Research Article

On Ordered Quasi-Gamma-Ideals of Regular Ordered Gamma-Semigroups

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We introduce the notion of ordered quasi- Γ -ideals of regular ordered Γ -semigroups and study the basic properties of ordered quasi- Γ -ideals of ordered Γ -semigroups. We also characterize regular ordered Γ -semigroups in terms of their ordered quasi- Γ -ideals, ordered right Γ -ideals, and left Γ -ideals. Finally, we have shown that (i) a partially ordered Γ -semigroup S is regular if and only if for every ordered bi- Γ -ideal B , every ordered Γ -ideal I , and every ordered quasi- Γ -ideal Q , we have $B \cap I \cap Q \subseteq (B\Gamma I\Gamma Q)$ and (ii) a partially ordered Γ -semigroup S is regular if and only if for every ordered quasi- Γ -ideal Q , every ordered left Γ -ideal L , and every ordered right- Γ -ideal R , we have that $R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L)$.

1. Introduction

Steinfeld [1–3] introduced the notion of a quasi-ideal for semigroups and rings. Since then, this notion has been the subject of great attention of many researchers and consequently a series of interesting results have been published by extending the notion of quasi-ideals to Γ -semigroups, ordered semigroups, ternary semigroups, semirings, Γ -semirings, regular rings, near-rings, and many other different algebraic structures [4–15].

It is a widely known fact that the notion of a one-sided ideal of rings and semigroups is a generalization of the notion of an ideal of rings and semigroups and the notion of a quasi-ideal of semigroups and rings is a generalization of a one-sided ideal of semigroups and rings. In fact the concept of ordered semigroups and Γ -semigroups is a generalization of semigroups. Also the ordered Γ -semigroup is a generalization of Γ -semigroups. So the concept of ordered quasi-ideals of ordered semigroups is a generalization of the concept of quasi-ideals of semigroups. In the same way, the notion of an ordered quasi-ideal of ordered semigroups is a generalization of a one-sided ordered ideal of ordered semigroups. Due to these motivating facts, it is naturally significant to generalize the results of semigroups to Γ -semigroups and of Γ -semigroups to ordered Γ -semigroups.

In 1998, the concept of an ordered quasi-ideal in ordered semigroups was introduced by Kehayopulu [16]. He studied theory of ordered semigroups based on ordered ideals analogous to the theory of semigroups based on ideals. The concept of po- Γ -semigroup was introduced by Kwon and Lee in 1996 [17] and since then it has been studied by several authors [18–22]. Our purpose in this paper is to examine many important classical results of ordered quasi- Γ -ideals in ordered Γ -semigroups and then to characterize the regular ordered Γ -semigroups through ordered quasi- Γ -ideals, ordered bi- Γ -ideals and ordered one-sided Γ -ideals.

2. Preliminaries

We note here some basic definitions and results that are relevant for our subsequent results.

Let S and Γ be two nonempty sets. Then S is called a Γ -semigroup if S satisfies $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. A nonempty subset N of a Γ -semigroup S is called a sub- Γ -semigroup of S if $a\alpha b \in N$ for all $a, b \in N$ and $\alpha \in \Gamma$. For any nonempty subsets A, B of S , $A\Gamma B = \{a\alpha b : a \in A, b \in B \text{ and } \alpha \in \Gamma\}$. We also denote $\{a\}\Gamma B$, $A\Gamma\{b\}$, and $\{a\}\Gamma\{b\}$, respectively, by $a\Gamma B$, $A\Gamma b$, and $a\Gamma b$. Many classical results of semigroups have been generalized and extended to Γ -semigroups [23–25]. By an ordered Γ -semigroup S

(also called po- Γ -semigroups), we mean an ordered set $(S, \leq)$, at the same time a Γ -semigroup satisfying the following conditions:

$$a \leq b \implies a\gamma c \leq b\gamma c, \quad c\gamma a \leq c\gamma b \quad \forall a, b, c \in S, \gamma \in \Gamma. \quad (1)$$

Throughout this paper, S will stand for an ordered Γ -semigroup unless otherwise stated. An ordered Γ -semigroup S is called regular if for each $s \in S$ and for each $\alpha, \beta \in \Gamma$ there exists $a \in S$ such that $s \leq s\alpha a\beta s$. Equivalent definitions of regular ordered Γ -semigroup are as follows: (i) $A \subseteq (A\Gamma S\Gamma A)$ for each $A \subseteq S$ and (ii) $s \in (s\Gamma S\Gamma s)$ for each $s \in S$. Let $(S; \leq)$ be an ordered Γ -semigroup and N a sub- Γ -semigroup of S ; then $(N; \leq)$ is an ordered Γ -semigroup. Let A be a nonempty subset of N . Then similarly to [26], we write $(A)_N = \{n \in N : n \leq a \text{ for some } a \in A\}$ and $A \cup a = A \cup \{a\}$. We also write $(A)_N$ by simply (A) if $N = S$ (see [27]). A nonempty subset I of an ordered Γ -semigroup S is called an ordered right- Γ -ideal (left- Γ -ideal) of S if $I\Gamma S \subseteq I(S\Gamma I \subseteq I)$, and for any $x \in I$, $(x) \subseteq I$. I is called an ordered Γ -ideal of S if it is both a left and a right Γ -ideals of S . Also for any $s \in S$, we have that $(S\Gamma s)$ is an ordered left Γ -ideal of S and $(s\Gamma S)$ is an ordered right Γ -ideal of S [18]. A nonempty subset Q of S is called an ordered quasi- Γ -ideal of S if (i) $(Q\Gamma S) \cap (S\Gamma Q) \subseteq Q$ and (ii) $(Q) \subseteq Q$. A sub- Γ -semigroup B of an ordered Γ -semigroup S is called an ordered bi- Γ -ideal of S if $B\Gamma S\Gamma B \subseteq B$ and for any $x \in B$, $(x) \subseteq B$.

