Research Article
Simplicity and Commutative Bases of Derivations in Polynomial and Power Series Rings

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The first part of the paper will describe a recent result of Retert in (2006) for \( k[x_1, \ldots, x_n] \) and \( k[[x_1, \ldots, x_n]] \). This result states that if \( \mathfrak{D} \) is a set of commute \( k \)-derivations of \( k[x, y] \) such that both \( \partial_y \in \mathfrak{D} \) and the ring is \( \mathfrak{D} \)-simple, then there is \( d \in \mathfrak{D} \) such that \( k[x, y] \) is \( [\partial_x, d] \)-simple. As for applications, we obtain relationships with known results of A. Nowicki on commutative bases of derivations.

1. Introduction

Let \( k \) be a field of characteristic zero and \( R \) denote either the ring \( k[x_1, \ldots, x_n] \) of polynomials over \( k \) or the ring \( k[[x_1, \ldots, x_n]] \) of formal power series over \( k \).

A \( k \)-derivation \( d : R \to R \) of \( R \) is a \( k \)-linear map such that \( d(ab) = d(a)b + ad(b) \) for any \( a, b \in R \). Denoting by \( \text{Der}_k(R) \) the set of all \( k \)-derivations of \( R \), let \( \mathfrak{D} \subset \text{Der}_k(R) \) be a nonempty family of \( k \)-derivations. An ideal \( I \) of \( R \) is called \( \mathfrak{D} \)-stable if \( d(I) \subset I \) for all \( d \in \mathfrak{D} \). For example, the ideals \( 0 \) and \( R \) are always \( \mathfrak{D} \)-stable. If \( R \) has no other \( \mathfrak{D} \)-stable ideal it is called \( \mathfrak{D} \)-simple. When \( \mathfrak{D} = \{d\} \), \( d \) is often called a simple derivation.

The commuting derivations have been studied by several authors: Li and Du [1], Maubach [2], Nowicki [3], Petravchuk [4], Retert [5], Van den Essen [6]. For example, it is well known that each pair of commuting linear operators on a finite dimensional vector space over an algebraically closed field has a common eigenvector; in [4], Petravchuk proved an analogous statement for derivation of \( k[x, y] \) over any field \( k \) of characteristic zero. More explicitly, if two derivations of \( k[x, y] \) are linearly independent over \( k \) and commute, then they have a common Darboux polynomial or they are Jacobian derivations; in [1], the authors proved the same result for \( k[x_1, \ldots, x_n] \) and \( k[[x_1, \ldots, x_n]] \). However, we observe that this result has already been proved by Nowicki in (Now86) for both rings. Another interesting result was proved by Nowicki in [3, Theorem 5] which says that the famous Jacobian conjecture in \( k[x_1, \ldots, x_n] \) is equivalent to the assertion that every commutative basis of \( \text{Der}_k(R) \) is locally nilpotent.

Let \( \mathfrak{D} \) be a set of commute \( k \)-derivations of \( k[x] \); then \( k[x] \) is \( \mathfrak{D} \)-simple if and only if it is \( d \)-simple for some \( k \)-derivation \( d \in \mathfrak{D} \) (see [5, Corollary 2.10]). In \( k[x, y] \), as pointed out in [5], up to scalar multiples, these are only sets \( \mathfrak{D} \) of two commuting, nonsimple \( k \)-derivations such that both \( \partial_y \in \mathfrak{D} \) and \( R \) is \( \mathfrak{D} \)-simple. Motivated by this, we analyze this result in [5] for \( R \) and then we propose some connections with known results on commutative bases of derivations in \( R \). More precisely, the derivations \( \partial_{x_1}, \ldots, \partial_{x_n} \) are not simple \( k \)-derivations of \( R \); however, as will be shown, they can be part of a set \( \mathfrak{D} \) of \( n \) commuting, nonsimple \( k \)-derivations such that \( R \) is \( \mathfrak{D} \)-simple. A trivial example is \( \mathfrak{D} = \{\partial_{x_1}, \ldots, \partial_{x_n}\} \). Using the notations in [3], we give a nontrivial commutative base containing only nonsimple \( k \)-derivations of the free \( R \)-module \( \text{Der}_k(R) \) such that \( R \) is \( \mathfrak{D} \)-simple and if the Jacobian conjecture is true in \( k[x_1, \ldots, x_n] \), as a consequence of [3, Theorem 5], we obtain a family of locally nilpotent derivations.
2. Commuting Derivations and Simplicity

Lemma 1. The set of all $k$-derivations of $R$ that commute with \( \partial_{x_1}, \ldots, \partial_{x_{n-1}} \) is
\[
\left\{ p(x_n) \partial_{x_1} + q_1(x_n) \partial_{x_2} + \cdots + q_{n-1}(x_n) \partial_{x_n} \right\}
\]
such that $p$, $q_1, \ldots, q_{n-1} \in k[x_n]$ (or $k[[x_n]]$ if $R = k[[x_1, \ldots, x_n]]$).

