Review Article

A Survey of Some Topics Related to Differential Operators

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This paper is the result of investigations suggested by recent publications and completes the work of Huet, 2010. The topics, which are dealt with, concern some spaces of functions and properties of solutions of linear and nonlinear, stationary and evolution differential equations, namely, existence, spectral properties, resonances, singular perturbations, boundary layers, and inertial manifolds. They are presented in the alphabetical order. The aim of this document and of Huet, 2010, is to be a useful reference for (young) researchers in mathematics and applied sciences.

1. Introduction

The article is divided into several sections entitled: Birman-Schwinger operators; BMO spaces; Bounded variation (functions of); Discrete energy; Dissipative operators; Dynamical systems; Equal- area condition; Inertial manifolds; Mathieu-Hill type equations; Memory (equations with); Nodes, Nodal; Resonances. The development of each entry includes indications on history, definitions, an overview of main results, examples, and applications but is, of course, nonexhaustive. Complements will be found in the references. A prepublication of some entries is presented in Huet [1].

2. Birman-Schwinger Operator

Definition 1. Consider the Schrödinger operator

$$H_\gamma = H_0 + \gamma V(x), \quad H_0 = (-\Delta)^l, \quad (1)$$

acting on $L^2(\mathbb{R}^d)$, $2l \geq d$, where $x \rightarrow V(x)$ is a real-valued continuous function defined on $\mathbb{R}^d$ which is nonnegative and tends to zero, sufficiently fast, as $|x| \rightarrow \infty$ and γ is a small negative coupling constant. The operator H_0 is self-adjoint and its spectrum is $\sigma_0 = [0, +\infty)$. The Birman-Schwinger operator associated with (1) is the operator

$$X_V(\lambda) = V^{1/2} R_\lambda(H_0) V^{1/2}, \quad (2)$$

where $R_\lambda(H_0)$ is the resolvent of H_0 in $(-\infty, 0)$. For each $\lambda \in (-\infty, 0)$, $X_V(\lambda)$ is self-adjoint and compact (cf. Arazy and Zelenko [2]).

Application. In [2], the authors consider the decomposition $X_V(\lambda) = \Phi(\lambda) + T(\lambda)$, where $\Phi(\lambda)$ is a finite rank operator and $T(\lambda)$ an Hilbert-Schmidt operator whose norm is uniformly bounded with respect to $\lambda \in (-\delta, 0)$ for some $\delta > 0$. An asymptotic expansion of the bottom virtual eigenvalue $\lambda_o(\gamma)$ of H_γ , as $\gamma < 0$ tends to zero, is deduced from this decomposition: if d is odd, it is of power type, while, when d is even, it involves the log function. Asymptotic estimates are obtained, as $\gamma \uparrow 0$, for the nonbottom virtual eigenvalues of H_γ , $\{\lambda_k(\gamma), k \in \mathbb{Z}_+^d; 0 < |k| \leq m\}$, where $m = l - ((d+1)/2)$ if d is odd and $m = l - (d/2)$ if d is even. If d is odd, $\Phi(-t^{2l})$ is a meromorphic operator function, and the leading terms of the asymptotic estimates of $\lambda_k(\gamma)$ are of power type. An algorithm, based on the Puiseux-Newton diagram (cf. Baumgärtel [3]), is proposed for an evaluation of the leading coefficients of these estimates. If d is even, two-sided estimates are obtained for eigenvalues with an exponential rate of decay; the rest of the eigenvalues have a power rate of decay. Estimates of Lieb-Thirring type are obtained for groups of eigenvalues which have the same rate of decay, when d is odd or even.

3. BMO Spaces

3.1. BMO and Related Spaces

Definition 2. Let $\sup_I$ be the supremum over all cubes $I \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, $\ell(I)$ the sidelength of I , and f_I the mean value of f over I . The John-Nirenberg's $\text{BMO} = \text{BMO}(\mathbb{R}^n)$ space (cf. John and Nirenberg [4]) is the space of locally integrable complex-valued functions f defined on $\mathbb{R}^n$, such that

$$\|f\|_{\text{BMO}} = \left(\sup_I (\ell(I))^{-n} \int_I |f(x) - f_I|^2 dx \right)^{1/2} < \infty. \quad (3)$$

The space of functions of bounded mean oscillation, modulo constants, equipped with the above norm, is a Banach space.

Definition 3 (real Hardy space $H^1(\mathbb{R}^n)$). A function $f \in H^1(\mathbb{R}^n)$ if and only if

$$f = \sum c_j \alpha_j, \quad (4)$$

where $c_j \in \mathbb{R}$, $\sum |c_j| < \infty$, each function α_j is supported on a ball B_j and has integral zero, and $\sup_{x \in B_j} |\alpha_j(x)| \leq 1/|B_j|$. Functions α that satisfy the above properties are called 1-atom (cf. Stein, [5]). With the norm

$$\|f\|_{H^1} = \inf \sum |c_j|, \quad (5)$$

where the inf is taken on all decompositions of f of the form (4), $H^1(\mathbb{R}^n)$ is a Banach space. In Fefferman and Stein [6, Theorem 2], it is proved that the dual of $H^1(\mathbb{R}^n)$ is BMO. For definitions and properties of Hardy-spaces $H^p(\mathbb{R}^n)$, see [6].

Related spaces are presented in [7].

3.2. BMO Nonlinearity (cf. Byun and Wang [8]). Let $a(\xi, x)$ be a nonlinear, real valued, function on $\mathbb{R}^n \times \mathbb{R}^n$. For $y \in \mathbb{R}^n$ and $\rho > 0$, $B_\rho(y)$ denotes the open ball of radius ρ centered at y . Set

$$\bar{a}_{B_\rho(y)}(\xi) = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} a(\xi, x) dx \quad (6)$$

and define the function $\beta(a, B_\rho(y))$ by

$$\beta[a, B_\rho(y)](x) = \sup_{\xi \in \mathbb{R}^n} \frac{|a(\xi, x) - \bar{a}_{B_\rho(y)}(\xi)|}{|\xi| + 1}. \quad (7)$$

Definition 4 ((δ, R)-BMO condition). The function $a = a(\xi, x)$ satisfies the (δ, R)-BMO condition if

$$\sup_{0 < \rho \leq R} \sup_{y \in \mathbb{R}^n} \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} |\beta[a, B_\rho(y)]|^2(x) dx \leq \delta^2. \quad (8)$$

Definition 5 (Reifenberg domain). A bounded open set Ω in $\mathbb{R}^n$ is (δ, R)-Reifenberg flat if, for every $x_o \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, \dots, y_n\}$ (which can depend on r and x_o) so that x_o is the origin in this coordinate system and that

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\} \quad (9)$$

(cf. [9]).

Applications. (i) Let Ω be a bounded, open subset of $\mathbb{R}^n$, $f \in L^p(\Omega; \mathbb{R}^n)$, $2 \leq p < \infty$, $a = a(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector field measurable in x for almost every ξ and continuous in ξ for each x . In [8], the authors consider the nonlinear boundary value problem

$$\text{div } a(\nabla u, x) = \text{div } f, \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega. \quad (10)$$

The following conditions are imposed on $a(\xi, x)$:

$$[a(\xi, x) - a(\eta, x)] \cdot (\xi - \eta) \geq c_0 |\xi - \eta|^2 \quad (11)$$

for all $\xi, \eta \in \mathbb{R}^n$ and almost every $x \in \Omega$,

$$|a(\xi, x)| \leq c_1 (1 + |\xi|) \quad (12)$$

for all $\xi \in \mathbb{R}^n$ and almost $x \in \mathbb{R}^n$, and

$$|\nabla_\xi a(\xi, x)| \leq c_2 \quad \text{for any } \xi \in \mathbb{R}^n \text{ and almost every } x \in \mathbb{R}^n \quad (13)$$

for some positive constants c_0, c_1 , and c_2 . Then it is proved that there exists $\delta > 0$ such that, if a satisfies the (δ, R)-BMO condition and Ω is (δ, R)-Reifenberg flat, the weak solution $u \in H_o^1(\Omega)$ to (10) belongs to $W_o^{1,p}(\Omega)$ with the estimate

$$\|\nabla u\|_{L^p(\Omega)} \leq C (\|f\|_{L^p(\Omega)} + 1), \quad (14)$$

where C is independent of u and f .

(ii) In [9], S-S Byun extends the previous results to Orlicz spaces. He recalls the following definitions.

Definition 6. A positive function ϕ defined on $[0, \infty)$ is called a Young function if it is increasing, convex and satisfies

$$\begin{aligned} \phi(0) &= 0, & \phi(\infty) &= \lim_{t \rightarrow \infty} \phi(t) = \infty, \\ \lim_{t \rightarrow 0} \frac{\phi(t)}{t} &= 0, & \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} &= \infty. \end{aligned} \quad (15)$$

Definition 7. One says that the Young function $\phi \in \Delta_2 \cup \nabla_2$ if it satisfies the following conditions:

$$\begin{aligned} \phi(2t) &\leq \kappa \phi(t), \quad \forall t \geq 0, \\ 2t_o \phi(t) &\leq \phi(t_o t), \quad \forall t > 0, \end{aligned} \quad (16)$$

for some numbers $\kappa, t_o > 1$.

Definition 8. Let $\phi \in \Delta_2 \cup \nabla_2$ be a Young function. The Orlicz space $L^\phi(\Omega)$ is the linear space of all measurable functions $v : \Omega \rightarrow \mathbb{R}$ satisfying

$$\int_\Omega \phi(|v|) dx < \infty. \quad (17)$$

Equipped with the norm

$$\|v\|_{L^\phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \phi\left(\frac{|v|}{\lambda}\right) dx \leq 1 \right\}, \quad (18)$$

$L^\phi(\Omega)$ is a Banach space.

The following result is proved in [9]: let $\phi \in \Delta_2 \cup \nabla_2$ be a Young function. There exist a small $\delta > 0$ and a positive constant C such that, if the nonlinearity a satisfies (11), (12), (13), and (8), if Ω satisfies (9), and if $|f|^2 \in L^\phi(\Omega)$, then the unique weak solution $u \in H_o^1(\Omega)$ to (10) satisfies $|\nabla u|^2 \in L^\phi(\Omega)$, with the estimate

$$\int_{\Omega} \phi(|\nabla u|^2) dx \leq C \left(1 + \int_{\Omega} \phi(|f|^2) dx \right). \quad (19)$$

Remark 9. If $p > 2$ and $\phi(t) = t^{p/2}$ inequality (19) reduces to only (14).

4. (Functions of) Bounded Variation

The notion of functions of bounded variation is closely related to the notion of *measure* and the following usual definitions are useful.

4.1. Measures

Definition 10. Let Ω be an open subset of $\mathbb{R}^n$, and $\mathcal{C}_c(\Omega, \mathbb{C})$ [resp., $\mathcal{C}_c(\Omega, \mathbb{R})$] the space of continuous functions $\varphi : \Omega \rightarrow \mathbb{C}$ [resp., $\mathbb{R}$] with compact support in Ω .

