, we prove the formula (56).

Similarly, we can prove the formula (56) for $M_1^{n+1}(c) = S_1^{n+1}(1)$ and $M_1^{n+1}(c) = H_1^{n+1}(-1)$. Thus, we complete the proof of Proposition 8. □

Theorem 9. *Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface without umbilical points. If conformal invariants of x satisfy*

$$(1),\ C = 0, \qquad (2),\ A = \mu B + \lambda g. \tag{59}$$

Then x is conformal equivalent to a space-like hypersurface with constant mean curvature and constant scalar curvature.

Proof. Since $C = 0$, from (20), we can take the local orthonormal basis $\{E_1, \ldots, E_n\}$ such that

$$\left(B_{ij}\right) = \operatorname{diag}\left(b_1, \ldots, b_n\right), \qquad \left(A_{ij}\right) = \operatorname{diag}\left(a_1, \ldots, a_n\right). \tag{60}$$

Since $A = \mu B + \lambda g$, from structure equations (14) we get that

$$dN - \lambda dY - \mu d\xi = 0. \tag{61}$$

Taking exterior differentiation of (61), we get that

$$d\lambda \wedge dY + d\mu \wedge d\xi = 0. \tag{62}$$

Writing $d\lambda = \sum_i \lambda_i\omega_i$, $d\mu = \sum_i \mu_i\omega_i$, from (62) we get that

$$\sum_{ijk}\left(\lambda_k\delta_{ij} + \mu_k B_{ij}\right)\omega_k \wedge \omega_j Y_i = 0, \tag{63}$$

which implies that

$$\lambda_k\delta_{ij} + \mu_k B_{ij} = \lambda_j\delta_{ik} + \mu_j B_{ik}. \tag{64}$$

Set $i = j$ in (64) and taking summation over i, we get that

$$\lambda_k - \frac{1}{n-1}\mu_k b_k = 0. \tag{65}$$

Set $i = j \neq k$ in (65), from (60), we get that

$$\lambda_k + \mu_k b_i = 0, \quad i \neq k. \tag{66}$$

Combining (65) and (66), we get that

$$\mu_k\left(b_i + \frac{1}{n-1}b_k\right) = 0, \quad i \neq k. \tag{67}$$

If $\mu_i = 0$, $1 \le i \le n$, then $\lambda_i = 0$, $1 \le i \le n$.

If $\mu_i \neq 0$, for some i; we can assume that $\mu_1 \neq 0$. Then, from (67), we have

$$b_i = -\frac{1}{n-1}b_1, \quad i \neq 1. \tag{68}$$

Since $\mathrm{tr}(B) = 0$ and $|B|^2 = (n-1)/n$, thus we have

$$b_1 = \frac{n-1}{n}, \qquad b_2 = \cdots = b_n = \frac{-1}{n}. \tag{69}$$

Using $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$ and (69), we get that

$$\omega_{1i} = 0, \quad 1 < i \le n. \tag{70}$$

Thus, we have

$$R_{1i1i} = 0, \quad 1 < i \le n, \quad a_2 = \cdots = a_n. \tag{71}$$

Since $a_i = \mu b_i + \lambda$, then from (71), we have

$$2\lambda + \frac{n-2}{n}\mu = \frac{n-1}{n^2}, \tag{72}$$

$$2\lambda_1 + \frac{n-2}{n}\mu_1 = 0. \tag{73}$$

From (66), we have

$$\lambda_1 + \frac{1}{n}\mu_1 = 0. \tag{74}$$

From (73) and (74), we get that

$$\mu_1 = 0. \tag{75}$$

This is a contradiction; so $\mu_i = 0$, $1 \le i \le n$. Since $\mu_i = 0$, $\lambda_i = 0$, $1 \le i \le n$, so λ and μ are constant.

From (61), we have

$$d\left(N - \lambda Y - \mu\xi\right) = 0. \tag{76}$$

Therefore, we can find a constant vector $e \in R_2^{n+3}$ such that

$$N - \lambda Y - \mu\xi = e. \tag{77}$$

From (11) and (77), we get that

$$\langle e, e\rangle = \mu^2 - 2\lambda, \qquad \langle Y, e\rangle = 1. \tag{78}$$

To prove the theorem, we consider the following three cases.

Case 1. e is light-like, that is, $\mu^2 - 2\lambda = 0$.

Case 2. e is space-like, that is, $\mu^2 - 2\lambda > 0$.

Case 3. e is time-like, that is, $\mu^2 - 2\lambda < 0$.

