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Research Article

Simplicity and Commutative Bases of Derivations in Polynomial and Power Series Rings

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The first part of the paper will describe a recent result of Retert in (2006) for $k[x_1, \ldots, x_n]$ and $k[[x_1, \ldots, x_n]]$. This result states that if $\mathfrak{D}$ is a set of commute k-derivations of $k[x, y]$ such that both $\partial_x \in \mathfrak{D}$ and the ring is $\mathfrak{D}$-simple, then there is $d \in \mathfrak{D}$ such that $k[x, y]$ is $\{\partial_x, d\}$-simple. As for applications, we obtain relationships with known results of A. Nowicki on commutative bases of derivations.

1. Introduction

Let k be a field of characteristic zero and R denote either the ring $k[x_1, \ldots, x_n]$ of polynomials over k or the ring $k[[x_1, \ldots, x_n]]$ of formal power series over k.

A *k-derivation* $d : R \to R$ of R is a k-linear map such that $d(ab) = d(a)b + ad(b)$ for any $a, b \in R$. Denoting by $\mathrm{Der}_k(R)$ the set of all k-derivations of R, let $\mathfrak{D} \subset \mathrm{Der}_k(R)$ be a nonempty family of k-derivations. An ideal I of R is called *$\mathfrak{D}$-stable* if $d(I) \subset I$ for all $d \in \mathfrak{D}$. For example, the ideals 0 and R are always $\mathfrak{D}$-stable. If R has no other $\mathfrak{D}$-stable ideal it is called *$\mathfrak{D}$-simple*. When $\mathfrak{D} = \{d\}$, d is often called a simple derivation.

The commuting derivations have been studied by several authors: Li and Du [1], Maubach [2], Nowicki [3], Petravchuk [4], Retert [5], Van den Essen [6]. For example, it is well known that each pair of commuting linear operators on a finite dimensional vector space over an algebraically closed field has a common eigenvector; in [4], Petravchuk proved an analogous statement for derivation of $k[x, y]$ over any field k of characteristic zero. More explicitly, if two derivations of $k[x, y]$ are linearly independent over k and commute, then they have a common Darboux polynomial or they are Jacobian derivations; in [1], the authors proved the same result for $k[x_1, \ldots, x_n]$ and $k[[x_1, \ldots, x_n]]$. However, we observe that this result has already been proved by Nowicki in (Now86) for both rings. Another interesting result was proved by Nowicki in [3, Theorem 5] which says that the famous Jacobian conjecture in $k[x_1, \ldots, x_n]$ is equivalent to the assertion that every commutative basis of $\mathrm{Der}_k(R)$ is locally nilpotent.

Let $\mathfrak{D}$ be a set of commute k-derivations of $k[x]$; then $k[x]$ is $\mathfrak{D}$-simple if and only if it is d-simple for some k-derivation $d \in \mathfrak{D}$ (see [5, Corollary 2.10]). In $k[x, y]$, as pointed out in [5], up to scalar multiples, these are only sets $\mathfrak{D}$ of two commuting, nonsimple k-derivations such that both $\partial_x \in \mathfrak{D}$ and R is $\mathfrak{D}$-simple. Motivated by this, we analyze this result in [5] for R and then we propose some connections with known results on commutative bases of derivations in R. More precisely, the derivations $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ are not simple k-derivations of R; however, as will be shown, they can be part of a set $\mathfrak{D}$ of n commuting, nonsimple k-derivations such that R is $\mathfrak{D}$-simple. A trivial example is $\mathfrak{D} = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$. Using the notations in [3], we give a nontrivial commutative base containing only nonsimple k-derivations of the free R-module $\mathrm{Der}_k(R)$ such that R is $\mathfrak{D}$-simple and if the Jacobian conjecture is true in $k[x_1, \ldots, x_n]$, as a consequence of [3, Theorem 5], we obtain a family of locally nilpotent derivations.

2. Commuting Derivations and Simplicity

Lemma 1. *The set of all k-derivations of R that commute with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ is*

$$\left\{p(x_n)\partial_{x_1} + q_1(x_n)\partial_{x_2} + \cdots + q_{n-1}(x_n)\partial_{x_n}\right\} \tag{1}$$

such that $p, q_1, \ldots, q_{n-1} \in k[x_n]$ (or $k[[x_n]]$ if $R = k[[x_1, \ldots, x_n]]$).