(i) A measure μ on Ω is a linear functional

$$\mu : \varphi \in \mathcal{C}_c(\Omega, \mathbb{C}) \longrightarrow \mu(\varphi) \text{ (or } \langle \mu, \varphi \rangle) \in \mathbb{C} \quad (20)$$

that is continuous; in the following sense, for all K compact in Ω , there exists a constant C_K such that

$$|\langle \mu, \varphi \rangle| \leq C_K \|\varphi\|_{\infty} \quad (21)$$

for all $\varphi \in \mathcal{C}_c(\Omega, \mathbb{C})$ whose support is contained in K . We write $\mu \in \mathcal{M}(\Omega)$. In [10], Schwartz introduces a topology on $\mathcal{C}_c(\Omega, \mathbb{C})$ and μ is continuous on this topological space.

(ii) μ is said to be *bounded* on Ω if, in (21), the constant is independent of K . The space of bounded measures on Ω is denoted by $\mathcal{M}^1(\Omega)$.

(iii) The conjugate $\bar{\mu}$ of μ is given by

$$\langle \bar{\mu}, \varphi \rangle = \overline{\langle \mu, \bar{\varphi} \rangle}. \quad (22)$$

(iv) μ is called *real* if $\langle \mu, \varphi \rangle \in \mathbb{R}$, for all $\varphi \in \mathcal{C}_c(\Omega, \mathbb{R})$.

(v) μ is called *positive* if $\langle \mu, \varphi \rangle \geq 0$, for all $\varphi \in \mathcal{C}_c(\Omega, \mathbb{R})$, $\varphi \geq 0$.

Definition 11 (absolute value of a measure). If μ is a complex or real measure, its absolute value, denoted by $|\mu|$, is the map

$$\begin{aligned} \psi &\in \mathcal{C}_c(\Omega, \mathbb{R}), \\ \psi \geq 0 &\longrightarrow \langle |\mu|, \psi \rangle = \sup_{\varphi \in \mathcal{C}_c(\Omega, \mathbb{C}), |\varphi| \leq \psi} \{ |\langle \mu, \varphi \rangle| \}. \end{aligned} \quad (23)$$

Definition 12 (total variation of a positive bounded measure). Let μ be a *positive, bounded measure* on Ω . Its total variation, denoted by $|\mu|_{\Omega}$ or $\int_{\Omega} \mu$, is

$$|\mu| = \sup_{\varphi \in \mathcal{C}_c(\Omega, \mathbb{R}), 0 \leq \varphi \leq 1} \mu(\varphi). \quad (24)$$

4.2. Space $BV(I, \mathbb{R})$ or $BV(I)$. Let $I = (a, b)$ be an interval in $\mathbb{R}$. The following alternative definitions are well known. A function $f : x \in I \rightarrow \mathbb{R}$ is of bounded variation:

- (i) if it can be expressed in the form $\phi - \psi$, where ϕ, ψ are nondecreasing bounded functions,
- (ii) if the interval (a, b) is divided up by points

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b, \quad (25)$$

then, there exists a constant $C > 0$, independent of the mode of division, such that

$$\sum_{v=0}^{n-1} |f(x_{v+1}) - f(x_v)| < C \quad (26)$$

and the upper bound of the sum (26) is called the total variation of f on I and is denoted by $V(f, I)$.

For these definitions, see Titchmarsh [11, page 355] and Riesz and Sz-Nagy [12, page 10].

We have also the following proposition: let $f \in L^1_{\text{loc}}(\mathbb{R})$. Then f defines a distribution $T = T_f \in \mathcal{D}'(\mathbb{R})$. In Schwartz [10, page 53] it is proved that, in order for the derivative of T to be a measure, it is necessary and sufficient that f is of bounded variation on every finite interval.

4.3. Space $BV(\Omega, \mathbb{R})$ or $BV(\Omega)$. Let Ω be an open set in $\mathbb{R}^n$. The following definitions are equivalent (cf. Brezis [13, page 153]).

A function $f \in W^1_1(\Omega)$ is of bounded variation

- (i) if all first derivatives of f , in the distributional sense, that is, in $\mathcal{D}'(\Omega)$, are bounded measures,
- (ii) if there exists a constant C such that

$$\left| \int_{\Omega} f \frac{\partial \phi}{\partial x_i} dx \right| \leq C \|\phi\|_{L^\infty(\Omega)} \quad \forall \phi \in C^\infty_c(\Omega), \quad i = 1, \dots, n, \quad (27)$$

(iii) if there exists a constant C such that

$$\|\tau_h f - f\|_{L^1(\omega)} \leq C |h| \quad (28)$$

for all open set $\omega \subset \Omega$ and all $h \in \mathbb{R}^n$ with $|h| < \text{dist}(\omega, \mathbb{C} \setminus \Omega)$. Moreover, in (27) and (28) we can take $C = \|\nabla f\|_{L^1(\Omega)}$. Here $\tau_h f(x) = f(x + h)$, and ∇f is the distributional gradient of f .

We have also the following proposition: a function $f \in L^1(\Omega)$ is of bounded variation if its first distributional derivatives are bounded measures. Then, the gradient ∇f is a bounded, vector-valued measure whose *absolute value* $|\nabla f|$ is the map

$$\begin{aligned} \psi &\in \mathcal{C}_c(\Omega, \mathbb{R}), \\ \psi \geq 0 &\longrightarrow \langle |\nabla f|, \psi \rangle = \sup_{\varphi \in \mathcal{C}_c(\Omega, \mathbb{C}^n), \sum_{i=1}^n |\varphi_i|^2 \leq \psi^2} \{ |\langle \nabla f, \varphi \rangle| \}, \end{aligned} \quad (29)$$

where $\langle \nabla f, \varphi \rangle = \sum_i^n (\partial f / \partial x_i) \varphi_i$ and $|\nabla f|$ is a bounded positive measure. The total variation of $|\nabla f|$ is

$$\int_{\Omega} |\nabla f| = \|\nabla f\|_{\Omega} = \sup_{\psi \in \mathcal{C}_c(\Omega, \mathbb{R}), 0 \leq \psi \leq 1} \langle |\nabla f|, \psi \rangle \quad (30)$$

(cf. F. Demengel and G. Demengel [14, page 303]).

4.4. Space $BV(\Omega, \mathbb{R}^n)$. This section is already presented in [7]. Let Ω be an open subset of $\mathbb{R}^d$ with a smooth boundary. A function $f \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$ has a bounded variation, that is, $f \in BV(\Omega, \mathbb{R}^n)$, if ∇f , in the distributional sense, is a vector-valued Radon measure of finite total mass. Let $\|f\|_{BV} = \int_{\Omega} |\nabla f|$ be a BV-seminorm. In Dávila [15], the following property of $|f|_{BV}$ is proved: there exists a positive constant K , which depends on d , such that, for every family of nonnegative radial mollifiers $\rho_{\epsilon} \in L^1_{\text{loc}}((0, \infty), \mathbb{R}_+)$ satisfying

$$\int_0^{\infty} \rho_{\epsilon}(r) r^{d-1} dr = 1, \quad \lim_{\epsilon \rightarrow 0} \int_{\delta}^{\infty} \rho_{\epsilon}(r) r^{d-1} dr = 0 \quad (31)$$

$\forall \delta > 0,$

we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega^2} \frac{|f(x) - f(y)|}{|x - y|} \rho_{\epsilon}(|x - y|) dx dy = K |f|_{BV}, \quad (32)$$

$\forall f \in BV(\Omega, \mathbb{R}^n).$

4.5. Application. In [16], B. Merlet shows, by means of the above property of $|f|_{BV}$, that, if $u \in BV(\Omega, S^1)$, there exists a lifting $\varphi \in BV(\Omega, \mathbb{R})$ of u (i.e., $u(x) = e^{i\varphi(x)}$, for all $x \in \Omega$) such that $|\varphi|_{BV} \leq 2|u|_{BV}$.

5. Discrete Energy

Let $\omega_N = (x_1, \dots, x_N)$, $N \geq 2$ be a set of N points on the unit-sphere $S^2 = \{x \in \mathbb{R}^3, |x| = 1\}$.

5.1. Discrete Energy of ω_N . Different discrete energies are associated with ω_N . Let $\alpha \in \mathbb{R}$, $-2 < \alpha < 2$.

Definition 13. The Coulomb [resp., logarithmic] energy associated with ω_N is

$$\mathcal{H}_C(\omega_N) = \sum_{i=1}^N \sum_{j=i+1}^N \frac{1}{|x_i - x_j|} \quad (33)$$

[resp., $\mathcal{H}_L(\omega_N) = -\sum_{i=1}^N \sum_{j=i+1}^N \log |x_i - x_j|$].

Definition 14. More generally, the α -energy associated with ω_N is

$$E(\alpha, \omega_N) = \begin{cases} \sum_{i=1}^N \sum_{j=i+1}^N |x_i - x_j|^{\alpha} & \text{if } \alpha \neq 0, \\ \mathcal{H}_L(\omega_N) & \text{if } \alpha = 0. \end{cases} \quad (34)$$

Remark 15. We have

$$\mathcal{H}_C(\omega_N) = E(-1, \omega_N). \quad (35)$$

Remark 16. $E(\alpha, \omega_N)$ is the energy of the N points $x_1, \dots, x_N$, on the surface of the sphere, interacting through a potential $V = r^{\alpha}$ (cf. Rakhmanov et al. [17]).

The article [17] is devoted to extremal energy for ω_N :

$$\mathcal{E}(\alpha, N) = \begin{cases} \inf_{\omega_N \in S^2} E(\alpha, \omega_N) & \text{if } \alpha \leq 0, \\ \sup_{\omega_N \in S^2} E(\alpha, \omega_N) & \text{if } \alpha > 0. \end{cases} \quad (36)$$

Bounds for $\mathcal{E}(\alpha, N)$, $-2 < \alpha < 2$ and explicit formula for N points on S^2 that yields good estimates for $\mathcal{E}(\alpha, N)$ are obtained. The authors point out important applications of the determination of $\mathcal{E}(\alpha, N)$ to geometry, chemistry, physics, and crystallography and give references for the history of related researches.

5.2. A More General Discrete Energy-Like Function on the Unit Sphere. In [18], Cheviakov et al. introduce the following definition.

