Research Article

New Subclasses of Analytic Functions with Respect to Symmetric and Conjugate Points

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We introduce new subclasses of close-to-convex and quasiconvex functions with respect to symmetric and conjugate points. The coefficient estimates for functions belonging to these classes are obtained.

1. Introduction

Let \(U\) be the class of functions which are analytic and univalent in the open unit disk \(E = \{z : |z| < 1\}\) given by

\[
\omega(z) = \sum_{k=0}^{\infty} c_k z^k
\]

and satisfying the conditions \(\omega(0) = 0, |\omega(z)| \leq 1, z \in E\).

Let \(S\) denote the class of functions \(f\) which are analytic and univalent in \(E\) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E.
\]

Let \(S^*_S\) be the subclass of functions \(f(z) \in S\) and satisfying the condition

\[
\text{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E.
\]

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1].

Also, let \(S^*_C\) be the subclass of functions \(f(z) \in S\) and satisfying the condition

\[
\text{Re}\left(\frac{zf'(z)}{f(z) + f(-z)}\right) > 0, \quad z \in E.
\]

These functions are called starlike with respect to conjugate points and were introduced by El-Ashwah and Thomas [2]. Further results on starlike functions with respect to symmetric points or conjugate points can be found in [3–5].

Then, Das and Singh [6] introduced another class \(C_s\), namely, convex functions with respect to symmetric points and satisfying the condition

\[
\text{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E.
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Then, Das and Singh [6] introduced another class \(C_s\), namely, convex functions with respect to symmetric points and satisfying the condition

\[
\text{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in E.
\]
Let $S^*_c(A, B)$ be the subclass of $S$ consisting of functions given by (2) satisfying the condition
\[
\frac{2zf'(z)}{f(z) + f'(z)} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E. \tag{7}
\]

Let $C_c(A, B)$ be the subclass of $S$ consisting of functions given by (2) satisfying the condition
\[
\frac{2(zf'(z))'}{(f(z) - f(-z))} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E. \tag{8}
\]

Let $C_s(A, B)$ be the subclass of $S$ consisting of functions given by (2) satisfying the condition
\[
\frac{2(zf'(z))'}{(f(z) + f'(z))} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E. \tag{9}
\]

Motivated by the previous classes, Selvaraj and Vasanthi [10] defined the following classes of functions with respect to symmetric and conjugate points.

**Definition 2.** Let $M_s(\alpha, A, B)$ be the subclass of $S$ consisting of functions given by (2) satisfying the condition
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(f(z) - f(-z)) + az(f(z) + f'(-z))} < \frac{1 + Az}{1 + Bz},
\]
\[-1 \leq B < A \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E. \tag{10}
\]

Let $M_c(\alpha, A, B)$ be the subclass of $S$ consisting of functions given by (2) satisfying the condition
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(f(z) + f'(-z)) + az(f(z) + f'(z))} < \frac{1 + Az}{1 + Bz},
\]
\[-1 \leq B < A \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E. \tag{11}
\]

In this paper, we introduce the class $K^*_s(\alpha, A, B; C, D)$ consisting of analytic functions $f$ of the form (2) and satisfying
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(g(z) - g(-z)) + az(g(z) - g'(-z))} < \frac{1 + Cz}{1 + Dz},
\]
\[-1 \leq D \leq B < A \leq C \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E, \tag{12}
\]

where $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B).$

In addition, we introduce the class $K^*_c(\alpha, A, B; C, D)$ consisting of analytic functions $f$ of the form (2) and satisfying
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(g(z) + g'(z)) + az(g(z) + g'(z))} < \frac{1 + Cz}{1 + Dz},
\]
\[-1 \leq D \leq B < A \leq C \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E, \tag{13}
\]

where $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B).$

By the definition of subordination, it follows that $f \in K^*_s(\alpha, A, B; C, D)$ if and only if
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(g(z) - g(-z)) + az(g(z) - g'(-z))} < \frac{1 + Cz}{1 + Dz},
\]
\[-1 \leq D \leq B < A \leq C \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E. \tag{14}
\]

and that $f \in K^*_c(\alpha, A, B; C, D)$ if and only if
\[
\frac{2zf'(z) + 2az^2f''(z)}{(1 - \alpha)(g(z) + g'(z)) + az(g(z) + g'(z))} < \frac{1 + Cz}{1 + Dz},
\]
\[-1 \leq D \leq B < A \leq C \leq 1, \quad 0 \leq \alpha \leq 1, \quad z \in E, \tag{15}
\]

where
\[
P(z) = 1 + \sum_{n=1}^\infty p_n z^n. \tag{16}
\]