Proof. It is clear that all derivations of this form commute with the derivations $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$. For the converse, let

$$\begin{aligned} d^* = {} & p^*(x_1, \ldots, x_n)\partial_{x_1} + q_1^*(x_1, \ldots, x_n)\partial_{x_2} \\ & + \cdots + q_{n-1}^*(x_1, \ldots, x_n)\partial_{x_n} \end{aligned} \tag{2}$$

be a k-derivation of R that commutes with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$. Then

$$0 = d^*\left(\partial_{x_i}(x_1)\right) = \partial_{x_i}(d^*(x_1)) = \partial_{x_i}(p^*(x_1, \ldots, x_n)) \tag{3}$$

for all $i = 1, \ldots, n-1$. Thus, $p^*(x_1, \ldots, x_n) \in k[x_n](k[[x_n]])$. Similarly, we can prove that $q_i^*(x_1, \ldots, x_n) \in k[x_n](k[[x_n]])$. □

Let $\mathfrak{D} = \{\delta_1, \ldots, \delta_s\}$ be any finite set of k-derivation of R that commutes with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ but not necessarily with each other. By Lemma 1, each δ_i is of the form

$$\delta_i = p_i(x_n)\partial_{x_1} + q_{(1,i)}(x_n)\partial_{x_2} + \cdots + q_{(n-1,i)}(x_n)\partial_{x_n}. \tag{4}$$

We denote by $v_{n-1}(x_n)$ the greatest common divisor of $q_{(n-1,1)}, \ldots, q_{(n-1,s)}$. We have the following characterization for the simplicity of R.

Lemma 2. *Using the notations above, R is $\mathfrak{D}$-simple if and only if $v_{n-1}(x_n)$ is a unit in $k[x_n]$ (or $k[[x_n]]$).*

Proof. If all $q_{(n-1,i)} = 0$, so all the k-derivations in $\mathfrak{D}$ stabilize the nonzero ideal $\langle x_n \rangle$; in this case, R is not $\mathfrak{D}$-simple. Then we assume that some $q_{(n-1,i)} \neq 0$. If $v_{n-1}(x_n)$ is not a unit, each δ_i stabilizes the nontrivial ideal $\langle v_{n-1}(x_n) \rangle$. Therefore R is not $\mathfrak{D}$-simple.

Conversely, assume that $v_{n-1}(x_n)$ is a unit and notice that, in this case, there are polynomials $r_i(x_n)$ such that

$$\sum_{i=1}^{s} r_i(x_n) q_{(n-1,i)}(x_n) = v_{n-1}(x_n), \tag{5}$$

multiplicand by the inverse of $v_{n-1}(x_n)$, we may assume that $\sum_{i=1}^{s} r_i(x_n) q_{(n-1,i)}(x_n) = 1$. Without loss of generality, let $\delta_1 = \partial_{x_1}, \ldots, \delta_{n-1} = \partial_{x_{n-1}}$; thus,

$$\sum_{i=n}^{s} r_i(x_n) q_{(n-1,i)}(x_n) = 1. \tag{6}$$

Let I be a $\mathfrak{D}$-ideal. Then since I is stabilized by each δ_i, I is stabilized by $r_i(x_n)\delta_i$ and, then, by the k-derivation

$$\begin{aligned} \sum_{i=n}^{s} r_i(x_n)\delta_i = {} & \left(\sum_{i=n}^{s} r_i(x_n) p_i(x_n)\right)\partial_{x_1} \\ & + \cdots + \left(\sum_{i=n}^{s} r_i(x_n) q_{(n-1,i)}(x_n)\right)\partial_{x_n}. \end{aligned} \tag{7}$$

Therefore, I is stabilized by $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ and a k-derivation of the form

$$u_1(x_n)\partial_{x_1} + \cdots + u_{n-1}(x_n)\partial_{x_{n-1}} + \partial_{x_n} \tag{8}$$

for $u_i(x_n) \in k[x_n](k[[x_n]])$. Thus I is stabilized by $\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \partial_{x_n}$ and, then, we deduce that I must be a trivial ideal. □

Note that until now we only assume that all the k-derivations commute with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ not that all elements commute with each other. Using the previous lemmas, the following theorem will show that if R is $\mathfrak{D}$-simple under a set $\mathfrak{D}$ of commuting k-derivations that contains $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$, then R is simple under a subset of n commuting nonsimple k-derivations.

Theorem 3. *Let $\mathfrak{D}$ be a set of k-derivations of R such that $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in \mathfrak{D}$. Then the derivations of $\mathfrak{D}$ commute with each other if and only if one of the following two cases holds.*

(a) *Each element δ_i of $\mathfrak{D}$ has the form $\delta_i = h_1^i(x_n)\partial_{x_1} + \cdots + h_{n-1}^i(x_n)\partial_{x_{n-1}}$, for some $h_j^i(x_n) \in k[x_n]$ (or $k[[x_n]]$).*

(b) *There exists $v_1(x_n), \ldots, v_n(x_n) \in k[x_n]$ (or $k[[x_n]]$), such that for each $\delta_i \in \mathfrak{D}$ there are scalars $\lambda_i, c_1^i, \ldots, c_{n-1}^i \in k$ such that*

$$\delta_i = \sum_{l=1}^{n-1} \left(\lambda_i v_l(x_n) + c_l^i\right)\partial_{x_l} + \lambda_i v_n(x_n)\partial_{x_n}. \tag{9}$$

If, in addition, R is $\mathfrak{D}$-simple, then $v_n(x_n)$ must be some nonzero scalar β and, also, some λ_j is not zero. In this case, R is also simple under the subset

$$\left\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \sum_{l=1}^{n-1} \left(\lambda_j v_l(x_n) + c_l^j\right)\partial_{x_l} + \lambda_j \beta \partial_{x_n}\right\}. \tag{10}$$

Proof. If either of the conditions is met, it is clear that all k-derivations of $\mathfrak{D}$ will commute.