Definition 17. The discrete energy-like function associated with ω_N is

$$\mathcal{H}(\omega_N) = \mathcal{H}_C(\omega_N) + \frac{1}{2} \mathcal{H}_L(\omega_N) - \frac{1}{2} \sum_{i=1}^N \sum_{j=i+1}^N \log(2 + |x_i - x_j|). \quad (37)$$

Application. In [18], $\mathcal{H}(\omega_N)$ is related to the mean first passage time (MFPT) $v(x)$ for a Brownian particle in the unit ball Ω in $\mathbb{R}^3$ that contains N small locally circular absorbing windows $\partial\Omega_{\epsilon_j}$ on its boundary $\partial\Omega = S^2$. Set $\partial\Omega_a = \bigcup_{j=1}^N \partial\Omega_{\epsilon_j}$. The function $v(x)$ is solution to the Dirichlet-Neumann problem:

$$\begin{aligned} \Delta v &= -\frac{1}{D} \quad \text{on } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega_a, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \frac{\partial\Omega}{\partial\Omega_a}, \end{aligned} \quad (38)$$

where D is a diffusivity coefficient. The authors obtain three-term asymptotic expansions for $v(x)$, for the average MFPT $\bar{v} = (1/|\Omega|) \int_{\Omega} v(x) dx$, and for the principal eigenvalue λ of the Laplacian associated with the boundary conditions (38), when the area $|\partial\Omega_{\epsilon_j}| \sim \pi \epsilon_j^2$, $\partial\Omega_{\epsilon_j} \rightarrow x_j \in S^2$ as $\epsilon \rightarrow 0$,

$j = 1, \dots, N$ and $|x_i - x_j| = \mathcal{O}(1)$, $i \neq j$. For instance, when the windows have common radius $\epsilon \ll 1$, they obtain

$$\begin{aligned} \bar{v} = \frac{|\Omega|}{4\epsilon DN} & \left[1 + \frac{\epsilon}{\pi} \log\left(\frac{2}{\epsilon}\right) \right. \\ & + \frac{\epsilon}{\pi} \left(-\frac{9N}{5} + 2(N-2) \log 2 + \frac{3}{2} \right) \\ & \left. + \frac{4}{N} \mathcal{H}(x_1, \dots, x_N) + \mathcal{O}(\epsilon^2 \log \epsilon) \right], \quad (39) \\ \lambda & \sim \frac{1}{D\bar{v}}. \end{aligned}$$

Moreover, $\bar{v}$ is minimized and the corresponding λ is maximized at the configuration ω_N that minimizes $\mathcal{H}(\omega_N)$. The optimum arrangements $\{x_1, \dots, x_N\}$ that minimize $\mathcal{H}(\omega_N)$ are numerically computed by different methods.

6. Dissipative Operators

I restrict myself to definitions (cf. Pazy [19]). Let X be a Banach space whose norm is denoted by $\|\cdot\|$, X^* its dual, and $x^* \in X^*$. The value of x^* at $x \in X$ is denoted by $\langle x^*, x \rangle = \langle x, x^* \rangle = x^*(x)$.

Definition 18. For $x \in X$, we define the *duality set* by

$$F(x) = \{x^* \in X^*; x^*(x) = \|x\|^2 = \|x^*\|^2\}. \quad (40)$$

Definition 19. A linear operator A , in X , with domain $D(A)$, is *dissipative* if, for every $x \in D(A)$, there exists $x^* \in F(x)$ such that $\Re \langle Ax, x^* \rangle \leq 0$ or, equivalently, if $\|(\lambda I - A)x\| \geq \lambda \|x\|$ for all $x \in D(A)$ and $\lambda > 0$. A dissipative operator A is called *m-dissipative* if, for all $\lambda > 0$, the operator $\lambda I - A : D(A) \rightarrow X$ is surjective.

The *Lumer-Phillips Theorem* states that a densely defined operator A in X is the generator of a C_0 -semigroup $T(t)$ (i.e., $\lim_{t \rightarrow 0} T(t)x = x$) of contractions if and only if it is *m-dissipative* (cf. [19]). I recall that the domain $D(A)$ of the infinitesimal generator A of $T(t)$ is

$$D(A) = \left\{ x \in X : \lim_{t \rightarrow 0} \frac{Tx - x}{t} \text{ exists} \right\} \quad (41)$$

and, for any $x \in D(A)$,

$$Ax = \lim_{t \rightarrow 0} \frac{Tx - x}{t}. \quad (42)$$

7. Dynamical Systems

In this section, I restrict myself to several definitions and examples of dynamical systems and also definitions and some properties of limit and invariant sets. More investigations into this large topic are in preparation.

Different definitions of dynamical systems are given in the literature.

7.1. Definitions

7.1.1. General Definitions. In (2011), Kloeden and Rasmussen give, in [20], the general definitions:

let $\mathbb{T} = \mathbb{R}$ [resp., $\mathbb{Z}$], $\mathbb{T}_o^+ = \mathbb{T} \cap [0, \infty)$, that is, $\mathbb{R}^+$ [resp., $\mathbb{N}$] and a *metric space* X .

Definition 20. A *dynamical system* is a *continuous* map $\Phi : \mathbb{T} \times X \rightarrow X$ which satisfies the initial value condition

$$\Phi(0, x) = x \quad \forall x \in X \quad (43)$$

and the group property

$$\Phi(s + t, x) = \Phi(s, \Phi(t, x)) \quad \forall s, t \in \mathbb{T}, x \in X. \quad (44)$$

When $\mathbb{T} = \mathbb{R}$ the dynamical system is called *continuous* and when $\mathbb{T} = \mathbb{Z}$ the dynamical system is called *discrete*.

Definition 21. A *semidynamical system* is a *continuous* map $\Phi : \mathbb{T}_o^+ \times X \rightarrow X$ which satisfies the initial value condition

$$\Phi(0, x) = x \quad \forall x \in X \quad (45)$$

and the semigroup property

$$\Phi(s + t, x) = \Phi(s, \Phi(t, x)) \quad \forall s, t \in \mathbb{T}_o^+, x \in X. \quad (46)$$

When $\mathbb{T}_o^+ = \mathbb{R}^+$ [resp., $\mathbb{N}$] the semidynamical system is called *continuous* [resp., *discrete*].

7.1.2. "Classical" Dynamical Systems. Let B be a *Banach space*. In 1969, Hale [21] gives the following definition.

Definition 22. A *classical dynamical system* on B is a continuous function $u : \mathbb{R}^+ \times B \rightarrow B$ which satisfies

$$u(0, \varphi) = \varphi, \quad u(t + \tau, \varphi) = u(t, u(\tau, \varphi)) \quad (47)$$

for all $t, \tau \geq 0$ and $\varphi \in B$.

Remark 23. In Definition 22, the mapping u is a continuous semidynamical system in terms of Definition 21.

7.1.3. "Abstract" Dynamical System. In 1991, Haraux [22] defines *abstract dynamical systems* in a *complete metric space* (E, d) in the following way.

Definition 24. An *abstract dynamical system* in E is a family $\{S(t)\}_{t \geq 0}$ of mappings on E which satisfy the following properties:

- (i) for each $t \geq 0$, $S(t) : E \rightarrow E$ is continuous,
- (ii) $S(0) = I$,
- (iii) $S(t + s) = S(t)S(s)$,
- (iv) for each $z \in E$, $t \in [0, \infty) \rightarrow S(t)z \in E$ is continuous.

Remark 25. In Definition 24, the mapping $u : (t, z) \in \mathbb{R}^+ \times E \rightarrow u(t, z) = S(t)z \in E$ satisfies conditions (45) and (46) but it is only *separately* continuous. Therefore it is not a semidynamical system in terms of Definition 21.

7.1.4. Dafermos Definition. In [23, Definition 4.1], Dafermos gives the following definition in a *metric space* X .

Definition 26. A Dafermos dynamical system in X is a map $u : X \times \mathbb{R}^+ \rightarrow X$ which satisfies the initial condition

$$u(x, 0) = 0 \quad \forall x \in X \quad (48)$$

the semigroup property

$$u(x, s + t) = u(u(x, s), t) \quad \forall t, s \in \mathbb{R}^+, x \in X, \quad (49)$$

and the continuity condition in x , for all $t \in \mathbb{R}^+$.

Remark 27. In the previous definitions the space X [resp. B , E] is called the *phase space*, and the set

$$\gamma^+(x) = \bigcup_{t \geq 0} \Phi(t, x) \left[\text{resp.} \bigcup_{t \geq 0} u(t, x) \right] \quad (50)$$

is called the *positive orbit* or *trajectory* through $x \in X$ [resp., $x \in B$, $x \in E$].

7.2. Examples

7.2.1. Continuous (Semi-)Dynamical Systems

Example 28 (cf. [21, Example 1]). Let f be a continuous function $\mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Consider the autonomous differential equation

$$\frac{dx}{dt} = f(x(t)), \quad t \geq 0, \quad (51)$$

with the initial condition

$$x(0) = \xi. \quad (52)$$

If there exists a unique solution $u(t, \xi)$ of (51) and (52) which depends continuously upon t, ξ , then the mapping

$$\Phi : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \text{defined by } \Phi(t, \xi) = u(t, \xi) \quad (53)$$

is a continuous semidynamical system on $\mathbb{R}^n$ (Definitions 21 and 22).

Example 29 (cf. [20, Example 1.6]). Let X be the Banach space of bounded continuous functions $\mu : [-1, 0] \rightarrow \mathbb{R}^d$, under the usual norm $\|\cdot\|_\infty$. Let $\mu \in X$ and f a continuous function $\mathbb{R}^d \rightarrow \mathbb{R}^d$ and suppose that there exists a unique solution $u(t, \mu)$ of the autonomous delay differential equation

$$\frac{dx}{dt} = f(x(t-1)), \quad t \geq 0, \quad (54)$$

with the condition

$$x(s) = \mu(s), \quad s \in [-1, 0]. \quad (55)$$

Then

$$\Phi : (t, \mu) \in \mathbb{R}^+ \times X \longrightarrow \Phi(t, \mu)(s) = u(t+s, \mu) \in X, \quad (56)$$

$$t \geq 0, s \in [0, 1],$$

is a continuous semidynamical system on X .

Example 30 (see cf. [22, page 12]). Let X be a real Banach space, A a m -dissipative operator with dense domain in X (cf. Section 5), $T(t)$ the semigroup of contractions generated by A , and $F : X \rightarrow X$ a Lipschitzian function on the bounded sets of X . Let $u(t, x) \in C([0, \tau(x)), X)$ be the unique maximal solution of

$$u(t, x) = T(t)x + \int_0^t T(t-s)F(u(s))ds, \quad (57)$$

$$t \in [0, \tau(x)), x \in X.$$

Set $u(t, x) = S(t)x$. Let $Y \subset X$ such that $\tau(y) = \infty$ for all $y \in Y$ and $\|S(t)y\| \leq M$, for all $t \geq 0, y \in Y$. For $y \in Y, u(t, y) = S(t)y$ is solution of (57) for $x = y$. Set $Z = \overline{\bigcup_{y \in Y} \bigcup_{t \geq 0} \{S(t)y\}}$ and denote by d the distance induced on Z by the norm in X . Then $\{S(t)\}_{t \geq 0}$ is an abstract dynamical system (Definition 24) on (Z, d) .

Example 31. Dafermos dynamical systems are adapted to equations of linear viscoelasticity. See Section 10.

Example 32 (see cf. [20]). Let X be a *metric space* and $f : X \rightarrow X$ be a *continuous* function. By iteration, $f^{n+1}(x) = f(f^n(x)) = f \circ f^n(x)$, for all $x \in X, n \in \mathbb{N}$, is well defined. The mapping

$$\Phi : (n, x) \in \mathbb{N} \times X \longrightarrow \Phi(n, x) = f^n(x) \in X \quad (58)$$

is a *discrete semidynamical system* (Definition 21). Now, suppose f is an homeomorphism; that is, it is continuous and invertible with continuous inverse. Then, the mapping Φ defined by (58) can be extended to $\mathbb{Z}^-$ by

$$\Phi(n, x) = (f^{-1})^{-n}(x), \quad \forall n \in \mathbb{Z}^-, x \in X, \quad (59)$$

and Φ is a *discrete dynamical system*.