Proof. It is clear that all derivations of this form commute with the derivations $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$. For the converse, let
\[
d^* = p^* (x_1, \ldots, x_n) \partial_{x_1} + q_1^* (x_1, \ldots, x_n) \partial_{x_2} + \cdots + q_{n-1}^* (x_1, \ldots, x_n) \partial_{x_n}
\]
be a $k$-derivation of $R$ that commutes with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$. Then
\[
0 = d^* (\partial_{x_i}(x_1)) = \partial_{x_i} (d^*(x_i)) = \partial_{x_i} (p^*(x_1, \ldots, x_n))
\]
for all $i = 1, \ldots, n-1$. Thus, $p^*(x_1, \ldots, x_n) \in k[x_n]$ (or $k[[x_n]]$). Similarly, we can prove that $q_i^*(x_1, \ldots, x_n) \in k[x_n]$ (or $k[[x_n]]$).

Let $D = \{ \delta_1, \ldots, \delta_k \}$ be any finite set of $k$-derivation of $R$ that commutes with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ but not necessarily with each other. By Lemma 1, each $\delta_i$ is of the form
\[
\delta_i = p_i(x_n) \partial_{x_1} + q_{i(1,j)}(x_n) \partial_{x_2} + \cdots + q_{i(n-1,1)}(x_n) \partial_{x_n}.
\]

We denote by $v_{n-1}(x_n)$ the greatest common divisor of $q_{i(n-1,1)}, \ldots, q_{i(n-1,j)}$. We have the following characterization for the simplicity of $R$.

Lemma 2. Using the notations above, $R$ is $D$-simple if and only if $v_{n-1}(x_n)$ is a unit in $k[x_n]$ (or $k[[x_n]]$).

Proof. If all $q_{i(n-1,1)} = 0$, so all the $k$-derivations in $D$ stabilize the nonzero ideal $(x_n)$; in this case, $R$ is not $D$-simple. Then we assume that some $q_{i(n-1,1)} \neq 0$. If $v_{n-1}(x_n)$ is not a unit, each $\delta_i$ stabilizes the nontrivial ideal $(v_{n-1}(x_n))$. Therefore, $R$ is not $D$-simple.

Conversely, assume that $v_{n-1}(x_n)$ is a unit and notice that, in this case, there are polynomials $r_i(x_n)$ such that
\[
\sum_{i=1}^s r_i(x_n) q_{i(n-1,1)}(x_n) = v_{n-1}(x_n),
\]
multiplicand by the inverse of $v_{n-1}(x_n)$, we may assume that
\[
\sum_{i=1}^s r_i(x_n) q_{i(n-1,1)}(x_n) = 1.
\]
Therefore, $R$ is $D$-simple.

Let $I$ be a $D$-ideal. Then since $I$ is stabilized by each $\delta_i$, $I$ is stabilized by $r_i(x_n) \delta_i$ and, then, by the $k$-derivation
\[
\sum_{i=1}^s r_i(x_n) \delta_i = \left( \sum_{i=1}^s r_i(x_n) p_i(x_n) \right) \partial_{x_i} + \cdots + \left( \sum_{i=1}^s r_i(x_n) q_{i(n-1,1)}(x_n) \right) \partial_{x_n}.
\]

Therefore, $I$ is stabilized by $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ and a $k$-derivation of the form
\[
u_1(x_n) \partial_{x_1} + \cdots + u_{n-1}(x_n) \partial_{x_{n-1}} + \partial_{x_n}
\]
for $u_i(x_n) \in k[x_n]$ (or $k[[x_n]]$). Thus $I$ is stabilized by $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$, $\partial_{x_n}$ and, then, we deduce that $I$ must be a trivial ideal.

Note that until now we only assume that all the $k$-derivations commute with $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$ not that all elements commute with each other. Using the previous lemmas, the following theorem will show that if $R$ is $D$-simple under a set $D$ of commuting $k$-derivations that contains $\partial_{x_1}, \ldots, \partial_{x_{n-1}}$, then $R$ is simple under a subset of $n$ commuting nonsimple $k$-derivations.