In the next section, we discuss the coefficient estimates for functions belonging to the classes $K^*_s(\alpha, A, B; C, D)$ and $K^*_c(\alpha, A, B; C, D).$
2. Some Preliminary Lemmas

We will require the following lemmas for proving our main results.

Lemma 3 (see [7]). If $P(z)$ is given by (14), (15) and (16), then for $-1 < D < C < 1$,

$$|p_n| \leq (C - D), \quad n = 1, 2, \ldots.$$ \hfill (17)

Lemma 4 (see [10]). Let $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B)$. Then, for $n \geq 1$, $0 \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A - B)}{2^n \cdot n! \left(1 + (2n - 1) \alpha\right)} \prod_{j=1}^{n-1} (A - B + 2j),$$

$$|b_{2n+1}| \leq \frac{(A - B)}{2^n \cdot n! \left(1 + 2n \alpha\right)} \prod_{j=1}^{n-1} (A - B + 2j).$$ \hfill (18)

Lemma 5 (see [10]). Let $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B)$. Then, for $n \geq 1$, $0 \leq \alpha \leq 1$,

$$|b_{2n}| \leq \frac{(A - B)}{(2n - 1)! \left(1 + (2n - 1) \alpha\right)} \prod_{j=1}^{2n-2} (A - B + j),$$

$$|b_{2n+1}| \leq \frac{(A - B)}{(2n)! \left(1 + 2n \alpha\right)} \prod_{j=1}^{2n-1} (A - B + j).$$ \hfill (19)

3. Main Results

Unless otherwise mentioned, we will assume in the reminder of this paper that $-1 \leq D \leq B < A \leq C \leq 1$, $0 \leq \alpha \leq 1$, and $z \in E$.

Theorem 6. Let $f \in K^*_c(\alpha, A, B, C, D)$, then for $n \geq 1$,

$$|a_{2n}| \leq \frac{(C - D)}{2^n \cdot n! \left(1 + (2n - 1) \alpha\right)} \prod_{j=1}^{n-1} (A - B + 2j),$$ \hfill (20)

$$|a_{2n+1}| \leq \frac{1}{(2n + 1) \left(1 + 2n \alpha\right)} \times \left\{ \left[ (C - D) + \frac{(A - B)}{2n} \right] \prod_{j=1}^{n-1} (A - B + 2j) \right\}.$$ \hfill (21)

Proof. Since $g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B)$, it follows that

$$2z g^{''}(z) + 2\alpha z^2 g^{''}(z)$$

$$= \left[ (1 - \alpha) \left( g(z) - g(-z) \right) + \alpha z \left( g(z) - g(-z) \right) \right] K(z),$$ \hfill (22)

for $z \in E$, with Re$(K(z)) > 0$, where $K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$.

On equating the coefficients of like powers of $z$ in (22), we get

$$2(1 + \alpha) b_2 = d_1,$$

$$2(1 + 2\alpha) b_3 = d_2,$$

$$4(1 + 3\alpha) b_4 = d_3 + (1 + 2\alpha) b_1 d_1,$$

$$4(1 + 4\alpha) b_5 = d_4 + (1 + 2\alpha) b_1 d_2,$$

and continuing in this way, we obtain

$$2n(1 + (2n - 1) \alpha) b_{2n} = d_{2n-1} + (1 + 2\alpha) b_3 d_{2n-3}$$

$$\quad + \cdots + (1 + (2n - 2) \alpha) b_{2n-1} d_1,$$

$$2n(1 + 2n \alpha) b_{2n+1} = d_{2n} + (1 + 2\alpha) b_3 d_{2n-2}$$

$$\quad + \cdots + (1 + (2n - 2) \alpha) b_{2n-1} d_2.$$ \hfill (23)