7.2.2. Discrete-Time Dynamical Systems and Fractals. What is known as a *discrete time dynamical system* in Pesin and Climenhaga [24] is related to a map from a set to itself. Let X be any set and f a map $f : X \rightarrow X$. By iteration, $f^{n+1}(x) = f(f^n(x)) = (f \circ f^n)(x)$ is well defined for every $n \in \mathbb{N}$ and we have the *semigroup property*:

$$f^{n+m} = f^m \circ f^n = f^n \circ f^m \quad (60)$$

for any integer $m, n \geq 0$.

Definition 33. Let $x \in X$. The sequence of points $f^o(x) = x, f(x), f^2(x), \dots$, is called the *trajectory* or *orbit* of x .

7.2.3. Examples with Chaotic Behavior

Example 34 (the Cantor set). Consider, in $\mathbb{R}$, the intervals $I_1 = [0, 1/3], I_2 = [2/3, 1]$, and $D = I_1 \cup I_2$ and the map f defined by $f(x) = 3x$ on I_1 and $3x - 2$ on I_2 . The range of f is $[0, 1]$ and does not lie in its domain of definition. The domain of definition of f^2 is $\{x; x \in D \text{ and } f(x) \in D\}$, that is, the union of the intervals:

$$\left[0, \frac{1}{9}\right], \quad \left[\frac{2}{9}, \frac{1}{3}\right], \quad \left[\frac{2}{3}, \frac{7}{9}\right], \quad \left[\frac{8}{9}, 1\right] \quad (61)$$

FIGURE 1: The Cantor set.

obtained by removing the open middle third of each interval I_1 and I_2 and so on. The domain on which every iterate f^n is defined is exactly the Cantor set (cf. Figure 1).

Example 35 (the Sierpinski triangle (or gasket)). An equilateral triangle is divided into four smaller triangles, each similar to the first and congruent to each other. Then, the middle triangle is removed, and the iterating procedure is iterated on the remaining three, and so on. The fractal T obtained as the limit of this procedure is the Sierpinski triangle (cf. Figure 2). With this procedure is associated the following algorithm. If $x \in T$, and only in this case, it is possible to define a function $x \in T \rightarrow f(x) \in \mathbb{R}^2$ such that $f^n(x) \in T$ for every n (cf. Schroeder [25] which calls this procedure “Sir Pinski game”).

The Cantor set and the Sierpinski triangle are examples of fractals (cf. [26, 27]).

7.3. Limit and Invariant Sets

Definition 36 (see cf. [20]). Let $\Phi : \mathbb{T} \times X \rightarrow X$ be a dynamical system in a metric space X . The ω -limit set [resp., α -limit set] of a point $\xi \in X$ is defined by

$$\omega(\xi) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t, \xi)} \quad (62)$$

$$\left[\text{resp., } \alpha(\xi) = \bigcap_{\tau \leq 0} \overline{\bigcup_{t \leq \tau} \phi(t, \xi)} \right].$$

In the case of a semidynamical system ($\Phi : \mathbb{T}_o^+ \times X \rightarrow X$) the notion of a α -limit set is not defined. In the same way, ω and α -limit sets of a subspace $M \subset X$ are defined by (62) where ξ is replaced by M .

Remark 37. The ω and α -limit sets may be defined as follows:

$$\omega(\xi) = \{x \in X : \text{there exists a sequence } t_n, t_n > 0,$$

$$t_n \rightarrow \infty \text{ such that } \Phi(t_n, \xi) \rightarrow x \text{ as } n \rightarrow \infty\},$$

$$\alpha(\xi) = \{x \in X : \text{there exists a sequence } t_n, t_n < 0,$$

$$t_n \rightarrow -\infty \text{ such that } \Phi(t_n, \xi) \rightarrow x \text{ as } n \rightarrow \infty\}.$$
(63)

Remark 38. Similar definitions are valid for classical and abstract dynamical systems.

Definition 39. Let $\Phi : \mathbb{T}_o^+ \times X \rightarrow X$ be a semidynamical system on the metric space X (Definition 21). A subset $M \subset X$ is called *invariant* under Φ if

$$\Phi(t, M) = M \quad \forall t \in \mathbb{T}_o^+. \quad (64)$$

It is called *positively* [resp., *negatively*] *invariant* if

$$\Phi(t, M) \subset M \quad [\text{resp., } M \subset \Phi(t, M)], \quad \forall t \in \mathbb{T}_o^+. \quad (65)$$

FIGURE 2: The Sierpinski gasket.

In the case of *discrete-time dynamical system* and of Example 32, invariance is equivalent to $f(M) = M$.

Properties of ω -Limit and Invariant Sets. Let (X, d) be a metric space, $\Phi : \mathbb{T}_o^+ \times X \rightarrow X$ a semidynamical system on X , and $x \in X$. If the trajectory $\gamma^+(x)$ through x is *precompact*, then $\omega(x)$ has the following properties:

$$\Phi(t, \omega(x)) = \omega(x) \neq \emptyset, \quad \forall t \in \mathbb{T}_o^+,$$

$$\omega(x) \text{ is a compact subset of } X, \quad (66)$$

$$\omega(x) \text{ is connected if } \mathbb{T}_o^+ = \mathbb{R}^+.$$

If $\mathbb{T}_o^+ = \mathbb{N}$, $\omega(x)$ does not need to be connected (cf. [20, Exercise 1.10]).

In the case of an abstract dynamical system $\{(S(t))_{t \geq 0}\}$, on a *complete metric space* (E, d) , previous results hold with $\Phi(t, x) = S(t)x$, for $x \in E$, if $\gamma^+(x)$ is *relatively compact* in E (see [28, page 122]). But in a complete metric space E a subset A is relatively compact if and only if it is precompact (cf. [29]).

8. Equal-Area Condition

Equal-area type conditions appear, as sufficient or necessary conditions, in the formation of layers (internal or superficial) in stationary solutions to various singularly perturbed reaction-diffusion systems. In the recent works do Nascimento [30], Crema and do Nascimento [31], and do Nascimento and de Moura [32], the authors prove the necessity of suitable equal-area condition for the formation of internal or (and) superficial transition layers in this type of problems.

Example 40. A simple particular case of problems studied in [30] is the elliptic boundary value problem

$$\epsilon \operatorname{div}(h(x) \nabla u) + f(u) = 0, \quad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (67)$$

where Ω is a smooth domain in $\mathbb{R}^N$, $N \geq 1$, $f: \mathbb{R} \rightarrow \mathbb{R}$ such that there exist $\alpha, \beta, \alpha > \beta$, with $f(\alpha) = f(\beta) = 0$.

Let $\Gamma \subset \Omega$ be a smooth $(N - 1)$ -dimensional compact manifold without boundary. It is proved that, if (67) has a family $\{u_\epsilon\}$ of solutions which develop an *internal* transition layer with interface Γ connecting the states α to β , then, necessarily, the simple *equal-area condition*

$$\int_{\alpha}^{\beta} f(s) ds = 0 \quad (68)$$

is satisfied.

Example 41. In [31], the following stationary system is considered:

$$\begin{aligned} \epsilon \operatorname{div}(h(x) \nabla u) + f(x, u, \mathbf{v}) &= 0, \quad x \in \Omega, \\ \operatorname{div}(\mathbf{k}(x) \nabla \mathbf{v}) + \mathbf{g}(x, u, \mathbf{v}) &= 0, \quad x \in \Omega, \end{aligned} \quad (69)$$

$$\frac{\partial \mathbf{v}}{\partial n} = 0 \quad (\text{or } \mathbf{v} = 0) \quad \text{on } \partial\Omega,$$

where Ω is a smooth domain in $\mathbb{R}^N$, $N \geq 1$, $f \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^N)$, $\mathbf{v}, \mathbf{k}$, and $\mathbf{g}$ are sufficiently smooth $\mathbb{R}^n$ -valued functions, and $\mathbf{k} \nabla \mathbf{v} = (k_1 \nabla v_1, \dots, k_n \nabla v_n)$.

Let $\mathcal{U}$ be an open connected set in Ω , $\Gamma \subset \overline{\mathcal{U}}$ be an $(N - 1)$ -dimensional compact connected orientable manifold whose boundary $\partial\Gamma$ is such that $\partial\Gamma \cap \partial\Omega$ is an $(N - 2)$ -dimensional submanifold of $\partial\Omega$. The authors give a definition of a family of *internal* transition layer solutions $\{(u_\epsilon, \mathbf{v}_\epsilon)\}$, $0 < \epsilon < \epsilon_o$ to (69) in $\mathcal{U}$ with interface Γ , depending on two functions $\alpha, \beta \in C^0(\mathcal{U})$, $\alpha(x) < \beta(x)$ on Γ . They show that, for such a family, there exists u_o [resp., $\mathbf{v}_o$] such that $u_\epsilon \rightarrow u_o$ [resp., $\mathbf{v}_\epsilon \rightarrow \mathbf{v}_o$] on compact sets of $\mathcal{U} \setminus \Gamma$ [resp., in $\overline{\mathcal{U}}$]. They also prove that, if a family of internal transition layer solutions to (69) exists, then $f(x, u_o(x), \mathbf{v}_o(x)) = 0$ on $\mathcal{U} \setminus \Gamma$ and necessarily the *equal-area condition*

$$\int_{\Gamma} \left(\int_{\alpha(x)}^{\beta(x)} f(x, s, \mathbf{v}_o(x)) ds \right) dS = 0 \quad (70)$$

is satisfied. Several concrete applications of these results are presented in the paper.

Example 42. In [32], Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with C^2 boundary $\partial\Omega$, S is a $C^2(N - 1)$ -dimensional surface with a boundary Σ which is assumed to be a $C^2(N - 2)$ -dimensional compact surface without boundary with $\Sigma = \overline{S} \cap \partial\Omega$, and $\overline{S}$ intersects $\partial\Omega$ transversally. The authors define a family of solutions $\{u_\epsilon\}$, $0 < \epsilon < \epsilon_o$, to the elliptic boundary value problem:

$$\begin{aligned} \epsilon \operatorname{div}(a(x) \nabla v_\epsilon) + f(x, v_\epsilon) &= 0, \quad x \in \Omega, \\ \epsilon a(x) \frac{\partial v_\epsilon}{\partial \hat{n}} &= g(x, v_\epsilon), \quad x \in \partial\Omega, \end{aligned} \quad (71)$$

which develops *internal* and *superficial* transition layers, depending on some smooth functions $\alpha, \beta, \alpha(x) < \beta(x)$, $x \in \overline{\Omega}$, with interfaces S and Σ , respectively.

Here $a \in C^1(\overline{\Omega})$, $a > 0$, $f: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, and $g: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are of class C^1 and $\hat{n}$ is the exterior normal vector field on $\partial\Omega$. It is proved that the *equal-area conditions*

$$\begin{aligned} \int_{\alpha(x)}^{\beta(x)} f(x, t) dt &= 0 \quad \forall x \in S, \\ \int_{\alpha(y)}^{\beta(y)} g(y, t) dt &= 0 \quad \forall y \in \Sigma \end{aligned} \quad (72)$$

are necessary for the existence of such solutions.