Theorem 3. Let $D$ be a set of $k$-derivations of $R$ such that $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in D$. Then the derivations of $D$ commute with each other if and only if one of the following two cases holds.

(a) Each element $\delta_i$ of $D$ has the form $\delta_i = h_i^0(x_n) \partial_{x_1} + \cdots + h_{n-1}^0(x_n) \partial_{x_{n-1}}$, for some $h_i^0(x_n) \in k[x_n]$ (or $k[[x_n]]$).

(b) There exists $v_1(x_n), \ldots, v_{n-1}(x_n) \in k[x_n]$ (or $k[[x_n]]$), such that for each $\delta_i \in D$ there are scalars $\lambda_i, \lambda_{ij}, \cdots, \lambda_{n-1}$ in $k$ such that
\[
\delta_i = \sum_{i=1}^{n-1} \left( \lambda_i v_i(x_n) + \lambda_{ij} v_j(x_n) \right) \partial_{x_i} + \lambda_{n-1} v_{n-1}(x_n) \partial_{x_n}.
\]
Since $\delta_1$ and $\delta_i$ commute,
\[
q_{(n-1, i)}(x_n) \partial_{x_n} (q_{(n-1, i)}(x_n)) = q_{(n-1, i)}(x_n) \partial_{x_n} (q_{(n-1, i)}(x_n)).
\]
Then, this equation must also hold in the ring of fractions; hence we deduce that
\[
\frac{q_{(n-1, j)}(x_n) \partial_{x_n} (q_{(n-1, i)}(x_n)) - q_{(n-1, i)}(x_n) \partial_{x_n} (q_{(n-1, j)}(x_n))}{(q_{(n-1, j)}(x_n))^2} = 0.
\]
In other words,
\[
\left(\frac{q_{(n-1, j)}(x_n)}{q_{(n-1, i)}(x_n)}\right)' = 0.
\]
Then there is some $\lambda_i \in K$ such that $q_{(n-1, i)}(x_n) = \lambda_i q_{(n-1, i)}(x_n)$.

Now, we observe that
\[
\delta_i (\delta_1(x_1)) = \delta_i (p_1(x_1)) = q_{(n-1, j)}(x_n) \partial_{x_n} (p_1(x_n)),
\]
\[
\delta_1 (\delta_i(x_1)) = \delta_1 (p_i(x_n)) = q_{(n-1, i)}(x_n) \partial_{x_n} (p_i(x_n)).
\]
Since $\delta_i$ and $\delta_1$ commute and both $k[x_n]$ and $k[[x_n]]$ are domains,
\[
\lambda_i \partial_{x_n} (p_1(x_n)) = \partial_{x_n} (p_i(x_n)).
\]
Then there is some $c^i_j \in K$ such that $p_j(x_n) = \lambda_i p_1(x_n) + c^i_j$. Finally, making the same argument for the other variables, we prove the desired result.

Now suppose, in addition, that $R$ is $\mathfrak{D}$-simple. By Lemma 2, the greatest common divisor of $q_{(n-1, 1)}(x_n), \ldots, q_{(n-1, n)}(x_n)$ must be a unit. However, we have demonstrated that all the $q_{(n-1, i)}(x_n)$ are scalar multiples; then at least one of the $q_{(n-1, i)}(x_n)$ must be a unit; we conclude that $q_{(n-1, i)}(x_n)$ is a unit. Since $I$ is stabilized by
\[
\{\partial_{x_1, \ldots, x_{n-1}}, p_j(x_n) \partial_{x_1} + q_{(1, j)}(x_n) \partial_{x_2} + \cdots + q_{(n-1, j)}(x_n) \partial_{x_n}\},
\]
$I$ must be trivial because, in this case, $I$ is stabilized by $\partial_{x_1, \ldots, x_{n-1}, \partial_{x_n}}$. Therefore, $R$ is $\{\partial_{x_1, \ldots, x_{n-1}, \partial_{x_n}} + q_{(1, j)}(x_n) \partial_{x_2} + \cdots + q_{(n-1, j)}(x_n) \partial_{x_n}\}$-simple which completes the proof. \qed