Conversely, let $\mathfrak{D} = \{\delta_1, \ldots, \delta_m\}$ be according to the hypotheses of the theorem. By Lemma 1, each δ_i is of the form $\delta_i = p_i(x_n)\partial_{x_1} + q_{(1,i)}(x_n)\partial_{x_2} + \cdots + q_{(n-1,i)}(x_n)\partial_{x_n}$.

If all $q_{(n-1,i)}(x_n)$ are zero, then the first case holds. So, without loss of generality, we assume that $q_{(n-1,1)}(x_n) \neq 0$ and observe that

$$\begin{aligned} \delta_i(\delta_1(x_n)) &= \delta_i(q_{(n-1,1)}(x_n)) \\ &= q_{(n-1,i)}(x_n)\partial_{x_n}(q_{(n-1,1)}(x_n)), \\ \delta_1(\delta_i(x_n)) &= \delta_1(q_{(n-1,i)}(x_n)) \\ &= q_{(n-1,1)}(x_n)\partial_{x_n}(q_{(n-1,i)}(x_n)). \end{aligned} \tag{11}$$

Since δ_i and δ_1 commute,

$$\begin{aligned} q_{(n-1,i)}(x_n)\,\partial_{x_n}(q_{(n-1,1)}(x_n)) \\ = q_{(n-1,1)}(x_n)\,\partial_{x_n}(q_{(n-1,i)}(x_n)). \end{aligned} \tag{12}$$

Then, this equation must also hold in the ring of fractions; hence we deduce that

$$\frac{q_{(n-1,i)}(x_n)\,\partial_{x_n}(q_{(n-1,1)}(x_n)) - q_{(n-1,1)}(x_n)\,\partial_{x_n}(q_{(n-1,i)}(x_n))}{(q_{(n-1,1)}(x_n))^2} = 0. \tag{13}$$

In other words,

$$\left(\frac{q_{(n-1,i)}(x_n)}{q_{(n-1,1)}(x_n)}\right)' = 0. \tag{14}$$

Then there is some $\lambda_i \in k$ such that $q_{(n-1,i)}(x_n) = \lambda_i q_{(n-1,1)}(x_n)$.

Now, we observe that

$$\begin{aligned} \delta_i(\delta_1(x_1)) &= \delta_i(p_1(x_n)) \\ &= q_{(n-1,i)}(x_n)\,\partial_{x_n}(p_1(x_n)), \\ \delta_1(\delta_i(x_1)) &= \delta_1(p_i(x_n)) \\ &= q_{(n-1,1)}(x_n)\,\partial_{x_n}(p_i(x_n)). \end{aligned} \tag{15}$$

Since δ_i and δ_1 commute and both $k[x_n]$ and $k[[x_n]]$ are domains,

$$\lambda_i \partial_{x_n}(p_1(x_n)) = \partial_{x_n}(p_i(x_n)). \tag{16}$$

Then there is some $c_1^i \in k$ such that $p_i(x_n) = \lambda_i p_1(x_n) + c_1^i$. Finally, making the same argument for the other variables, we prove the desired result.

Now we suppose, in addition, that R is $\mathfrak{D}$-simple. By Lemma 2, the greatest common divisor of $q_{(n-1,1)}(x_n), \ldots, q_{(n-1,m)}(x_n)$ must be a unit. However, we have demonstrated that all the $q_{(n-1,i)}(x_n)$ are scalar multiples; then at least one of the $q_{(n-1,i)}(x_n)$ must be a unit; we assume that $q_{(n-1,j)}(x_n)$ is a unit. Since I is stabilized by

$$\begin{aligned} \{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, p_j(x_n)\,\partial_{x_1} + q_{(1,j)}(x_n)\,\partial_{x_2} \\ + \cdots + q_{(n-1,j)}(x_n)\,\partial_{x_n}\}, \end{aligned} \tag{17}$$

I must be trivial because, in this case, I is stabilized by $\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \partial_{x_n}$. Therefore, R is $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, p_j(x_n)\partial_{x_1} + q_{(1,j)}(x_n)\partial_{x_2} + \cdots + q_{(n-1,j)}(x_n)\partial_{x_n}\}$-simple which completes the proof. □