9. Inertial Manifolds

9.1. Setup. Let H be a Hilbert space and $S(t)$ the semigroup associated with an evolution equation of the form

$$u_t' + Au + Ru = 0 \quad (73)$$

with the initial condition $u(0) = u_o \in H$, where A is a linear operator and R a nonlinear one. When an inertial manifold M exists for problem (73), the restriction of (73) to M reduces to a finite dimensional ordinary differential equation (80), which is an exact copy of the initial system (cf. [33]). The manifold M is usually viewed as the graph of a suitable smooth function $\Phi: PH \rightarrow QH$, where P is an orthogonal projection on H with finite-dimensional range, and $Q = I - P$.

After some definitions, the following sections will be devoted to the existence of an inertial manifold for (73) and the presentation of results on the existence and the behavior of inertial manifolds for phase-field equations.

9.2. Definitions. Let (E, d) be a metric space and $S(t)$ a continuous semigroup on E .

Definition 43. A set $M \subset E$ is an inertial manifold for $S(t)$ if

- (i) M is a finite-dimensional Lipschitz manifold in E ,
- (ii) M is of class C^1 ,
- (iii) M is positively invariant under the flow that is, $S(t)M \subset M$, for all $t \geq 0$,
- (iv) M is exponentially attracting; that is, there exists a constant c_o such that, for any $u_o \in E$, there exists a constant $c_1(u_o) > 0$ such that

$$\operatorname{dist}_E(S(t)u_o, M) \leq c_1 e^{-c_o t}, \quad \forall t \geq 0, \quad (74)$$

where dist_E is the Hausdorff semidistance (see Section 1.8 in Huet [7]); compare Bonfoh et al. [34, page 164].

Definition 44 (see Temam [33]). Let $B \subset E$ and U an open subset of E such that $B \subset U$; B is said to be *absorbing* for $S(t)$ in U if, for any bounded set B_o in U , there exists $t_1(B_o)$ such that $S(t)B_o \subset B$, for all $t \geq t_1$. In particular, if B is bounded, $S(t)B \subset B$, for $t \geq t_1(B)$.

Remark 45. Similar definitions were given in [33] and Luskin and Sell [35], in the case of Hilbert spaces. In [33] the smoothness of M and the continuity of the semigroup $S(t)$ are not parts of the definition of an inertial manifold.

9.3. *Existence of an Inertial Manifold for Problem (73) (cf. [33]).* In a Hilbert space H whose norm is denoted by $|\cdot|$, consider problem (73) where A is an unbounded linear operator in H , strictly positive (i.e., $\exists c > 0$ such that $(Au, u) \geq c \|u\|_H^2$ for all u in the domain $D(A)$ of A), and self-adjoint. It is assumed that A^{-1} is compact. Thanks to assumptions on A , it is possible to define its powers A^s defined on $D(A^s)$, for all $s \in \mathbb{R}$. When $0 < s < 1$ see also Huet [36], Kato [37], Schechter [38]. For $r > 0$, we denote by $B_{s,r}$ the ball, in $D(A^s)$, with center 0 of radius r ; that is,

$$B_{s,r} = \{v \in D(A^s); \|v\|_s = |A^s v| \leq r\}. \quad (75)$$

It is assumed that R satisfies the following properties.

H_1 . There exists $\alpha \in \mathbb{R}$ such that R is Lipschitz on the bounded sets of $D(A^\alpha)$ with values in $D(A^{\alpha-1/2})$ and

$$\begin{aligned} |A^{\alpha-1/2} R(u) - A^{\alpha-1/2} R(v)| &\leq C_K |A^\alpha(u - v)|, \\ \forall u, v \in D(A^\alpha) \text{ satisfying } |A^\alpha u| &\leq K, \quad |A^\alpha v| \leq K. \end{aligned} \quad (76)$$

H_2 . If $u_o \in D(A^\alpha)$ problem (73) has a unique solution $u \in C(\mathbb{R}^+; D(A^\alpha)) \cap L^2((0, T); D(A^{\alpha+1/2}))$, for all $T \geq 0$, and the map $S(t) : u_o \rightarrow u(t)$ is continuous from $D(A^\alpha)$ into itself, for all $t \geq 0$.

H_3 . The semigroup $S(t)$ possesses an absorbing set $B_o \subset D(A^\alpha)$ such that

- (1) $S(t)B_o \subset B_o$, for all $t \geq 0$,
- (2) the ω -limit set of B_o

$$\omega(B_o) = \bigcap_{\beta \geq 0} \overline{\bigcup_{t \geq \beta} S(t)B_o} \quad (77)$$

is the maximal attractor for $S(t)$ in $D(A^\alpha)$.

- (3) $\rho > 0$ is chosen such that B_o is included in the ball $B_{\alpha, \rho/2}$ of $D(A^\alpha)$ (cf. (75)).

9.3.1. *The Prepared Equation.* The prepared equation is equivalent to the original one for t large. Let $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ be a C^∞ function such that

$$\begin{aligned} \theta(s) &= \begin{cases} 1 & \text{for } 0 \leq s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \\ \sup_{s \geq 0} |\theta'(s)| &\leq 2. \end{aligned} \quad (78)$$

The aim of the prepared equation is to avoid the difficulties related to the behavior of the nonlinear term $R(u)$ for large values of $|A^\alpha(u)|$. Let $\theta_r(s) = \theta(s/r)$, $r > 0$, and

$$R_r(u) = \theta_r(|A^\alpha(u)|) R(u) \quad \forall u \in D(A^\alpha). \quad (79)$$

The prepared equation associated with (73) is of the form

$$u_t' + Au + R_\rho u = 0, \quad (80)$$

where ρ is chosen in $H_3(3)$. Let $S_\rho(t)$ be the semigroup associated with (80). The following is assumed.

H_4 . The ball $B_{\alpha, \rho}$ is absorbing for $S_\rho(t)$.

Thanks to the assumptions on R , R_ρ is a bounded operator and a Lipschitz mapping from $D(A^\alpha)$ into $D(A^{\alpha-1/2})$. In particular

$$\begin{aligned} |R_\rho(u_1) - R_\rho(u_2)| &\leq C_\rho |A^\alpha(u_1 - u_2)| \\ \forall u_1, u_2 \in D(A^\alpha), \end{aligned} \quad (81)$$

where C_ρ is a constant which depends on ρ .

9.3.2. *Space $\mathcal{F}_{Nbl}^\alpha$.* Let $N \in \mathbb{N}$, $b, l > 0$ and α defined in H_1 . Under the assumptions of A , there exists an orthonormal basis w_j in H , where w_j is the eigenvector of A corresponding to the eigenvalue λ_j , with $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow \infty$, as $j \rightarrow \infty$. Set

$$P = P_N \quad Q = I - P = I - P_N, \quad (82)$$

where P_N is the orthogonal projector, in H , onto the space spanned by $w_1, \dots, w_N$. The projections P and Q commute with A^β , for all $\beta \in \mathbb{R}$.

Definition 46. Let $\mathcal{F}$ be the space defined by

$$\mathcal{F} = \mathcal{F}_{Nbl}^\alpha = \{\varphi : PD(A^\alpha) \rightarrow QD(A^\alpha)\}, \quad (83)$$

where φ is a Lipschitz function with $\text{supp}(\varphi) \subset \{p \in PD(A^\alpha), |A^\alpha p| \leq 2\rho\}$, that satisfies

$$|A^\alpha \varphi(p)| \leq b, \quad (84)$$

$$|A^\alpha \varphi(p_1) - A^\alpha \varphi(p_2)| \leq l |A^\alpha(p_1 - p_2)|$$

for all $p \in PD(A^\alpha)$, $p_1, p_2 \in D(A^\alpha)$.

Remark 47. (1) $PD(A^\alpha)$ is finite dimensional.

(2) $\mathcal{F}$ is a complete metric space for the distance

$$d(\varphi_1, \varphi_2) = \sup_{p \in PD(A^\alpha)} |A^\alpha(\varphi_1(p) - \varphi_2(p))|. \quad (85)$$

9.3.3. *Construction of a Map $\mathcal{T} : \mathcal{F} \rightarrow \mathcal{F}$.* Let $\varphi \in \mathcal{F}$, $p_o \in PD(A^\alpha)$. Thanks to assumptions on φ and R_ρ , the problem

$$\frac{dp}{dt} + Ap + PR_\rho(p + \varphi(p)) = 0, \quad p(0) = p_o, \quad (86)$$

has a unique solution $p_{p_o, \varphi}(t) \in PD(A^\alpha)$. Then, the problem

$$\frac{dq}{dt} + Aq = -QR_\rho(p_{p_o, \varphi}(t) + \varphi(p_{p_o, \varphi}(t))) \quad (87)$$

has a unique solution $q_{p_o, \varphi}(t) \in C_b(\mathbb{R}; QD(A^\alpha))$. Therefore $q_{p_o, \varphi}(0) \in QD(A^\alpha)$ (cf. Lemma 2.3, page 420 in [33]). The mapping $\mathcal{T}$ is defined by $\psi = \mathcal{T}(\varphi)$ with

$$\psi : p_o \in PD(A^\alpha) \rightarrow q_{p_o, \varphi}(0) \in QD(A^\alpha). \quad (88)$$

Let $\sigma(t)$ denote the right-hand side of (87). Then

$$\psi(p_o) = - \int_{-\infty}^0 e^{tA} \sigma(\tau) d\tau. \quad (89)$$

9.3.4. *Existence of an Inertial Manifold for (73) or (80).* Under assumptions $H_1 - H_4$, if it is possible to find N , b , and l , such that $\mathcal{T}$ is a strict contraction of $\mathcal{F}$ into itself, that is, $\exists L < 1$ with $d(\mathcal{T}\varphi_1, \mathcal{T}\varphi_2) \leq Ld(\varphi_1, \varphi_2)$, for all $\varphi_1, \varphi_2 \in \mathcal{F}$, then, $\mathcal{T}$ has a fixed point $\Phi \in \mathcal{F}$, and the graph of Φ is an inertial manifold for (73) or (80).