Let X be a nonempty subset of S . Then the least right (left) ordered Γ -ideal of S containing X is given by $R(X) = (X \cup X\Gamma S)(L(X) = (S\Gamma X \cup X))$. If $X = \{s\}$, $s \in S$, we write $R\{s\}$ and $L\{s\}$, respectively, by $R(s)$ and $L(s)$, and $R(s) = (s \cup s\Gamma S)$, $L(s) = (S\Gamma s \cup s)$ and the ideal generated by $s \in S$ is given by $I(s) = (s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S)$. Also, the least quasi- Γ -ideal of S containing X is denoted by $Q(X)$. Moreover, we will need some notations as follows: (i) $N_Q = \{Q : Q \neq \emptyset, \text{ where } Q \subseteq S \text{ and } (Q) \subseteq Q\}$, (ii) R_I is a set of ordered right Γ -ideals of S , (iii) L_I is a set of ordered left Γ -ideals of S , and (iv) I_T is a two-sided Γ -ideal of S .

Now for any two elements $Q_1, Q_2 \in N_Q$, we define an operation $*$ in N_Q as follows:

$$Q_1 * \Gamma * Q_2 = (Q_1\Gamma Q_2). \quad (2)$$

Further, let N be a sub- Γ -semigroup of S . Then we can easily observe here the following (see [16, 18, 21, 28–30]):

- (i) $A \subseteq (A)_N \subseteq (A) = ((A))$ for $A \subseteq N$,
- (ii) for $A \subseteq N$ and $B \subseteq N$, we have $(A \cup B) = (A) \cup (B)$,
- (iii) for $A \subseteq N$ and $B \subseteq N$, we have $(A \cap B) \subseteq (A) \cap (B)$,
- (iv) for a and $b \in N$ with $a \leq b$, we have $(a\Gamma N) \subseteq (b\Gamma N)$ and $(NTa) \subseteq (NTb)$,
- (v) $(A)\Gamma(B) \subseteq (A\Gamma B)$,
- (vi) for every left (right, two-sided) ideal L of S , $(L) = L$,
- (vii) if A and B are ordered Γ -ideals of S , then $(A\Gamma B)$ and $A \cup B$ are also ideals of S ,
- (viii) for any $s \in S$, $(S\Gamma s\Gamma S)$ is an ideal of S .

3. Ordered Γ -Semigroups and Ordered Quasi- Γ -Ideals

In this section, we study some classical properties of the ordered Γ -semigroup S . We start with the following lemma.

Lemma 1. *Let S be an ordered Γ -semigroup. Then,*

- (i) $(N_Q, *, \subseteq)$ is an ordered Γ -semigroup;
- (ii) $(L_I, *, \subseteq)$, $(R_I, *, \subseteq)$, and $(I_T, *, \subseteq)$ are sub- Γ -semigroups of $(N_Q, *, \subseteq)$.

Proof. (i) Suppose $P, Q, R \in N_Q$. Since $P\Gamma Q \in (P\Gamma Q)$, we obtain $((P\Gamma Q)\Gamma R) \subseteq ((P\Gamma Q)\Gamma R)$. Next, we have $(P * \Gamma * Q) * \Gamma * R = (P\Gamma Q\Gamma R)$ by using $(P * \Gamma * Q) * \Gamma * R = (P\Gamma Q) * \Gamma * R = ((P\Gamma Q)\Gamma R) \subseteq ((P\Gamma Q)\Gamma R) = (P\Gamma Q\Gamma R)$. In a similar way, we can show that $P * \Gamma * (Q * \Gamma * R) = (P\Gamma Q\Gamma R)$ and therefore $(P * \Gamma * Q) * \Gamma * R = P * \Gamma * (Q * \Gamma * R)$. Hence $(N_Q, *)$ is a Γ -semigroup. Suppose $P \subseteq Q$. Then $P * \Gamma * R = (P\Gamma Q) \subseteq (Q\Gamma R) = Q * \Gamma * R$ and $R * \Gamma * P = (R\Gamma P) \subseteq (R\Gamma Q) = R * \Gamma * Q$. Hence $(N_Q, *, \subseteq)$ is an ordered Γ -semigroup.