Remark 4. Nowicki in [7, Theorem 2.5.5] proved that every k-derivation of a commutative base of $\mathrm{Der}_k(k[x_1, \ldots, x_n])$ is a special k-derivation. This means that the divergence d^* of d is 0. Moreover, it is easy to prove that the set

$$\left\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, \sum_{l=1}^{n-1}\left(\lambda_j \nu_l(x_n) + c_l^j\right)\partial_{x_l} + \lambda_j \beta \partial_{x_n}\right\} \tag{18}$$

obtained by the previous theorem is a commutative base of $\mathrm{Der}_k(k[x_1, \ldots, x_n])$. Thus, in particular, $\sum_{l=1}^{n-1}(\lambda_j \nu_l(x_n) + c_l^j)\partial_{x_l} + \lambda_j \beta \partial_{x_n}$ is a special derivation. However, this is easily verified since

$$d^* = \sum_{l=1}^{n-1}\partial_{x_l}\left(\lambda_j \nu_l(x_n) + c_l^j\right) + \partial_{x_n}\left(\lambda_j \beta\right) = 0. \tag{19}$$

Corollary 5. *Let $\mathfrak{D}$ be a set of commute k-derivations of R such that R is $\mathfrak{D}$-simple and $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in \mathfrak{D}$. Then, there exists $d \in \mathfrak{D}$ and there exist elements $f_1, \ldots, f_n \in R$ such that $d(f_n) = 1$ and $d(f_i) = 0$, for any $i = 1, \ldots, n-1$, so that R is $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$-simple.*

Proof. By the previous theorem, we know that there is $d \in \mathfrak{D}$ of the form

$$d = \sum_{l=1}^{n-1}\left(\lambda_j \nu_l(x_n) + c_l^j\right)\partial_{x_l} + \lambda_j \beta \partial_{x_n} \tag{20}$$

such that R is $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$-simple. Since β and λ_j, in the theorem, are nonzero scalars, we denote $f_n = (\lambda_j \beta)^{-1} x_n$; thus $d(f_n) = 1$.

Let $f_l^* \in k[x_n]$ (or $k[[x_n]]$) such that $d(f_l^*) = \lambda_j \nu_l(x_n) + c_l^j$. Since $c(k) = 0$, f_l^* exist. Then, let $f_l = x_l - f_l^*$; hence $d(f_l) = 0$, for any $l = 1, \ldots, n-1$. This completes the proof. □

Remark 6. The previous corollary is a particular case of an important theorem about the characterization of commutative basis of $\mathrm{Der}_k(R)$ However, in our case the proof is more evident (see [3, Theorem 2] (Now86)).

For the remainder of this note we assume that R is the ring $k[x_1, \ldots, x_n]$ of polynomials over k and $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$ is as in the previous theorem and also we recall the following definitions.

We recall from [7] that a k-derivation d of $k[x_1, \ldots, x_n]$ is called locally nilpotent if for each $f \in R$ exists a natural number n such that $d^n(f) = 0$ and we say that a basis $\{d_1, \ldots, d_n\}$ of $\mathrm{Der}_k(k[x_1, \ldots, x_n])$ is locally nilpotent if every derivation d_i is locally nilpotent for $i = 1, \ldots, n$.

We remember also that the Jacobian conjecture states that if $F = (F_1, \ldots, F_n)$ is a polynomial map such that the Jacobian matrix is invertible, then F has a polynomial inverse (see [7]).

Theorem 7 (see [3, Theorem 5]). *Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. The following conditions are equivalent.*

(1) *The Jacobian conjecture is true in the n-variable case.*

(2) *Every commutative basis of the R-module $\mathrm{Der}_k(R)$ is locally nilpotent.*

(3) *Every commutative basis of the R-module $\mathrm{Der}_k(R)$ is locally finite.*

Corollary 8. *Let $\mathfrak{D}$ be a set of commute k-derivations of R such that R is $\mathfrak{D}$-simple, $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in \mathfrak{D}$, and the Jacobian*

conjecture is true in R. Then, there exists $d \in \mathfrak{D}$ *such that* $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$ *is a locally nilpotent commutative base of the R-module* $Der_k(R)$. *In particular,* d *is a* k*-derivation locally nilpotent.*

Proof. The proof is immediate consequence of [3, Theorem 5]. □

Question. A ring is called *w-differentially simple* if it is a simple relative to a family with w derivations. Recall that we are assuming $R = k[x_1, \ldots, x_n]$; then we know that R is 1-differentially simple and $\dim(R)$-differentially simple as well. However, $n = \dim(R)$ is not necessarily the smallest w for which such a ring can be w-differentially simple (see [8]). Thus, one may ask the following: what is the smallest positive integer $w \neq 1$ such that R is w-differentially simple and all w derivations are nonsimple and commute?

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