9.4. Inertial Manifolds for Phase-Field Equations

Example 48 (cf. Bonfioh [39]). Set $\Omega = \prod_{i=1}^n (0, L_i)$, $L_i > 0$, and $n \leq 2$ and denote by N the operator $-\Delta : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$. Consider the phase-field equations (P_ϵ) ,

$$\begin{aligned} \delta \phi_t + N\phi + g(\phi) &= u, \\ \epsilon u_t + \phi_t + Nu &= 0 \end{aligned} \quad (90)$$

in Ω , with the boundary conditions

$$\phi, u|_{\partial\Omega} = 0 \quad (91)$$

and the initial conditions

$$\phi|_{t=0} = \phi_o, \quad u|_{t=0} = u_o, \quad (92)$$

where $\delta > 0$, $\epsilon \in (0, \epsilon_o]$, and $g \in C^2(\mathbb{R})$ satisfies the conditions

$$\begin{aligned} |g'(s)| &\leq C_1(|s|^p + 1), \quad g'(s) \geq -C_2, \\ C_1 > 0, \quad C_2 &\geq 0, \quad p > 0, \quad \forall s \in \mathbb{R}. \end{aligned} \quad (93)$$

The limit problem (P_o) is given by (90) where $\epsilon = 0$:

$$\begin{aligned} \delta \phi_t + N\phi + g(\phi) &= u, \\ \phi_t + Nu &= 0, \end{aligned} \quad (94)$$

the boundary conditions (91), and the initial condition $\phi|_{t=0} = \phi_o$. Set

$$\begin{aligned} H_1 &= H_o^1(\Omega), \quad H_2 = H^2(\Omega) \cap H_o^1(\Omega), \\ H_3 &= \{q \in H_3(\Omega) \cap H_o^1(\Omega); \Delta q \in H_o^1(\Omega)\}. \end{aligned} \quad (95)$$

Let $\|\cdot\|$ denote the usual norm in $L^2(\Omega)$ and set $\|q\|_r = \|N^{r/2}q\|$, $r = 1, 2, 3$. The norms $\|\cdot\|_r$ are equivalent to the usual Sobolev $H^r(\Omega)$ norms on H_r . Several Banach spaces whose norms depend on ϵ are introduced. In particular

$$\mathcal{H}_\epsilon^o = H_1 \times L^2(\Omega), \quad (96)$$

with the norm $\|(p, q)\|_{\mathcal{H}_\epsilon^o} = (\|p\|_1^2 + \epsilon\|q\|_2^2)^{1/2}$.

Under additional assumptions on the regularity of the data, there exist semigroups $S_\epsilon(t)$, global attractors $\mathcal{A}_\epsilon$, and exponential attractors $\mathcal{M}_\epsilon$ for problems (P_ϵ) , $0 \leq \epsilon \leq \epsilon_o$ (cf. Chepyzhov and Vishik [40]). It is proved that $\mathcal{A}_\epsilon$ and $\mathcal{M}_\epsilon$ converge as $\epsilon \rightarrow 0$, to some lifting of $\mathcal{A}_o$ and $\mathcal{M}_o$, in $\mathcal{H}_1^o$

(cf. (96)). Moreover, there exists $r > 0$, independent of ϵ , such that

$$\begin{aligned} B &= \{u \in H_3, \|u\|_3 \leq r\}, \\ B_\epsilon &= \{(\phi, u) \in \mathcal{H}_\epsilon^2, \|(\phi, u)\|_{\mathcal{H}_\epsilon^2} \leq r\} \end{aligned} \quad (97)$$

are bounded absorbing sets for $S(t)$ in H_3 and $S_\epsilon(t)$ in $\mathcal{H}_\epsilon^2$, respectively, where

$$\mathcal{H}_\epsilon^2 = H_3 \times H_2 \quad \text{with the norm } \|(p, q)\|_{\mathcal{H}_\epsilon^2}^2 = \|p\|_3^2 + \epsilon\|q\|_2^2 \quad (98)$$

and there exists $t_1 > 0$ such that $S(t)B \subset B$, $S_\epsilon(t)B_\epsilon \subset B_\epsilon$, for $t \geq t_1$. In particular

$$\tilde{B} = S(t_1)B \subset B, \quad \tilde{B}_\epsilon = S_\epsilon(t_1)B_\epsilon \subset B_\epsilon. \quad (99)$$

Following the classical method presented in the above section for (73), the authors show the existence of an inertial manifold M^r to problem (P_o) and of a family of inertial manifolds M_ϵ^r to problems (P_ϵ) such that

$$M^r \subset \tilde{B}, \quad M_\epsilon^r \subset \tilde{B}_\epsilon. \quad (100)$$

Let $(M^r)_o$ defined by

$$(M^r)_o = \{(\phi, (I + \delta N)^{-1}(N\phi + g(\phi))), \phi \in M_r\}. \quad (101)$$

Moreover, it is proved that

- (1) for all $0 < \epsilon_2 < \epsilon_1$ small enough, $M_{\epsilon_2}^r$ is lower and upper semicontinuous at ϵ_2 , with respect to the metric induced by the $\mathcal{H}_{\epsilon_1}^0$ norm,
- (2) M_ϵ^r converges, in a suitable sense, to $(M^r)_o$ with respect to the metric induced by the $\mathcal{H}_1^0$ norm, as ϵ goes to 0.

10. Mathieu-Hill Type Equations

10.1. *Mathieu and Hill Equations.* The real Mathieu [resp., Hill] equation has the form

$$y'' + (a + b \cos 2\pi x)y = 0, \quad (102)$$

where a and b are constants,

$$[\text{resp., } y'' + (a + bq(x))y = 0], \quad (103)$$

where q is any smooth periodic function of period 1 with mean 0 (see Coddington and Levinson [41] and Wang and Guo [42]). Physical problems leading to Mathieu or Hill equations often require solutions with periodicity, called oscillatory solutions. Therefore, to find conditions on the data for which the above equations have a fundamental system of periodic solutions is a central problem.

10.2. Its Equation. In [43], A. R. Its considers the Schrödinger equation on the positive semiaxis

$$y'' - [x^\beta p(x^{1+\alpha}) + cx^{-2}]y = 0, \quad (104)$$

where p is a smooth periodic function, with period 1 and mean 0, c is a real number, and the parameters α, β satisfy the relations $\beta - \alpha \geq -1$ and $2\alpha - \beta > 0$ (cf. Figure 3).

He proves that (104) has oscillatory solutions when $\beta > \alpha - 1$. If $\beta = \alpha - 1$, the solutions are oscillatory or not. In all cases, asymptotic formulas for the solutions are stated, as $x \rightarrow \infty$. His method is based on a transformation which leads to a Hill-type equation and Floquet functions.

10.3. Method of Diagonalisation. In [44], Bodine and Lutz investigate (104) by a general method of diagonalization with a L^1 -regularity for p . Set

$$\begin{aligned} t = x^{1+\alpha}, \quad \rho(t) &= \int_0^t \frac{p(s)}{1+\alpha} ds, \\ \mu &= \int_0^1 \rho(t) dt, \quad \nu = \int_0^1 (\rho(t))^2 dt \quad (\mu^2 < \nu). \end{aligned} \quad (105)$$

Let

$$\begin{aligned} \lambda_1 &= \frac{1/2 + \sqrt{c + \mu^2 - \nu + 1/4}}{1 + \alpha}, \\ \lambda_2 &= \frac{1/2 - \sqrt{c + \mu^2 - \nu + 1/4}}{1 + \alpha} \end{aligned} \quad (106)$$

be the eigenvalues of a suitable matrix which depends on α, c, μ , and ν , and Λ the diagonal matrix $\text{diag}\{\lambda_1, \lambda_2\}$. The matrix Λ is a crucial tool in the diagonalizable method used by the authors. When $\beta = \alpha - 1$, three cases are studied separately as $\lambda_1 - \lambda_2$ belongs to $\mathbb{N}$, or as it does not belong to $\mathbb{N}_0$ or as it is equal to 0. In each case, very sharp asymptotic formulas are obtained for a fundamental system of solutions to (104), as $t \rightarrow \infty$. In particular, when $\lambda_1 - \lambda_2 \in \mathbb{N}_0$, a logarithmic term appears in the formulas. For instance, when $\lambda_1 - \lambda_2 \in \mathbb{N}$ the following formulas are obtained:

$$\begin{aligned} y_1(x) &= x^{\lambda_1(1+\alpha)} \left[1 + \sum_{r=1}^{N-1} \frac{p_r^1(x^{1+\alpha})}{x^{r(1+\alpha)}} + \mathcal{O}\left(\frac{1}{x^{N(1+\alpha)}}\right) \right] \\ &\quad + \eta(\ln x) y_2(x), \\ y_2(x) &= x^{\lambda_2(1+\alpha)} \left[1 + \sum_{r=1}^{N-1} \frac{p_r^2(x^{1+\alpha})}{x^{r(1+\alpha)}} + \mathcal{O}\left(\frac{1}{x^{N(1+\alpha)}}\right) \right], \end{aligned} \quad (107)$$

where $p_r^{1,2}$ are scalar valued, bounded, continuous, periodic functions with period 1 which are recursively calculated and $\eta \neq 0$ is a suitable number. This logarithmic term is missing in Its formulas.

FIGURE 3: The α - β plane.

11. (Equations with) Memory

Equations with memory are differential equations which are influenced by the past of one or more variables. In this section, I restrict myself to a few examples. I present the equation with memory studied by Dafermos [45] in the context of linear viscoelasticity and examples of singularly perturbed equations with memory.

11.1. Dafermos Results

Definition 49 (cf. Coleman and Noll [46]). A function h is called an *influence function* if

- (i) $h(s)$ is defined for $0 \leq s < \infty$, $h(s) > 0$,
- (ii) there exists $r > 0$ such that $\lim_{s \rightarrow \infty} s^r h(s) = 0$, monotonically for large s .

The function $h(s)$ is said to be of order r .

For example, $h(s) = (s+1)^{-p}$ is an influence function for $r < p$ and $h(s) = e^{-\beta s}$; $\beta > 0$ is an influence function of any order.

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. In [45], Dafermos investigates solutions $t \in (-\infty, +\infty) \rightarrow u(t) \in H$, on $[0, \infty)$, to the integrodifferential equation related to problems in viscoelasticity:

$$\frac{d}{dt}(\rho \dot{u}(t)) + Cu(t) + \int_{-\infty}^t G(t-\tau)u(\tau) d\tau = 0 \quad (108)$$

with specified “history”

$$u(\tau) = v(\tau), \quad \tau \in (-\infty, 0]. \quad (109)$$

Here ρ , C , and $G(t)$ are self-adjoint operators in H , ρ is bounded, and C and $G(t)$ are unbounded with dense domains

$\mathcal{D}(C)$ and $\mathcal{D}[G(t)]$, respectively. Moreover, they satisfy the following conditions:

$$\begin{aligned} \langle \rho w, w \rangle &\geq \rho_o \|w\|^2, \quad \forall w \in H, \\ \langle Cw, w \rangle &\geq c \|w\|^2, \quad \forall w \in \mathcal{D}(C), \\ \langle G(t)w, w \rangle &\leq 0, \quad \forall w \in \mathcal{D}[G(t)], \quad t \in [0, \infty), \end{aligned} \quad (110)$$

with $\rho_o > 0$, $c > 0$. Let H_2 [resp., H_1] be the Hilbert space defined as $\mathcal{D}(C)$ [resp., $\mathcal{D}(C^{1/2})$], equipped with the natural norms, and H_{-1} the dual of H_1 . The injection $H_2 \rightarrow H$ is assumed to be compact. More regularity assumptions on $G(t)$ and $\dot{G}(t)$ imply the existence of a decreasing influence function $h \in C^0[0, \infty)$ such that

$$\int_0^\infty \left[\|G(t)\|_{\mathcal{L}(H_1, H_{-1})} + \|\dot{G}(t)\|_{\mathcal{L}(H_1, H_{-1})} \right] h^{-2}(t) dt < \infty. \quad (111)$$