Remark 4. Nowicki in [7, Theorem 2.5.5] proved that every $k$-derivation of a commutative base of $\text{Der}_k(k[x_1, \ldots, x_n])$ is a special $k$-derivation. This means that the divergence $d^*$ of $d$ is 0. Moreover, it is easy to prove that the set
\[
\{\partial_{x_1, \ldots, x_{n-1}} \sum_{j=1}^{n-1} (\lambda_j \eta_j(x_n) + \zeta^i_j) \partial_{x_2} + \lambda \beta \partial_{x_n}\}
\]
obtained by the previous theorem is a commutative base of $\text{Der}_k(k[x_1, \ldots, x_n])$. Thus, in particular, $\sum_{j=1}^{n-1} (\lambda_j \eta_j(x_n) + \zeta^i_j) \partial_{x_2} + \lambda \beta \partial_{x_n}$ is a special derivation. However, this is easily verified since
\[
d^* = \sum_{i=1}^{n-1} (\lambda_j \eta_i(x_n) + \zeta^i_j) \partial_{x_2} + \lambda \beta \partial_{x_n} = 0.
\]

Corollary 5. Let $\mathfrak{D}$ be a set of commute $k$-derivations of $R$ such that $R$ is $\mathfrak{D}$-simple and $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in \mathfrak{D}$. Then, there exists $d \in \mathfrak{D}$ and there exist elements $f_1, \ldots, f_n \in R$ such that $d(f_n) = 1$ and $d(f_i) = 0$, for any $i = 1, \ldots, n-1$, so that $R$ is $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$-simple.

Proof. By the previous theorem, we know that there is $d \in \mathfrak{D}$ of the form
\[
d = \sum_{i=1}^{n-1} (\lambda_j \eta_i(x_n) + \zeta^i_j) \partial_{x_2} + \lambda \beta \partial_{x_n}
\]
such that $R$ is $\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$-simple. Since $\beta$ and $\lambda_j$, in the theorem, are nonzero scalars, we denote $f_n = (\lambda_j \beta)^{-1} x_n$; thus $d(f_n) = 1$.

Let $f^*_i \in k[x_n]$ (or $k[[x_n]]$) such that $d(f^*_i) = \lambda_j \eta_i(x_n) + \zeta^i_j$. Since $c(k) = 0$, $f^*_i$ exist. Then, let $f_i = x_2 - f^*_i$; hence $d(f_i) = 0$, for any $i = 1, \ldots, n$. This completes the proof. \qed

Remark 6. The previous corollary is a particular case of an important theorem about the characterization of commutative basis of $\text{Der}_k(R)$ However, in our case the proof is more evident (see [3, Theorem 2] (Now86)).

For the remainder of this note we assume that $R$ is the ring $k[x_1, \ldots, x_n]$ of polynomials over $k$ and $[\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d]$ is as in the previous theorem and also we recall the following definitions.

We recall from [7] that a $k$-derivation $d$ of $k[x_1, \ldots, x_n]$ is called locally nilpotent if for each $f \in R$ exists a natural number $n$ such that $d^n(f) = 0$ and we say that a basis $\{d_1, \ldots, d_n\}$ of $\text{Der}_k(k[x_1, \ldots, x_n])$ is locally nilpotent if every derivation $d_i$ is locally nilpotent for $i = 1, \ldots, n$.

We remember also that the Jacobian conjecture states that if $F = (x_1, \ldots, x_n)$ is a polynomial map such that the Jacobian matrix is invertible, then $F$ has a polynomial inverse (see [7]).

Theorem 6 [see [3, Theorem 5]). Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $k$. The following conditions are equivalent.

(1) The Jacobian conjecture is true in the $n$-variable case.

(2) Every commutative basis of the $R$-module $\text{Der}_k(R)$ is locally nilpotent.

(3) Every commutative basis of the $R$-module $\text{Der}_k(R)$ is locally finite.

Corollary 8. Let $\mathfrak{D}$ be a set of commute $k$-derivations of $R$ such that $R$ is $\mathfrak{D}$-simple, $\partial_{x_1}, \ldots, \partial_{x_{n-1}} \in \mathfrak{D}$, and the Jacobian
conjecture is true in $R$. Then, there exists $d \in D$ such that 
$\{\partial_{x_1}, \ldots, \partial_{x_{n-1}}, d\}$ is a locally nilpotent commutative base of the $R$-module $\text{Der}_k(R)$. In particular, $d$ is a $k$-derivation locally nilpotent.

**Proof.** The proof is immediate consequence of [3, Theorem 5].

**Question.** A ring is called $w$-differentially simple if it is a simple relative to a family with $w$ derivations. Recall that we are assuming $R = k[x_1, \ldots, x_n]$; then we know that $R$ is 1-differentially simple and $\dim(R)$-differentially simple as well. However, $n = \dim(R)$ is not necessarily the smallest $w$ for which such a ring can be $w$-differentially simple (see [8]). Thus, one may ask the following: what is the smallest positive integer $w \neq 1$ such that $R$ is $w$-differentially simple and all $w$ derivations are nonsimple and commute?

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**References**