From (14) and (16), we have

$$(z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots + 2na_{2n} z^{2n} + \cdots)$$

$$+ \alpha \left( 2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 + 20a_5 z^5 + \cdots \right)$$

$$\quad + (2n - 1) 2na_{2n} z^{2n} + \cdots$$

$$= \left[ (1 - \alpha) \left( z + b_3 z^2 + b_5 z^5 + \cdots + b_{2n-1} z^{2n-1} \right)$$

$$\quad + b_{2n+1} z^{2n+1} + \cdots \right]$$

$$\times \left[ (1 + p_1 z + p_2 z^2 + p_3 z^3 + p_4 z^4 + p_5 z^5 + \cdots) \right]$$

$$\times \cdots + P_{2n-1} z^{2n-1} + P_{2n} z^{2n} + \cdots).$$ \hfill (24)

On equating the coefficients, we obtain

$$2(1 + \alpha) a_2 = p_1,$$

$$3(1 + 2\alpha) a_3 = p_2 + (1 + 2\alpha) b_3,$$

$$4(1 + 3\alpha) a_4 = p_3 + (1 + 2\alpha) b_3 p_1,$$

$$5(1 + 4\alpha) a_5 = p_4 + (1 + 2\alpha) b_3 p_2 + (1 + 4\alpha) b_5,$$ \hfill (27)
and so
\[2n (1 + (2n - 1) \alpha) a_{2n} = p_{2n-1} + (1 + 2\alpha) b_3 p_{2n-3}
+ \cdots + (1 + (2n - 2) \alpha) b_{2n-1} p_1,
\]
\[(2n + 1) (1 + 2n \alpha) a_{2n+1} = p_{2n} + (1 + 2\alpha) b_3 p_{2n-2}
+ \cdots + (1 + (2n - 2) \alpha) b_{2n-1} p_2
(2n-1) (1 + 2n \alpha) a_{2n-1}
+ \cdots + (1 + (2n - 2) \alpha) b_{2n-1} p_1,\]
\[(2n+1)(1 + 2n \alpha) a_{2n+1} = p_{2n} + (1 + 2\alpha) b_3 p_{2n-2}
+ \cdots + (1 + (2n - 2) \alpha) b_{2n-1} p_2
(2n-1) (1 + 2n \alpha) a_{2n} = p_{2n-1} + (1 + 2\alpha) b_3 p_{2n-3}
+ \cdots + (1 + (2n - 2) \alpha) b_{2n-1} p_1,\]  
(29)

By using Lemma 3 and (27), we have
\[|a_2| \leq \frac{(C - D)}{2 \cdot 1 \cdot (1 + \alpha)}, \quad |a_3| \leq \frac{(A - B) + 2 (C - D)}{3 \cdot 2 \cdot (1 + 2\alpha)}.\]
(30)

Again, by applying Lemma 3 and using (23), we obtain from (28)
\[|a_4| \leq \frac{(C - D) (A - B + 2)}{4 \cdot 2 \cdot (1 + 3\alpha)}, \quad |a_5| \leq \frac{(A - B + 2) [(A - B) + 4 (C - D)]}{5 \cdot 8 \cdot (1 + 4\alpha)}.\]
(31)

It follows that (20) and (21) hold for \(n = 1, 2\). We now prove (20) and (21) by induction.

Equation (29) together with Lemma 3 yield
\[|a_{2n}| \leq \frac{(C - D)}{2n (1 + (2n - 1) \alpha)} \left[ 1 + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right],\]
(32)

\[|a_{2n+1}| \leq \frac{1}{(2n + 1) (1 + 2n\alpha)} \times \left[ (C - D) \left[ 1 + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right]
+ (1 + 2n\alpha) |b_{2n+1}| \right].\]
(33)

Again, using Lemma 3 in (25), we have
\[|b_{2n+1}| \leq \frac{(A - B)}{2n (1 + 2n\alpha)} \left[ 1 + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right].\]
(34)

Using (34) in (33), we obtain
\[|a_{2n+1}| \leq \frac{1}{(2n + 1) (1 + 2n\alpha)} \times \left[ (C - D) + \frac{(A - B)}{2n} \right] \left[ 1 + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right].\]
(35)