In the following, h is a fixed influence function which satisfies (111). Set $I = (-\infty, 0]$. For $k = 0, 1, \dots$, in [45], the author introduces the Banach spaces of *fading memory type* (cf. [46]) $\mathcal{E}_k$ [resp., $\mathcal{B}_k$] of sufficiently smooth functions $w : I \rightarrow H_1$ [resp., $w : I \rightarrow H_2$] such that

$$\begin{aligned} \|w\|_{\mathcal{E}_k} &= \sum_{j=0}^1 \sum_{i=0}^{k+1-j} \sup_I \left[h(-t) \|w^{(i)}(t)\|_j \right] < \infty \\ \left[\text{resp., } \|w\|_{\mathcal{B}_k} &= \sum_{j=0}^2 \sum_{i=0}^{k+2-j} \sup_I \|w^{(i)}(t)\|_j < \infty \right]. \end{aligned} \quad (112)$$

Then, $\mathcal{B}_k \subset \mathcal{E}_{k+1} \subset \mathcal{E}_k$, and the injection $\mathcal{B}_k \rightarrow \mathcal{E}_k$ is compact. Let $T > 0$ and set $I_T = (-\infty, T]$. If $v \in \mathcal{E}_k$ [resp., $\mathcal{B}_k$], he proves the existence of a unique solution u to (108) on $[0, T]$ and (109) on $(-\infty, 0]$. Moreover, si $v \in \mathcal{E}_k$ [resp., $\mathcal{B}_k$], u satisfies

$$\begin{aligned} \sum_{j=0}^1 \sum_{i=0}^{k+1-j} \sup_{[0, T]} \|u^{(i)}(t)\|_j &\leq c \|v\|_{\mathcal{E}_k} \\ \left[\text{resp., } \sum_{j=0}^2 \sum_{i=0}^{k+2-j} \sup_{[0, T]} \|u^{(i)}(t)\|_j &\leq c \|v\|_{\mathcal{B}_k} \right]. \end{aligned} \quad (113)$$

The following *stability results* are obtained. Set

$$A = C + \int_0^\infty G(t) dt. \quad (114)$$

It is assumed that

$$\begin{aligned} \langle \dot{G}(t)w, w \rangle &\geq 0 \quad \forall w \in H_1, \quad t \in [0, \infty), \\ \int_0^\infty \|G(t)\|_{\mathcal{L}(H_2; H)} dt &\equiv M_\infty < 1, \\ \langle Aw, w \rangle &\geq a_o \|w\|_1^2, \quad a_o > 0, \quad \forall w \in H_1. \end{aligned} \quad (115)$$

The new function

$$\begin{aligned} \omega : (v, \xi) \in \mathcal{E}_k \times [0, \infty) &\longrightarrow \omega[(v, \xi)](\tau) \\ &= u(\xi + \tau) \in \mathcal{E}_k, \quad \tau \in I, \end{aligned} \quad (116)$$

where u is the solution of (108) on $[0, \infty)$ and (109) on $(-\infty, 0]$, satisfying

$$\begin{aligned} \omega(v, 0) &= v \quad \forall v \in \mathcal{E}_k, \\ \omega(v, \xi + \xi') &= \omega[\omega(v, \xi), \xi'], \quad \forall \xi, \xi' \in [0, \infty), \quad v \in \mathcal{E}_k, \end{aligned} \quad (117)$$

that is, ω looks like a dynamical system on $\mathcal{E}_k$ (or $\mathcal{B}_k$), but it is not necessarily continuous on $\mathcal{E}_k \times [0, \infty)$. Therefore, ω is not a classical dynamical system (cf. Section 6). Nevertheless, it is possible to define the following Lyapunov function $V_k(v)$, for ω , on $\mathcal{E}_k$:

$$\begin{aligned} V_k(v) &= \frac{1}{2} \langle \rho u^{k+1}(t), u^{k+1}(t) \rangle + \frac{1}{2} \langle Au^k(t), u^k(t) \rangle \\ &\quad - \frac{1}{2} \int_{-\infty}^t \langle G(t-\tau)(u^k(t) - u^k(\tau)), \\ &\quad \quad u^k(t) - u^k(\tau) \rangle d\tau \geq 0 \end{aligned} \quad (118)$$

whose derivative, with respect to t , satisfies $\dot{V}_k(v)(t) \leq 0$. Here u is solution of (108) and (109). Under suitable spectral conditions on G , via an adaptation of Hale's results [21], the following stability results are obtained for the solution u of (108) on $[0, \infty)$ and of (109) on $(-\infty, 0]$, when $v \in \mathcal{E}_m$, $m = 0, 1, \dots$:

$$\begin{aligned} u^{(i)}(t) &\longrightarrow 0 \quad \text{in } H_1 \text{ as } t \longrightarrow \infty, \quad i = 0, 1, \dots, m, \\ u^{m+1}(t) &\longrightarrow 0 \quad \text{in } H \text{ as } t \longrightarrow \infty. \end{aligned} \quad (119)$$

11.2. Equations with Memory and Singular Perturbations

Example 50. In [47], Gatti et al. consider a family of Banach spaces H_ϵ and a family of semigroups $S_\epsilon(t) : H_\epsilon \rightarrow H_\epsilon$, $\epsilon \in [0, 1]$. They state sufficient conditions on H_ϵ and S_ϵ which imply the existence of exponential attractors $\mathcal{E}_\epsilon$ for $S_\epsilon(t)$, $\epsilon \in [0, 1]$ such that

$$\begin{aligned} \text{dist-sym}_{H_\epsilon}(\mathcal{E}_\epsilon, \mathcal{E}_o) &\leq C\epsilon^\delta, \\ \text{dist-sym}_{H_{\epsilon_2}}(\mathcal{E}_{\epsilon_1}, \mathcal{E}_{\epsilon_2}) &\leq C(\epsilon_1 - \epsilon_2)^\delta, \quad \forall \epsilon_1 \geq \epsilon_2 > 0, \end{aligned} \quad (120)$$

with $C \geq 0$ and $\delta \in (0, 1)$. The results are applied to evolution equations with *memory* of the form

$$\begin{aligned} \partial_t x(t) + \int_0^\infty k_\epsilon(s) B_o(x(t-s)) ds + B_1(x(t)) &= 0, \\ t &\in [0, \infty), \end{aligned} \quad (121)$$

with specified history

$$x(t) = v(t), \quad t \in (-\infty, 0], \quad (122)$$

where v is a given function, $k : [0, \infty) \rightarrow [0, \infty)$ is a convex summable function with $\int_0^\infty k(s)ds = 1$ and $k_\epsilon(s) = (1/\epsilon)k(s/\epsilon)$, and B_0, B_1 are densely defined, possibly nonlinear operators in a suitable Banach space. As $\epsilon \rightarrow 0$, the formal limit to problem (121) and (122) is

$$\partial_t x(t) + B_0(x(t)) + B_1(x(t)) = 0, \quad x(0) = v(0). \quad (123)$$

Example 51. This example is already presented in [7, Example 4]. In [48], Conti et al. are interested in the convergence, in an appropriate sense, as $\epsilon \rightarrow 0$, of the solution $u_\epsilon(x, t)$, $x \in \Omega$, $t \in \mathbb{R}$ to problem (p_ϵ) :

$$\begin{aligned} u_t - \omega \Delta u - (1 - \omega) \int_0^\infty k_\epsilon(s) \Delta u(t - s) ds \\ + \varphi(u) = f, \quad t > 0, \end{aligned} \quad (124)$$

with Dirichlet boundary conditions on the boundary of the smooth bounded domain Ω in $\mathbb{R}^3$, $\omega > 0$; φ is a suitable nonlinearity, f is a time-independent source term, the memory kernel k_ϵ converges to the Dirac distribution at the origin, $u(x, t)$ is supposed to be a given datum for $t \leq 0$. Let $A = -\Delta$ on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$, $H^r = \mathcal{D}(A^{r/2})$, $r \in \mathbb{R}$. Following Dafermos [45] and Grasselli and Pata [49], they introduce, under additional assumptions on k, φ, f , ($\varphi(x) = x^3 - x$ is allowed), the auxiliary variable $\eta^t(x, s) = \int_0^s u(x, t - y)dy$, the functions $\mu(s) = -(1 - \omega)k'(s)$, $\mu_\epsilon(s) = (1/\epsilon^2)\mu(s/\epsilon)$, and the Hilbert spaces $\mathcal{M}_\epsilon^r = L_\mu^2(\mathbb{R}^+, H^{r+1})$ and $\mathcal{H}_\epsilon^r = H^r \times \mathcal{M}_\epsilon^r$ for $\epsilon > 0$, $\mathcal{H}_0^r = H^r$. The correct reformulation of p_ϵ , in the framework of dynamical systems, is (P_ϵ) : find $(u_\epsilon, \eta_\epsilon) \in C([0, \infty), \mathcal{H}_\epsilon^0)$ solution to

$$\begin{aligned} u_t + \omega A u + \int_0^\infty \mu_\epsilon(s) A \eta(s) ds + \varphi(u) = f, \\ \partial_t \eta = -\partial_s \eta + u \end{aligned} \quad (125)$$

for $t > 0$, associated with the initial condition $(u_0, \eta_0) \in \mathcal{H}_\epsilon^0$. The existence, for $\epsilon > 0$, of a strongly continuous semigroup $S_\epsilon(t)$ on $\mathcal{H}_\epsilon^0$ corresponding to P_ϵ and of an exponential attractor $\mathcal{E}_\epsilon$ for $S_\epsilon(t)$ is proved. The convergence of $\mathcal{E}_\epsilon$ is also studied.

12. Nodes and Nodal

12.1. Nodal Points, Curves, Surfaces, or Nodes

Definition 52. In Courant and Hilbert [50], they are defined as points [resp., curves, surfaces], on which some eigenfunctions of a differential problem, in a domain $G \in \mathbb{R}^N$, $N = 1$ [resp., $N = 2, 3$], vanish.

Example 53. In the 1970 edition of [50, page 452], a second-order Sturm-Liouville problem is considered. It is proved that the nodes of the n th eigenfunction u_n divide the domain into no more than n subdomains.

12.2. Nodal Sets, Nodal Domains, and Nodal Solutions

Example 54. In [51], Hoffmann-Ostenhof et al. study the Dirichlet eigenvalue problem

$$-\Delta u_i = \lambda_i u_i, \quad i = 1, 2, \dots, \quad u \in H_0^1(D), \quad (126)$$

where D is a bounded domain in $\mathbb{R}^n$. The nodal set u_i is defined as

$$\mathcal{N}(u_i) = \overline{\{x \in D : u_i(x) = 0\}}. \quad (127)$$

The nodal domains of u_i are the connected components of $D \setminus \mathcal{N}(u_i)$. The authors construct a domain $D \subset \mathbb{R}^2$ on which the second eigenvalue has a nodal set disjoint from the boundary, whereas the nodal line conjecture, first mentioned by Payne [52], stated that

$$\mathcal{N}(u_2) \cap \partial D \neq \emptyset. \quad (128)$$

Example 55. In [53], Bartsch and Weth consider the nonlinear elliptic Dirichlet problem:

$$-\epsilon^2 \Delta u + au = f(u) \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad (129)$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $a > 0$, ϵ is a small positive parameter, and f grows superlinearly and subcritically. They study the number of nodal solutions, that is, sign-changing solutions of (129) and their nodal domains. They show that the number of nodal solutions can be expressed as a Lyusternik-Schnirelman category (cf. Lyusternik and Schnirelman [54]), of a suitable inclusion between two spaces which involve the shape of $\partial\Omega$.