First, we consider Case 1; e is light-like, that is, $\mu^2 - 2\lambda = 0$. Then, there exists a $T \in O(n+2, 2)$ such that

$$\overline{e} = \left(-1, \vec{0}, 1\right) = eT = \left(N - \lambda Y - \mu\xi\right)T. \tag{79}$$

Let $\overline{x} : M^n \to R_1^{n+1}$ be a hypersurface that its conformal position vector is $\overline{Y} = YT$, then $\overline{N} = NT$, $\overline{\xi} = \xi T$, and

$$\overline{e} = \overline{N} - \lambda\overline{Y} - \mu\overline{\xi}. \tag{80}$$

Writing

$$\begin{aligned} \overline{Y} &= e^{\overline{\tau}}\left(\frac{\langle\overline{x},\overline{x}\rangle + 1}{2}, \overline{x}, \frac{\langle\overline{x},\overline{x}\rangle - 1}{2}\right), \\ \overline{\xi} &= -\overline{H}\left(\frac{\langle\overline{x},\overline{x}\rangle + 1}{2}, \overline{x}, \frac{\langle\overline{x},\overline{x}\rangle - 1}{2}\right) + \overline{y}_{n+1}, \end{aligned} \tag{81}$$

then from

$$\left\langle\overline{Y}, \overline{e}\right\rangle = 1, \qquad \left\langle\overline{\xi}, \overline{e}\right\rangle = -\mu, \tag{82}$$

we obtain that

$$e^{\overline{\tau}} = 1, \qquad \overline{H} = \mu. \tag{83}$$

Since $\overline{Y} = ((\langle\overline{x},\overline{x}\rangle + 1)/2, \overline{x}, (\langle\overline{x},\overline{x}\rangle - 1)/2)$, then $g = \langle d\overline{x}, d\overline{x}\rangle = \overline{I}$. From (24) we have

$$\mathrm{tr}(A) = n\lambda = \frac{1}{2n}\left(n^2\kappa - 1\right). \tag{84}$$

Since $\kappa_M = \kappa$, so the mean curvature and scalar curvature of hypersurface $\overline{x}$ are constant.

Next, we consider Case 2; e is space-like, that is, $\mu^2 - 2\lambda > 0$. Then, there exists a $T \in O(n+2, 2)$ such that

$$\overline{e} = \left(\vec{0}, \sqrt{\mu^2 - 2\lambda}\right) = eT = \left(N - \lambda Y - \mu\xi\right)T. \tag{85}$$

Let $\overline{x} : M^n \to H_1^{n+1}(-1)$ be a hypersurface which its conformal position vector is $\overline{Y} = YT$; then $\overline{N} = NT$, $\overline{\xi} = \xi T$, and

$$\overline{e} = \overline{N} - \lambda\overline{Y} - \mu\overline{\xi}. \tag{86}$$

Writing $\overline{Y} = e^{\overline{\tau}}(\overline{x}, 1)$, $\overline{\xi} = -\overline{H}(\overline{x}, 1) + \overline{y}_{n+1}$, then from

$$\left\langle\overline{Y}, \overline{e}\right\rangle = 1, \qquad \left\langle\overline{\xi}, \overline{e}\right\rangle = -\mu, \tag{87}$$

we obtain that

$$e^{\overline{\tau}} = \frac{1}{\sqrt{\mu^2 - 2\lambda}}, \qquad \overline{H} = \mu. \tag{88}$$

Since $\langle d\overline{x}, d\overline{x}\rangle = (\mu^2 - 2\lambda)g$,

$$\kappa_M = \frac{1}{\mu^2 - 2\lambda}\kappa. \tag{89}$$

Therefore, the mean curvature and scalar curvature of hypersurface $\overline{x}$ are constant.

Finally, we consider Case 3; e is time-like, that is, $\mu^2 - 2\lambda < 0$. Then, there exists a $T \in O(n+2, 2)$ such that

$$\overline{e} = \left(-\sqrt{2\lambda - \mu^2}, \vec{0}\right) = eT = \left(N - \lambda Y - \mu\xi\right)T. \tag{90}$$

Let $\overline{x} : M^n \to S_1^{n+1}(1)$ be a hypersurface that its conformal position vector is $\overline{Y} = YT$; then $\overline{N} = NT$, $\overline{\xi} = \xi T$, and

$$\overline{e} = \overline{N} - \lambda\overline{Y} - \mu\overline{\xi}. \tag{91}$$

Writing $\overline{Y} = e^{\overline{\tau}}(1, \overline{x})$, $\overline{\xi} = -\overline{H}(1, \overline{x}) + \overline{y}_{n+1}$, then from

$$\left\langle \overline{Y}, \overline{e} \right\rangle = 1, \qquad \left\langle \overline{\xi}, \overline{e} \right\rangle = -\mu, \tag{92}$$

we obtain that

$$e^{\overline{\tau}} = \frac{1}{\sqrt{2\lambda - \mu^2}}, \qquad \overline{H} = \mu. \tag{93}$$