(ii) We have that L_I, R_I , and I_T are nonempty subsets of N_Q . Suppose $L_1, L_2 \in L_I$. Then, obviously, we have $(L_1 * \Gamma * L_2) = ((L_1\Gamma L_2)) = (L_1\Gamma L_2)$. Moreover, using

$$\begin{aligned} S\Gamma(L_1 * \Gamma * L_2) &= S\Gamma(L_1\Gamma L_2) \\ &\subseteq (S\Gamma(L_1\Gamma L_2)) \\ &\subseteq ((S\Gamma L_1)\Gamma L_2) \\ &\subseteq (L_1\Gamma L_2) \\ &= L_1 * \Gamma * L_2, \end{aligned} \quad (3)$$

we infer that $L_1 * \Gamma * L_2$ is a left Γ -ideal of S ; that is, $L_1 * \Gamma * L_2 \in L_I$. Thus $(L_I, *, \subseteq)$ is a sub- Γ -semigroup of $(N_Q, *, \subseteq)$.

Dually, we can prove that $(R_I, *, \subseteq)$ is a sub- Γ -semigroup of $(N_Q, *, \subseteq)$. Since $I_T = L_I \cap R_I$, it follows that $(I_T, *, \subseteq)$ is a sub- Γ -semigroup of $(N_Q, *, \subseteq)$. $\square$

Let $Q_I = \{Q : Q \text{ is an ordered quasi-}\Gamma\text{-ideal of } S\}$. Then, obviously we have $L_I \cup R_I \subseteq Q_I \subseteq N_Q$. This implies that every one-sided Γ -ideal of an ordered Γ -semigroup is a quasi- Γ -ideal of S . Thus the class of ordered quasi- Γ -ideals of S is a generalization of the class of one-sided ordered Γ -ideals of S .

Lemma 2. *Each ordered quasi- Γ -ideal Q of an ordered Γ -semigroup S is a sub- Γ -semigroup of S .*

Proof. Proof is straightforward. In fact, we have $Q\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q \subseteq (Q\Gamma S) \cap (S\Gamma Q) \subseteq Q$. $\square$

Lemma 3. *For every ordered right Γ -ideal R and an ordered left Γ -ideal L of an ordered Γ -semigroup S , $R \cap L$ is an ordered quasi- Γ -ideal of S .*

Proof. As $R\Gamma L \subseteq S\Gamma L \subseteq L$ and $R\Gamma L \subseteq R\Gamma S \subseteq R$, we obtain $R\Gamma L \subseteq R \cap L$, so $R \cap L \neq \emptyset$. Now the fact that $R \cap L$ is an ordered quasi- Γ -ideal of S follows from the following:

$$(i) (R \cap L) \subseteq (R) \cap (L) \subseteq R \cap L,$$

$$(ii) ((R \cap L) \Gamma S] \cap (S \Gamma (R \cap L))] \subseteq (R \Gamma S] \cap (S \Gamma L] \subseteq (R] \cap (L] \subseteq R \cap L.$$

□

Lemma 4. Let Q be an ordered quasi- Γ -ideal of S , then one obtains $Q = L(Q) \cap R(Q) = (S \Gamma Q \cup Q] \cap (Q \cup Q \Gamma S]$.

Proof. The following relation

$$Q \subseteq (S \Gamma Q \cup Q] \cap (Q \cup Q \Gamma S] \text{ is obvious.} \quad (4)$$

Conversely, suppose $a \in (S \Gamma Q \cup Q] \cap (Q \cup Q \Gamma S]$. Then $a \leq b$ or $a \leq x \alpha u$ and $a \leq v \beta y$ for some $b, u, v \in Q, x, y \in S$, and $\alpha, \beta \in \Gamma$. As Q is an ordered quasi- Γ -ideal of S , the former case implies that $a \in (Q] \subseteq Q$ and the latter case implies that $a \in (S \Gamma Q] \cap (Q \Gamma S] \subseteq Q$. Therefore $(S \Gamma Q \cup Q] \cap (Q \cup Q \Gamma S] = Q$. □

We recall here that if X is a nonempty subset of an ordered Γ -semigroup S , then we write the least quasi-ideal of S containing X by $Q(X)$. If $X = \{a\}$, we write $Q(\{a\})$ by $Q(a)$.

Theorem 5. Suppose S is an ordered Γ -semigroup. Then one has the following:

- (i) for every $s \in S$, $Q(s) = L(s) \cap R(s) = (S \Gamma s \cup s] \cap (s \cup s \Gamma S]$,
- (ii) let $\emptyset \neq X \subseteq S$, $Q(X) = L(X) \cap R(X) = (S \Gamma X \cup X] \cap (X \cup X \Gamma S]$.

Proof of (i). Suppose $s \in S$. Using Lemma 3, $L(s) \cap R(s)$ is a quasi- Γ -ideal of S containing s ; therefore $Q(s) \subseteq L(s) \cap R(s)$, and by Lemma 4, we obtain

$$\begin{aligned} L(s) \cap R(s) &= (S \Gamma s \cup s] \cap (s \cup s \Gamma S] \\ &\subseteq (S \Gamma Q(s) \cup Q(s)] \cap (Q(s) \cup Q(s) \Gamma S] \quad (5) \\ &= Q(s). \end{aligned}$$

Hence $Q(s) = L(s) \cap R(s)$. □

Proof of (ii). Its proof can be given as (i). □

The notion of a bi- Γ -ideal of Γ -semigroups is a generalization of the notion of a quasi- Γ -ideal of Γ -semigroups. Similarly, the class of ordered quasi- Γ -ideals of ordered Γ -semigroups is a particular case of the class of ordered bi- Γ -ideals of ordered Γ -semigroups. This is what we have shown in the following result.