12.3. N-Mode Solution. In [55], Urano et al. consider the following ordinary differential problem:

$$\begin{aligned} \epsilon^2 u''(x) + f(x, u(x)) &= 0, \\ x \in]0, 1[; \quad u'(0) &= u'(1) = 0, \end{aligned} \quad (130)$$

where $f(x, u) = u(1 - u)(u - a(x))$; a is a $C^2[0, 1]$ function such that $0 < a(x) < 1$, $a'(0) = a'(1) = 0$, and the subsets Σ and Λ of $(0, 1)$, where $a(x) = 1/2$ and $a'(x) = 0$, respectively, are finite sets and $a'(x) \neq 0$ for any $x \in \Sigma$. The authors study the asymptotic behavior, as $\epsilon \rightarrow 0$, of N -mode solutions u_ϵ of (130); that is, solutions such that $v_\epsilon = u_\epsilon - a$ have exactly N zero-points in $(0, 1)$, and show that any transition layer [resp., spike] is located in a neighborhood of a point of Σ [resp., Λ].

13. Resonances

Only a few aspects of this large topic are presented here. More aspects are in preparation.

13.1. The Case of a Vibrating String

13.1.1. Free Motion (cf. Schwartz [56]). Consider a homogeneous vibrating string, with linear density ρ , subjected to a

constant tension μ . The position of the string, fixed at the end points $x = 0, x = L$, is given by solutions of the wave equation

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \text{ in } [0, L] \quad (131)$$

with the boundary conditions

$$u(t, 0) = u(t, L) = 0, \quad (132)$$

where $v = \sqrt{\mu/\rho}$ has the dimension of a velocity. To solve problem (131) and (132) the initial values of the position and of the velocity of the string are needed; that is,

$$u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = u_1(x). \quad (133)$$

Solutions $u(x, t)$ of the above initial value problem of the form $u(x, t) = U(x)V(t)$ are

$$u_k(x, t) = \sin \frac{k\pi x}{L} \left(A_k \cos \frac{k\pi vt}{L} + B_k \sin \frac{k\pi vt}{L} \right), \quad (134)$$

where $k > 0$ is an integer and A_k, B_k are constants. The numbers $\lambda = k^2\pi^2/L^2$, such that

$$U''(x) + \lambda U(x) = 0, \quad U(0) = U(L) = 0, \quad (135)$$

are called the *eigenvalues* and $\sin(k\pi x/L)$ the *eigenfunctions* of the problem. The constants A_k, B_k are the Fourier coefficients of suitable extensions of u_0 and u_1 to periodic functions.

13.1.2. The Case of a Forced Motion: Resonance (cf. Courant and Hilbert [50]). We suppose that the vibrating string is under the influence of an external force which has the Fourier expansion:

$$\mathcal{Q}(x, t) = \sum_{k=1}^{\infty} Q_k(t) \sin \frac{k\pi x}{L}. \quad (136)$$

Then, the deflection of the string is solution to

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \mathcal{Q}(x, t), \quad u(0, t) = u(L, t) = 0 \quad (137)$$

with the initial conditions (133). We look for a solution of the form

$$u(x, t) = \sum_{k=1}^{\infty} N_k(t) \sin \frac{k\pi x}{L}. \quad (138)$$

The coefficients $N_k(t)$ must be solutions of the equations

$$\frac{1}{v^2} N_k''(t) + \frac{k^2\pi^2}{L^2} N_k(t) = Q_k(t). \quad (139)$$

The general solution of the homogeneous equation associated with (139) is

$$a_k \cos \frac{k\pi vt}{L} + b_k \sin \frac{k\pi vt}{L}. \quad (140)$$

A particular solution of (139) is obtained by the method of variation of constants (cf. Coddington [57]). Finally,

$$N_k(t) = \frac{1}{(k\pi/L)} \int_0^t Q_k(u) \sin \frac{k\pi}{L} (t-u) du + a_k \cos \frac{k\pi vt}{L} + b_k \sin \frac{k\pi vt}{L}, \quad (141)$$

where the constants a_k, b_k are determined by the initial conditions (133).

Now, suppose $\mathcal{Q}_k(t) = \alpha \cos \omega t + \beta \sin \omega t$. Then, if $\omega \neq k\pi/L$, $u(x, t)$, in (138), is a linear combination of a sinusoidal function of frequency ω and one of frequencies $k\pi/L$. But, if $\omega = k\pi/L$, $N_k(t)$ contains terms of the forms $t \sin t$ and $t \cos t$ which are unbounded, we say that *resonance* occurs.

13.2. Resonances as Poles of Complex Functions (cf. Zworski [58]). Set $X = [0, L]$ with $L = \pi$; for simplicity, $P_V = -(\partial^2/\partial x^2) + V(x)$, $V \in C^\infty(X)$, $V(x) \geq 0$, and

$$U(x, t) = \begin{pmatrix} u(t, x) \\ -i\partial_t u(t, x) \end{pmatrix}, \quad \mathcal{P}_V = \begin{pmatrix} 0 & 1 \\ P_V & 0 \end{pmatrix}. \quad (142)$$

Consider the system

$$\begin{aligned} \frac{1}{i} \partial_t U &= \mathcal{P}_V U, & U(t, 0) &= U(t, \pi) = 0, \\ U(0, x) &= \begin{pmatrix} u_0(x) \\ -iu_1(x) \end{pmatrix}. \end{aligned} \quad (143)$$

The solution to system (143) involves *eigenvalues* and *eigenfunctions* of

$$\mathcal{P}_V W = \lambda W, \quad W|_{\partial X} = 0. \quad (144)$$

Consider the Hilbert space $\mathcal{H}$, closure of $C_c^\infty(X) \times C_c^\infty(X)$ for the inner product

$$\begin{aligned} \langle \Phi, \Psi \rangle &= \langle P_V \phi_1, \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \\ \Phi &= \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned} \quad (145)$$

Then $\mathcal{P}_V$ is an unbounded self-adjoint operator on $\mathcal{H}$ and its eigenvalues are real. Let $\sigma(\mathcal{P}_V)$ be the spectrum of $\mathcal{P}_V$. It is pointed out, in [58], that

$$\lambda \in \sigma(\mathcal{P}_V) \iff \lambda \text{ is a pole of } (\mathcal{P}_V - z)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}. \quad (146)$$

Remark 56. When $V(x) = 0$, system (143) is equivalent to problem (131)–(133), and the eigenvalues of

$$\mathcal{P}_0 W = \lambda W, \quad W|_{\partial X} = 0 \quad (147)$$

are $\lambda = k \in \mathbb{Z} - \{0\}$ which correspond to the eigenvalues k^2 of problem (135).

More generally, consider the problem with a dissipative term $a\partial_t, a > 0$:

$$\left(-\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} - P_V\right)u(t, x) = 0, \quad t > 0, \quad x \in X, \quad u|_{\partial X} = 0. \quad (148)$$

It can be rewritten as

$$\frac{1}{i}\partial_t U(t, x) = \mathcal{P}_a U(t, x), \quad u|_{\partial X} = 0, \quad (149)$$

where U is defined in (142) and

$$\mathcal{P}_a = \begin{pmatrix} 0 & 1 \\ P_V & ia \end{pmatrix}. \quad (150)$$

The operator $\mathcal{P}_a$ is not self-adjoint on $\mathcal{H}$. The *eigenvalues* and the *eigenfunctions* of $\mathcal{P}_a$ are still defined by

$$\mathcal{P}_a W = \lambda W, \quad W|_{\partial X} = 0, \quad (151)$$

and the solutions of (149) are given by superpositions of solutions of the form

$$U(t, x) = \exp(-it\lambda) W(x), \quad (152)$$

but the eigenvalues $\lambda = \alpha + i\beta$ are not real any more. An elementary calculus shows that $\lambda(\lambda - ia) = \mu^2$, where $\mu \in \mathbb{R}$ is an eigenvalue of P_V ; that is, $\beta = a/2$, $\alpha^2 = \mu^2 - (a^2/4)$. Here, again, the eigenvalues of $\mathcal{P}_a$ are poles of $(\mathcal{P}_a - z)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$. Now, since $\alpha \notin \sigma(P_a)$, for a given f , the equation

$$(\mathcal{P}_a - \alpha)u = f \quad (153)$$

has a solution. But, if a is small enough, $\lambda = \alpha + (ia/2)$ is a pole of $(\mathcal{P}_a - z)^{-1}$ very close to α , and, for a suitable choice of f , the solution of (153) can be “enormous” (cf. [58]). We say that a *resonance* occurs.

13.3. Quasi-Periodic Schrödinger Operator, the Resonant Case. In their papers [59–61] Fedotov and Klopp study the spectrum of quasi-periodic Schrödinger operators of the form

$$H_{z,\epsilon} = -\frac{d^2}{dx^2} + V(x - z) + \alpha \cos(\epsilon x), \quad (154)$$

where $\alpha > 0$, $V \in L^2_{\text{loc}}(\mathbb{R})$ is periodic with period 1, z is a real parameter indexing the equations of the family, and $\epsilon > 0$ is such that $2\pi/\epsilon$ is irrational. Let

$$H_0 = -\frac{d^2}{dx^2} + V(x) \quad (155)$$

be the periodic Schrödinger unperturbed operator. Its spectrum, on $L^2(\mathbb{R})$, consists of the union of intervals $[E_{2n-1}, E_{2n}]$, $n \geq 1$, such that $E_n \rightarrow \infty$, as $n \rightarrow \infty$. It is assumed that all the spectral gaps (E_{2n}, E_{2n+1}) are open. The spectrum of (154), at an energy E , depends on the relative position of the spectral window $\mathcal{F}(E) = [E - \alpha, E + \alpha]$ with respect to the spectrum of H_0 . In [59] four cases are distinguished as $\mathcal{F}(E)$ lies inside

FIGURE 4: Spectral band.

or as it completely covers a spectral band of H_0 or it contains exactly one edge or two edges (Figure 4) of a spectral band.

In each case, under specific additional assumptions, the authors describe the nature of the spectrum of (154) and state asymptotic formulas, as $\epsilon \rightarrow 0$.

In [60], they are interested in the spectrum of (154), in intervals J such that, for all $E \in J$, $\mathcal{F}(E)$ covers the edges of two neighboring spectral bands of H_0 and the spectral gap located between them (Figure 4). Let Γ_R be the real iso-energy curve associated with (154) and γ_0, γ_π the connected components of Γ_R in a periodicity cell. To each of these loops, one associates a sequence of energies in J , E_a^n , $a = \{0, \pi\}$, and, near each E_a^n , an exponentially small interval I_a^n such that the spectrum of (154), in J , is contained in the union of these intervals. The location and the nature of the spectrum of (154) are investigated in the union $I_0^n \cup I_\pi^m$, in the *resonant case*, that is, when I_0^n and I_π^m intersect with each other.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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