We suppose that (20) and (21) hold for \(k = 3, 4, \ldots, (n - 1)\). Using Lemma 4 in (32) and (35), we get
\[|a_{2n}| \leq \frac{(C - D)}{2n (1 + (2n - 1) \alpha)} \times \left[ 1 + \sum_{k=1}^{n-1} (A - B) \frac{k-1}{2k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right],\]
(36)

\[|a_{2n+1}| \leq \frac{1}{(2n + 1) (1 + 2n\alpha)} \times \left[ (C - D) + \frac{(A - B)}{2n} \right] \times \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B) k-1}{2k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right].\]
(37)

In order to prove (20), it is sufficient to show that
\[\frac{(C - D)}{2m (1 + (2m - 1) \alpha)} \left[ 1 + \sum_{k=1}^{m-1} \frac{(A - B) k-1}{2k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right]
= \frac{(C - D)}{2^m \cdot m! (1 + (2m - 1) \alpha)} \prod_{j=1}^{m-1} (A - B + 2j),\]
\((m = 3, 4, \ldots, n).\)
(38)

Thus, (38) is valid for \(m = 3\).

Let us assume that (38) is true for all \(m, 3 < m \leq (n - 1)\). Then, from (36), we have
\[\frac{(C - D)}{2n (1 + (2n - 1) \alpha)} \left[ 1 + \sum_{k=1}^{n-1} \frac{(A - B) k-1}{2k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right]
= \frac{(n - 1)}{n} \left\{ \frac{(C - D)}{2 (n - 1) (1 + (2n - 1) \alpha)} \times \left[ 1 + \sum_{k=1}^{n-2} \frac{(A - B) k-1}{2k \cdot k!} \prod_{j=1}^{k-1} (A - B + 2j) \right] + \frac{(C - D)}{2n (1 + (2n - 1) \alpha)} \cdot \frac{(A - B)}{2^{n-1} \cdot (n - 1)!} \times \prod_{j=1}^{n-2} (A - B + 2j) \right\}\]
\[
\begin{align*}
&= \frac{(n - 1)}{n} \cdot \frac{(C - D)}{2^{n-1} \cdot (n - 1)!} \cdot \frac{1}{(1 + (2n - 1)\alpha)} \times \\
&\times \prod_{j=1}^{n-2} (A - B + 2j) \\
&+ \frac{(C - D)}{2n \cdot (1 + (2n - 1)\alpha)} \cdot \frac{1}{2^{n-1} \cdot (n - 1)!} \times \prod_{j=1}^{n-2} (A - B + 2j) \\
&\times (A - B + 2(n - 1)) \\
&= \frac{(C - D)}{2^n \cdot n! \cdot (1 + (2n - 1)\alpha)} \prod_{j=1}^{n-1} (A - B + 2j) \\
&\times (A - B + 2(n - 1)) \\
&= \frac{(C - D)}{2^n \cdot n! \cdot (1 + (2n - 1)\alpha)} \prod_{j=1}^{n-1} (A - B + 2j).
\end{align*}
\]

Thus, (38) holds for \(m = n\), and, hence, (20) follows. Next, we prove (21).

From (38), we have
\[
1 + \sum_{k=1}^{\frac{n-1}{2^k \cdot k!}} (A - B + 2j) = \frac{1}{2^{n-1} \cdot (n - 1)!} \prod_{j=1}^{n-1} (A - B + 2j).
\]

By using (40) in (37), we obtain
\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)^2} \left( C - D + \frac{A - B}{2n} \right) \times \left( C - D + \frac{A - B}{2n} \right) \times \left( 2 + \frac{A - B}{2n} \right) \times \left( 2 + \frac{A - B}{2n} \right)
\]

which proves (21).

By taking \(\alpha = 0\) and \(\alpha = 1\) in Theorem 6, respectively, we can readily deduce the following corollaries.