Theorem 6. Suppose I is a two-sided ordered Γ -ideal of an ordered Γ -semigroup S and Q is a quasi- Γ -ideal of I ; then Q is an ordered bi- Γ -ideal of S .

Proof. Since Q is an ordered quasi- Γ -ideal of I and $Q \subseteq I$, we obtain

$$\begin{aligned} Q \Gamma Q &\subseteq Q \Gamma S \Gamma I \\ &= Q \Gamma (S \Gamma I) \subseteq Q \Gamma I \\ &\subseteq (Q \Gamma I] \subseteq (S \Gamma I] \subseteq (I] \subseteq I, \end{aligned}$$

$$Q \Gamma S \Gamma Q \subseteq I \Gamma Q \Gamma S$$

$$= (I \Gamma S) \Gamma Q \subseteq I \Gamma Q \subseteq (I \Gamma Q]$$

$$\subseteq (I \Gamma S] \subseteq (I] \subseteq I,$$

(6)

and $q \in (Q] \Rightarrow$ There exists $q_1 \in Q \subseteq I$ such that $q \leq q_1 \Rightarrow q \in (I] = I$ and $q \in (Q] \Rightarrow q \in I \cap (Q] = (Q)_I \subseteq Q$.

Therefore,

$$\begin{aligned} Q \Gamma S \Gamma Q &\subseteq (I \cap (I \Gamma Q]) \cap (I \cap (Q \Gamma I]) \\ &= (I \Gamma Q)_I \cap (Q \Gamma I)_I \subseteq Q, \quad (Q] \subseteq Q. \end{aligned} \quad (7)$$

Hence applying these facts together with Lemma 2, we have shown that Q is an ordered bi- Γ -ideal of S . □

4. Regular Ordered Γ -Semigroups and Ordered Quasi- Γ -Ideals

In this section, we use the concept of ordered quasi- Γ -ideals to characterize regular ordered Γ -semigroups.

Lemma 7. Let S be an ordered Γ -semigroup. Then the ordered sub- Γ -semigroup of $(N_Q, *)$ generated by $(L_I, *)$ and $(R_I, *)$ is in the following form:

$$\langle L_I \cup R_I \rangle = L_I \cup R_I \cup (R_I * \Gamma * L_I). \quad (8)$$

Proof. One can easily see that

$$\begin{aligned} \langle L_I \cup R_I \rangle &= \{Y_1 * \Gamma_1 * Y_2 * \cdots * Y_{n-1} * \Gamma_{n-1} * Y_n \mid Y_j \in L_I \\ &\text{or } Y_j \in R_I, \quad j = 1 \cdots n, n \in \mathbb{Z}^+, \Gamma_j \in \Gamma\}. \end{aligned} \quad (9)$$

Suppose $Y_j, Y_{j+1} \in L_I \cup R_I$. Then the conditions that arise are as follows: (i) $Y_j, Y_{j+1} \in L_I$: in this condition by Lemma 1, we obtain $Y_j * \Gamma * Y_{j+1} \in L_I$; (ii) $Y_j, Y_{j+1} \in R_I$: in this condition, $Y_j * \Gamma * Y_{j+1} \in R_I$ by also Lemma 1; (iii) $Y_j \in L_I, Y_{j+1} \in R_I$: in this condition, $Y_j * \Gamma * Y_{j+1} = (Y_j \Gamma Y_{j+1})$ is an ordered Γ -ideal of S , so $Y_j * \Gamma * Y_{j+1} \in L_I = L_I \cap R_I$; (iv) $Y_j \in R_I, Y_{j+1} \in L_I$: in this condition, $Y_j * \Gamma * Y_{j+1} \in R_I * \Gamma * L_I$ in $(N_Q, *)$. Therefore for any $Y_1, \dots, Y_n \in L_I \cup R_I$, where $n \in \mathbb{Z}^+$, using (i)–(iv), there arise three conditions as follows.

- (i)' If $Y_1 \in L_I$, then $Y_1 * \Gamma_1 * Y_2 * \cdots * Y_{n-1} * \Gamma_{n-1} * Y_n \in L_I$.
- (ii)' If $Y_n \in R_I$, then $Y_1 * \Gamma_1 * Y_2 * \cdots * Y_{n-1} * \Gamma_{n-1} * Y_n \in R_I$.
- (iii)' If $Y_1 \in R_I$ and $Y_n \in L_I$, where $n \geq 2$, then $Y_1 * \Gamma_1 * Y_2 * \cdots * Y_{n-1} * \Gamma_{n-1} * Y_n \in R_I * \Gamma * L_I$. Hence the lemma holds. □

Theorem 8. Let S be an ordered Γ -semigroup. Then the following assertions on S are equivalent.

- (i) S is a regular ordered Γ -semigroup.
(ii) For every ordered left Γ -ideal L and every ordered right Γ -ideal R , one has

$$(R\Gamma L) = R \cap L. \quad (10)$$

- (iii) For every ordered right Γ -ideal R and ordered left Γ -ideal L of S ,

- (1) $(R\Gamma R) = R$,
(2) $(L\Gamma L) = L$,
(3) $(R\Gamma L)$ is an ordered quasi- Γ -ideal of S .