Since $\langle d\overline{x}, d\overline{x} \rangle = (2\lambda - \mu^2) g$,

$$\kappa_M = \frac{1}{2\lambda - \mu^2} \kappa. \tag{94}$$

Therefore, the mean curvature and scalar curvature of hypersurface $\overline{x}$ are constant. Thus, we complete the proof the Theorem 9. □

Corollary 10. *Let $x : M^n \to M_1^{n+1}(c)$ be a space-like hypersurface without umbilical points. If conformal invariants of x satisfy*

$$(1),\ C = 0, \qquad (2),\ A = \lambda g, \tag{95}$$

then x is conformal equivalent to a maximal space-like hypersurface with constant scalar curvature in $H_1^{n+1}(-1)$.

Proof. From the process of proof of Theorem 9, we have that x is conformal equivalent to a maximal space-like hypersurface with constant scalar curvature. Next, we prove that x is a maximal space-like hypersurface with constant scalar curvature in $H_1^{n+1}(-1)$. In fact, using covariant derivative of ∇B and $C = 0$, we have

$$\begin{aligned} \frac{1}{2}\Delta |B|^2 &= \sum_{ijk} \left| B_{ij,k} \right|^2 + \sum_{ij} \Delta B_{ij} \\ &= \sum_{ijk} \left| B_{ij,k} \right|^2 + \left(\frac{n-1}{n} \right)^2 + \frac{(n-1)}{n} \operatorname{tr}(A) \\ &\quad + n \sum_{ijk} B_{ik} B_{kj} A_{ji}. \end{aligned} \tag{96}$$

Since $|B|^2 = (n-1)/n$ and $A = \lambda g$, we get that

$$\sum_{ijk} \left| B_{ij,k} \right|^2 + \left(\frac{n-1}{n} \right)^2 + 2(n-1)\lambda = 0. \tag{97}$$

Thus, $\lambda \le -(n-1)/2n^2 < 0$. From the process of proof of the Theorem 9, we have that x is conformal equivalent to a maximal space-like hypersurface with constant scalar curvature in $H_1^{n+1}(-1)$. □

5. Some Special Willmore Space-Like Hypersurfaces

In this section, we consider some special Willmore hypersurfaces. First, we give an example of the Willmore space-like hypersurface with constant mean curvature and constant scalar curvature in $H_1^{n+1}(-1)$.

Example 11. Let

$$\begin{aligned} H^k(r_1) &= \left\{ u = (u_0, \vec{u}_1) \in R_1^{k+1} \mid \langle u, u \rangle = -r_1^2 \right\}, \\ H^{n-k}(r_2) &= \left\{ v = (v_0, \vec{v}_1) \in R_1^{n-k+1} \mid \langle v, v \rangle = -r_2^2 \right\}, \end{aligned} \tag{98}$$

where $r_1^2 + r_2^2 = 1$. Define space-like hypersurface

$$\begin{aligned} x &: H^k(r_1) \times H^{n-k}(r_2) \longrightarrow H_1^{n+1}(-1), \\ x(u, v) &= (u_0, v_0, \vec{u}_1, \vec{v}_1) \in H_1^{n+1}(-1) \subset R_2^{n+2}. \end{aligned} \tag{99}$$

Then, unit normal vector of x is $e_{n+1} = ((r_2/r_1)u_0, -(r_1/r_2)v_0, (r_2/r_1)\vec{u}_1, -(r_1/r_2)\vec{v}_1)$; thus, the induced metric and the second fundamental form of x are, respectively,

$$\begin{aligned} I &= I_u + I_v, \qquad II = \frac{r_2}{r_1} I_u - \frac{r_1}{r_2} I_v, \\ nH &= k\frac{r_2}{r_1} - (n-k)\frac{r_1}{r_2}, \\ |II|^2 &= k\left(\frac{r_2}{r_1}\right)^2 + (n-k)\left(\frac{r_1}{r_2}\right)^2. \end{aligned} \tag{100}$$

Now, we assume that x is Willmore. Since H and $|II|^2$ are constant, from (33), we get that the Euler-Lagrange equation (51) is

$$k\left(\frac{r_2}{r_1}\right)^6 + (3k - 2n)\left(\frac{r_2}{r_1}\right)^4 + (3k - n)\left(\frac{r_2}{r_1}\right)^2 + k = 0. \tag{101}$$