- (iv) $(L_I, *)$ and $(R_I, *)$ are ordered idempotent Γ -semigroups and $(Q_I, *)$ is the sub- Γ -semigroup of $(N_Q, *)$ generated by $(L_I, *)$ and $(R_I, *)$.

- (v) $(Q_I, *)$ is a regular ordered sub- Γ -semigroup of the Γ -semigroup $(N_Q, *)$.

- (vi) Every ordered quasi- Γ -ideal Q of S is given by $Q = (Q\Gamma S\Gamma Q)$.

- (vii) $(Q_I, *, \subseteq)$ is a regular sub- Γ -semigroup of the ordered Γ -semigroup of $(N_Q, *, \subseteq)$.

Proof. (i) $\Rightarrow$ (ii) Suppose R and L are ordered right and left Γ -ideals of S , respectively; then we have

$$(R\Gamma L) \subseteq R \cap L. \quad (11)$$

Let S be regular; we need to prove only that $R \cap L \subseteq (R\Gamma L)$. Suppose $a \in R \cap L$. Since S is regular, we obtain $a \leq a\alpha x\beta a$ for some $x \in S$ and $\alpha, \beta \in \Gamma$, and so $a \in R$ and $x\alpha a \in L$; therefore $a\alpha x\beta a \in R\Gamma L$. Therefore $a \in (R\Gamma L)$, and thus $R \cap L \subseteq (R\Gamma L)$.

(ii) $\Rightarrow$ (iii) $(R\Gamma L)$ is an ordered quasi- Γ -ideal of S that follows directly from Lemma 3 and the condition (ii). As the ordered two-sided Γ -ideal of S is generated by $R = (R \cup S\Gamma R)$, the condition (ii) implies that

$$\begin{aligned} R &= R \cap (R \cup S\Gamma R) = (R\Gamma (R \cup S\Gamma R)); \\ \text{therefore } (R\Gamma R) &\subseteq (R\Gamma (R \cup S\Gamma R)) = R. \end{aligned} \quad (12)$$

Conversely, suppose $a \in (R\Gamma (R \cup S\Gamma R))$. Then $a \leq r\alpha b$ for $r \in R$ and $b \in (R \cup S\Gamma R)$. From $b \in (R \cup S\Gamma R)$, we have $b \leq c$, where $c = r' \in R$ or $c = s\alpha r''$ for some $s \in S$ and $r'' \in R$. Therefore $a \leq rac = rar' \in R\Gamma R$ or $a \leq rac = r\alpha(s\beta r'') = (ras)\gamma r'' \in R\Gamma R$ for $\alpha, \beta, \gamma \in \Gamma$; thus $a \in (R\Gamma R)$. Thus $R \subseteq (R\Gamma R)$, so that $(R\Gamma R) = R$. Similarly we can prove that $(L\Gamma L) = L$ dually.

(iii) $\Rightarrow$ (iv) The conditions (1), (2) in (iii) and Lemma 7 show that $(L_I, *)$ and $(R_I, *)$ are idempotent Γ -semigroups, respectively. Applying (iii) (3), we obtain $R_I * \Gamma * L_I \subseteq Q_I$; therefore $\langle L_I \cup R_I \rangle \subseteq Q_I$ in $(N_Q, *)$.

Conversely, suppose $Q \in Q_I$. Then $(Q \cup S\Gamma Q)$ is the ordered left Γ -ideal of S generated by Q . The condition (iii) (2) implies that

$$\begin{aligned} Q &\subseteq (Q \cup S\Gamma Q) = ((Q \cup S\Gamma Q) \Gamma (Q \cup S\Gamma Q)) \\ &\subseteq (Q\Gamma Q \cup S\Gamma Q\Gamma Q \cup Q\Gamma S\Gamma Q \\ &\quad \cup (S\Gamma Q) \Gamma (S\Gamma Q)) \subseteq (S\Gamma Q). \end{aligned} \quad (13)$$

We can dually prove that $Q \subseteq (Q\Gamma S)$. Therefore using these facts and Lemma 4, it follows that

$$(a) \quad Q \subseteq (S\Gamma Q) \cap (Q\Gamma S) \subseteq (S\Gamma Q \cup Q) \cap (Q \cup Q\Gamma S) = Q. \quad (14)$$

Therefore for $Q \in Q_I$, we have $Q = (S\Gamma Q) \cap (Q\Gamma S)$, and the condition (iii) (3) together with (a) implies that

$$(b) \quad (R\Gamma L) = (S\Gamma (R\Gamma L)) \cap ((R\Gamma L) \Gamma S). \quad (15)$$

Moreover, by the assertion (iii) (2), we have $S = (S\Gamma S)$ and

$$\begin{aligned} (S\Gamma Q) &= ((S\Gamma Q)^2] \\ &= ((S\Gamma Q) \Gamma (S\Gamma Q)) = ((S\Gamma Q) \Gamma ((S\Gamma S) \Gamma Q)) \\ &\subseteq (S\Gamma Q\Gamma S\Gamma Q) \subseteq (S\Gamma (Q\Gamma S) \Gamma (S\Gamma Q)) \\ &\subseteq (S\Gamma ((Q\Gamma S) \Gamma (S\Gamma Q))) \subseteq (S\Gamma (Q\Gamma S\Gamma S\Gamma Q)) \\ &\subseteq (S\Gamma Q). \end{aligned} \quad (16)$$

Therefore $(S\Gamma Q) = (S\Gamma((Q\Gamma S)\Gamma(S\Gamma Q)))$. Dually, we can prove that

$$(Q\Gamma S) = (((Q\Gamma S) \Gamma (S\Gamma Q)) \Gamma S). \quad (17)$$

From these facts, (a) and (b), we obtain

$$\begin{aligned} (c) \quad Q &= (Q\Gamma S) \cap (S\Gamma Q) \\ &= (((Q\Gamma S) \Gamma (S\Gamma Q)) \Gamma S) \cap (S\Gamma ((Q\Gamma S) \Gamma (S\Gamma Q))) \\ &= ((Q\Gamma S) \Gamma (S\Gamma Q)) \\ &= (Q\Gamma S) * \Gamma * (S\Gamma Q) \in R_I * \Gamma * L_I \\ &\subseteq \langle L_I \cup R_I \rangle \end{aligned} \quad (18)$$

by Lemma 7. Therefore $Q_I \subseteq \langle L_I \cup R_I \rangle$. Hence $Q_I = \langle L_I \cup R_I \rangle$ in $(N_Q, *)$.

(iv) $\Rightarrow$ (iii) It is a consequence of Lemma 7.

(iii) $\Rightarrow$ (v) By (iii) $\Rightarrow$ (iv), we have (b) and (c). Suppose Q_1, Q_2 are two ordered quasi- Γ -ideals of S . Then $(S\Gamma(Q_1\Gamma Q_2) \cup (Q_1\Gamma Q_2))$ is the least ordered left Γ -ideal of S containing $(Q_1\Gamma Q_2)$. Then the condition (iii) (2) implies that

$$\begin{aligned} (Q_1\Gamma Q_2) &\subseteq (S\Gamma(Q_1\Gamma Q_2) \cup (Q_1\Gamma Q_2)) \\ &= ((S\Gamma(Q_1\Gamma Q_2) \cup (Q_1\Gamma Q_2))^2] \\ &\subseteq (S\Gamma(Q_1\Gamma Q_2)) = ((S\Gamma S) \Gamma (Q_1\Gamma Q_2)) \\ &\subseteq (S\Gamma(S\Gamma(Q_1\Gamma Q_2))). \end{aligned} \quad (19)$$

Dually one can prove that $(Q_1 \Gamma Q_2] \subseteq ((Q_1 \Gamma Q_2] \cup (Q_1 \Gamma Q_2] \Gamma S] \subseteq (((Q_1 \Gamma Q_2] \Gamma S] \Gamma S]$. These facts together with (b) show that

$$\begin{aligned}
 (Q_1 \Gamma Q_2] &\subseteq (S \Gamma (Q_1 \Gamma Q_2] \cup (Q_1 \Gamma Q_2]) \\
 &\cap ((Q_1 \Gamma Q_2] \cup (Q_1 \Gamma Q_2] \Gamma S] \\
 &\subseteq (S \Gamma (S \Gamma (Q_1 \Gamma Q_2])) \\
 &\cap (((Q_1 \Gamma Q_2] \Gamma S] \Gamma S] \\
 &= (((Q_1 \Gamma Q_2] \Gamma S] \Gamma (S \Gamma (Q_1 \Gamma Q_2])) \\
 &\subseteq ((Q_1 \Gamma (Q_2 \Gamma S \Gamma S) \Gamma Q_1) \Gamma Q_2] \subseteq (Q_1 \Gamma Q_2].
 \end{aligned} \tag{20}$$

By Theorem 5 (ii), $(Q_1 \Gamma Q_2] = (S \Gamma (Q_1 \Gamma Q_2] \cup (Q_1 \Gamma Q_2]) \cap ((Q_1 \Gamma Q_2] \cup (Q_1 \Gamma Q_2] \Gamma S]$ is an ordered quasi- Γ -ideal of S ; therefore $Q_1 * \Gamma * Q_2 \in Q_I$. Hence $(Q_I, *)$ is a sub- Γ -semigroup of $(N_Q, *)$. For every $Q \in Q_I$, by (c), we obtain $Q = ((Q \Gamma S] \Gamma (S \Gamma Q]) \subseteq (Q \Gamma S \Gamma S \Gamma Q] \subseteq (Q \Gamma S \Gamma Q] \subseteq Q$, and so $Q = (Q \Gamma S \Gamma Q] = Q * \Gamma * S * \Gamma * Q$, where $S \in Q_I$. Thus $(Q_I, *)$ is a regular sub Γ -semigroup of $(N_Q, *)$.

(v) $\Rightarrow$ (vi) Suppose Q is an ordered quasi- Γ -ideal of S . Applying the condition (iv), there is an ordered quasi- Γ -ideal Q_1 of S so that, by Lemma 4,

$$\begin{aligned}
 Q &= Q * \Gamma * Q_1 * \Gamma * Q = (Q \Gamma Q_1 \Gamma Q] \\
 &\subseteq (Q \Gamma S \Gamma Q] \subseteq (S \Gamma Q] \cap (Q \Gamma S] \\
 &\subseteq (S \Gamma Q \cup Q] \cap (Q \cup Q \Gamma S] = Q,
 \end{aligned} \tag{21}$$

and therefore $Q = (Q \Gamma S \Gamma Q]$.