Since $r_1^2 + r_2^2 = 1$, we get that

$$r_1 = -\sqrt{\frac{n-k}{n}}, \qquad r_2 = -\sqrt{\frac{k}{n}}. \tag{102}$$

Corollary 12. *Space-like hypersurface $x : H^k(r_1) \times H^{n-k}(r_2) \to H_1^{n+1}(-1)$ is Willmore if and only if*

$$r_1 = -\sqrt{\frac{n-k}{n}}, \qquad r_2 = -\sqrt{\frac{k}{n}}. \tag{103}$$

Remark 13. Space-like hypersurface $x : H^k(r_1) \times H^{n-k}(r_2) \to H_1^{n+1}(-1)$ is maximal if and only if

$$r_1 = -\sqrt{\frac{k}{n}}, \qquad r_2 = -\sqrt{\frac{n-k}{n}}. \tag{104}$$

Thus, maximal hypersurfaces are not Willmore in general.

Theorem 14. *Let $x : M^n \to M_1^{n+1}(c)$ be a Willmore space-like hypersurface without umbilical points. If conformal invariants of x satisfy*

$$(1),\ C = 0, \qquad (2),\ A = \mu B + \lambda g, \tag{105}$$

then x is conformal equivalent to the space-like hypersurface in $H_1^{n+1}(-1)$ with constant mean curvature and constant scalar curvature, and

$$2|h|^2H = \operatorname{tr}\left(h^3\right) + nH^3, \qquad |h|^2 - nH^2 \le n, \tag{106}$$

where $h = (h_{ij})$. If $|h|^2 - nH^2 = n$, then x is

$$H^k\left(-\sqrt{\frac{n-k}{n}}\right) \times H^{n-k}\left(-\sqrt{\frac{k}{n}}\right). \tag{107}$$

Proof. Let $x : M^n \to M_1^{n+1}(c)$ be a Willmore space-like hypersurface without umbilical points. Let $\{E_1, \ldots, E_n\}$ be the local orthonormal basis for g such that

$$\left(B_{ij}\right) = \operatorname{diag}\left(b_1, \ldots, b_n\right), \tag{108}$$

and corresponding principal curvatures are $(k_1, \ldots, k_2)$. Since

$$(1),\ C = 0, \qquad (2),\ A = \mu B + \lambda g, \tag{109}$$

and since x is Willmore, from (51), we get that

$$\frac{n-1}{n}\mu = \sum_i b_i^3. \tag{110}$$

From (33), we have

$$2|h|^2H = \operatorname{tr}\left(h^3\right) + nH^3. \tag{111}$$

By calculation of $(1/2)\Delta|h|^2$, we have

$$\begin{aligned}\frac{1}{2}\Delta|h|^2 &= \sum_{ijk}\left|h_{ij,k}\right|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{ij}R_{ijij}\left(k_i - k_j\right)^2 \\ &= \sum_{ijk}\left|h_{ij,k}\right|^2 - n^2|\nabla H|^2 + nc|h|^2 - n^2cH^2 + |h|^4 \\ &\quad - nH\sum_i k_i^3.\end{aligned} \tag{112}$$

Since (1), $C = 0$, (2), $A = \mu B + \lambda g$, so x is conformal equivalent to a space-like hypersurface with constant mean curvature and constant scalar curvature. We can assume that mean curvature and scalar curvature of x are constant. Thus, from the Gauss equation of x, we get that

$$|h|^2 = \text{constant}, \qquad H = \text{constant}. \tag{113}$$

From (112), we have

$$0 = \sum_{ijk}\left|h_{ij,k}\right|^2 + nc|h|^2 - n^2cH^2 + |h|^4 - nH\sum_i k_i^3. \tag{114}$$

Combining (111) and (114), we get that

$$\sum_{ijk}\left|h_{ij,k}\right|^2 + nc\left(|h|^2 - nH^2\right) + \left(|h|^2 - nH^2\right)^2 = 0. \tag{115}$$

Since $|h|^2 - nH^2 \ge 0$ and from (115), we get that

$$c = -1. \tag{116}$$

Thus, x is a space-like hypersurface in $H_1^{n+1}(-1)$. From (115), we have

$$\sum_{ijk}\left|h_{ij,k}\right|^2 + \left(|h|^2 - nH^2\right)\left(|h|^2 - nH^2 - n\right) = 0. \tag{117}$$

Since $|h|^2 - nH^2 \ge 0$, from (117), we have

$$|h|^2 - nH^2 \le n. \tag{118}$$

If $|h|^2 - nH^2 = n$, then $h_{ij,k} = 0$; thus, the principal curvatures of x are constant. It is well known that space-like isoparametric hypersurfaces in $H_1^{n+1}(-1)$ are either totally umbilical hypersurfaces or $H^k(r_1) \times H^{n-k}(r_2)$ (see [11]).

Thus, we complete the proof of Theorem 14. □

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