(vi) $\Rightarrow$ (vii) It is straightforward.

(vii) $\Rightarrow$ (i) For every $s \in S$, using Theorem 5, $R(s) \cap L(s)$ is an ordered quasi- Γ -ideal of S containing s . By (vii), there exists $Q \in Q_S$ so that

$$\begin{aligned}
 s \in R(s) \cap L(s) &\subseteq (R(s) \cap L(s)) * \Gamma * Q * \Gamma * (R(s) \cap L(s)) \\
 &= ((R(s) \cap L(s)) \Gamma Q \Gamma (R(s) \cap L(s))) \\
 &\subseteq (R(s) \Gamma S \Gamma L(s)) \\
 &= ((s \cup s \Gamma S] \Gamma S \Gamma (S \Gamma s \cup s]) \subseteq (s \Gamma S \Gamma s].
 \end{aligned} \tag{22}$$

Hence S is a regular ordered Γ -semigroup. $\square$

Lemma 9. Every two-sided ordered Γ -ideal I of a regular ordered Γ -semigroup S is a regular sub- Γ -semigroup of S .

Proof. Suppose $i \in I$. As S is regular, there exists $s \in S$ so that, for $\alpha, \beta, \gamma, \delta \in \Gamma$, we have

$$i \leq i \alpha s \beta i \leq i \alpha s \beta i \gamma s \delta i = i \alpha (s \beta i \gamma s) \delta i. \tag{23}$$

As $s \alpha i \beta s \in S \Gamma I \Gamma S \subseteq I$, we observe that $i \in (i \Gamma I \Gamma i)_I$. $\square$

Theorem 10. Suppose S is a regular ordered Γ -semigroup. Then the following statements are true.

(i) Every ordered quasi- Γ -ideal of S can be expressed as follows:

$$Q = R \cap L = (R \Gamma L], \tag{24}$$

where R and L are, respectively, the ordered right and left Γ -ideals of S generated by Q .

(ii) Let Q be an ordered quasi- Γ -ideal of S ; then $(Q \Gamma Q] = (Q \Gamma Q \Gamma Q]$.

(iii) Every ordered bi- Γ -ideal of S is an ordered quasi- Γ -ideal of S .

(iv) Every ordered bi- Γ -ideal of any ordered two sided- Γ -ideal of S is a quasi- Γ -ideal of S .

(v) For every $L_1, L_2 \in L_I$ and $R_1, R_2 \in R_I$, one obtains

$$\begin{aligned}
 L_1 \cap L_2 &\subseteq (L_1 \Gamma L_2], \\
 R_1 \cap R_2 &\subseteq (R_1 \Gamma R_2].
 \end{aligned} \tag{25}$$

Proof. Because S is a regular ordered Γ -semigroup, then by Lemma 4 and Theorem 8, the statement (i) is done. Since $(Q \Gamma Q \Gamma Q] \subseteq (Q \Gamma Q]$ is always true, we need to show that $(Q \Gamma Q] \subseteq (Q \Gamma Q \Gamma Q]$. We have that $(Q \Gamma Q]$ is also an ordered quasi- Γ -ideal of S by Theorem 8. Moreover we have the following equation:

$$\begin{aligned}
 (Q \Gamma Q] &= (Q \Gamma Q \Gamma S \Gamma Q \Gamma Q] \\
 &= (Q \Gamma (Q \Gamma S \Gamma Q) \Gamma Q] \subseteq (Q \Gamma Q \Gamma Q].
 \end{aligned} \tag{26}$$

Suppose Q_1 is an ordered bi- Γ -ideal of S . Then $(S \Gamma Q_1]$ is an ordered left Γ -ideal and $(Q_1 \Gamma S]$ is an ordered right Γ -ideal of S . Applying Theorem 8, we obtain

$$\begin{aligned}
 (S \Gamma Q_1] \cap (Q_1 \Gamma S] &= ((Q_1 \Gamma S] \Gamma (S \Gamma Q_1]) \\
 &\subseteq (Q_1 \Gamma S \Gamma Q_1] \subseteq (Q_1] \subseteq Q_1.
 \end{aligned} \tag{27}$$

Therefore Q_1 is an ordered quasi- Γ -ideal of S .

Suppose I is a two-sided ordered Γ -ideal of S and B is an ordered bi- Γ -ideal of I . By the relation (iii) and Lemma 9, B is an ordered quasi- Γ -ideal of I ; therefore using Theorem 6, B is an ordered bi- Γ -ideal of S . Also from the relation (iii) again, we obtain B as an ordered quasi- Γ -ideal of S .

Lastly, suppose $L_1, L_2 \in L_I$. Because S is regular and $L_1 \cap L_2$ is an ordered quasi- Γ -ideal of S , using Theorem 8, we obtain

$$\begin{aligned}
 L_1 \cap L_2 &= ((L_1 \cap L_2) \Gamma S \Gamma (L_1 \cap L_2)) \\
 &\subseteq (L_1 \Gamma (S \Gamma L_2)) \subseteq (L_1 \Gamma L_2].
 \end{aligned} \tag{28}$$

Dually, we can prove that $R_1 \cap R_2 \subseteq (R_1 \Gamma R_2]$ for all $R_1, R_2 \in R_I$. $\square$

Theorem 11. A partially ordered Γ -semigroup S is regular if and only if for every ordered bi- Γ -ideal B , every ordered Γ -ideal I , and every ordered quasi- Γ -ideal Q , one has

$$B \cap I \cap Q \subseteq (B \Gamma I \Gamma Q]. \tag{29}$$

Proof. Let S be regular. Then for any $a \in B \cap I \cap Q$ there exists $s \in S$ such that

$$\begin{aligned} a &\leq a\alpha s\beta a \leq (a\alpha s\beta a)\gamma s\delta (a\alpha s\beta a) \\ &= (a\alpha s\beta a)\gamma (s\alpha a\beta s)\delta a \in (B\Gamma B)\Gamma (S\Gamma I\Gamma S)\Gamma Q \\ &\subseteq B\Gamma I\Gamma Q. \end{aligned} \quad (30)$$

Hence $a \in (B\Gamma I\Gamma Q)$, where $\alpha, \beta, \gamma, \delta \in \Gamma$.

Conversely, let $B \cap I \cap Q \subseteq (B\Gamma I\Gamma Q)$ for every ordered bi- Γ -ideal B , every ordered Γ -ideal I , and every ordered quasi- Γ -ideal Q of S . Suppose $s \in S$. Let $B(s)$ and $Q(s)$ be the ordered bi- Γ -ideal and ordered quasi- Γ -ideal of S generated by s , respectively. So we have the following:

$$\begin{aligned} s &\in B(s) \cap I(s) \cap Q(s) \\ &\subseteq (B(s)\Gamma I(s)\Gamma Q(s)) \\ &\subseteq ((s \cup s\Gamma S\Gamma s)\Gamma S\Gamma (s \cup (S\Gamma s \cap s\Gamma S))) \\ &\subseteq ((s \cup s\Gamma S\Gamma s)\Gamma S\Gamma (s \cup S\Gamma s)) \subseteq (s\Gamma S\Gamma s). \end{aligned} \quad (31)$$

Hence S is regular. $\square$

Next consider R in place of Q in Theorem 11 to obtain the following.

Corollary 12. *An ordered Γ -semigroup S is regular if and only if for every ordered bi- Γ -ideal B , every ordered Γ -ideal I , and every right Γ -ideal R of S ,*

$$B \cap I \cap R \subseteq (B\Gamma I\Gamma R). \quad (32)$$

Theorem 13. *A partially ordered Γ -semigroup S is regular if and only if for every ordered quasi- Γ -ideal Q , every ordered left Γ -ideal L , and every ordered right- Γ -ideal R , one has*

$$R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L). \quad (33)$$

Proof. Let S be regular; then for any $a \in R \cap Q \cap L$, there exists $s \in S$ such that $a \leq a\alpha s\beta a \leq (a\alpha s\beta a)\gamma s\delta (a\theta s\lambda a) = (a\alpha s)\beta a\gamma (s\delta a\theta s\lambda a) \in (R\Gamma S)\Gamma Q\Gamma (S\Gamma L\Gamma S\Gamma L) \subseteq R\Gamma Q\Gamma L$, for $\alpha, \beta, \gamma, \theta, \delta, \lambda \in \Gamma$. Hence $a \in (R\Gamma Q\Gamma L)$.

Conversely, let

$$R \cap Q \cap L \subseteq (R\Gamma Q\Gamma L), \quad (34)$$

for every ordered right Γ -ideal R , every ordered quasi- Γ -ideal Q , and every ordered left Γ -ideal L of S . Suppose $s \in S$. So we have

$$\begin{aligned} s &\in R(s) \cap Q(s) \cap L(s) \\ &\subseteq (R(s)\Gamma Q(s)\Gamma L(s)) \subseteq (R(s)\Gamma S\Gamma L(s)) \\ &\subseteq (R(s)\Gamma L(s)) \subseteq ((s \cup s\Gamma S)\Gamma (s \cup S\Gamma s)) \\ &\subseteq ((s\Gamma s \cup s\Gamma S\Gamma s)). \end{aligned} \quad (35)$$

So for $\alpha, \beta, \gamma \in \Gamma$, $s \leq sas$ or $s \leq sax\beta s$ for some $x \in S$. If $s \leq sas$, then $s \leq sas \leq (sas)\beta(s\gamma s) = sa(s\beta s)^2\gamma s \in s\Gamma S\Gamma s$. If $s \leq sax\beta s$ for some $x \in S$, then $s \in s\Gamma S\Gamma s$. So, finally we obtain $s \in (s\Gamma S\Gamma s)$. Hence S is regular. $\square$

Corollary 14. *If one considers an ordered left Γ -ideal L (or an ordered right Γ -ideal R) in place of the ordered quasi- Γ -ideal Q in Theorem 13, one obtains*

$$L \cap R \subseteq (R\Gamma L). \quad (36)$$

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