**Corollary 7** (see [11]). Let \(f \in K_{\lambda}(A, B; C, D)\), then, for \(n \geq 1\),
\[
|a_{2n}| \leq \frac{(C - D)\prod_{j=1}^{n-1} (A - B + 2j)}{2^n \cdot n!},
\]
\[
|a_{2n+1}| \leq \frac{1}{2n + 1} \left[ C - D + \frac{A - B}{2n} \right] \times \left[ \frac{1}{2^{n-1} \cdot (n - 1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right].
\]

**Corollary 8.** Let \(f \in K_{\lambda}^+(A, B; C, D)\), then, for \(n \geq 1\),
\[
|a_{2n}| \leq \frac{(C - D)\prod_{j=1}^{n-1} (A - B + 2j)}{2^n \cdot n!},
\]
\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)^2} \left[ C - D + \frac{A - B}{2n} \right] \times \left[ \frac{1}{2^{n-1} \cdot (n - 1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right].
\]

Further, putting \(C = 1\) and \(D = -1\) in Corollary 8, one obtains the following.

**Corollary 9.** Let \(f \in K_{\lambda}^+(A, B)\), then, for \(n \geq 1\),
\[
|a_{2n}| \leq \frac{1}{2^n \cdot n!} \prod_{j=1}^{n-1} (A - B + 2j),
\]
\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)^2} \left[ 2 + \frac{A - B}{2n} \right] \times \left[ \frac{1}{2^{n-1} \cdot (n - 1)!} \prod_{j=1}^{n-1} (A - B + 2j) \right].
\]

**Remark 10.** Corollary 9 improves the result obtained by Janteng and Halim [13, Theorem 3.1].
Theorem 11. Let \( f \in K^*_c(\alpha, A, B; C, D) \), then, for \( n \geq 1 \),

\[
|a_{2n}| \leq \frac{1}{2n(1 + (2n - 1)\alpha)} \left[(C - D) + \frac{(A - B)}{2n - 1}\right] \left[\frac{1}{(2n - 2)!} \prod_{i=1}^{2n-2} (A - B + j)\right] \tag{45}
\]

\[
|a_{2n+1}| \leq \frac{1}{(2n + 1)(1 + 2n\alpha)} \left[(C - D) + \frac{(A - B)}{2n}\right] \left[\frac{1}{(2n-1)!} \prod_{j=1}^{2n-1} (A - B + j)\right]. \tag{46}
\]

Proof. Since \( g(z) = z + \sum_{n=2}^\infty b_n z^n \in M_c(\alpha, A, B) \), it follows that

\[
2z g'(z) + 2ax^2 g''(z) = \left[(1 - \alpha) \left(g(z) + \frac{g(z)}{z}\right) + az \left(g(z) + \frac{g(z)}{z}\right)\right] K(z), \tag{47}
\]

where \( K(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots \).

On equating the coefficients of like powers of \( z \) in (47), we get

\[
(1 + \alpha) b_2 = d_1, \tag{48}
\]

\[
2(1 + 2\alpha) b_3 = d_2 + (1 + \alpha) b_2 d_1, \tag{49}
\]

\[
3(1 + 3\alpha) b_4 = d_3 + (1 + \alpha) b_2 d_2 + (1 + 2\alpha) b_3 d_1, \tag{50}
\]

\[
4(1 + 4\alpha) b_5 = d_4 + (1 + \alpha) b_2 d_3 + (1 + 2\alpha) b_3 d_2 + (1 + 3\alpha) b_4 d_1, \tag{51}
\]

and continuing in this way, we obtain

\[
(2n - 1)(1 + (2n - 1)\alpha) b_{2n} = d_{2n-1} + (1 + \alpha) b_2 d_{2n-2} + \cdots + (1 + (2n - 2)\alpha) b_{2n-1} d_1, \tag{52}
\]

\[
2n(1 + 2n\alpha) b_{2n+1} = d_{2n} + (1 + \alpha) b_2 d_{2n-1} + \cdots + (1 + (2n - 1)\alpha) b_{2n} d_1. \tag{53}
\]

From (15) and (16), we have

\[
( z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + 5a_5 z^5 + \cdots + 2na_{2n} z^{2n} + \cdots \) + \alpha \left( 2a_2 z^2 + 6a_3 z^3 + 12a_4 z^4 + 20a_5 z^5 + \cdots \right) + \cdots + (2n - 1) 2na_{2n} z^{2n} + \cdots
\]

\[
= \left[(1 - \alpha) \left(z + b_2 z^2 + b_3 z^3 + \cdots + b_{2n} z^{2n} + \cdots \right) + \alpha \left(z + 2b_2 z^2 + 3b_3 z^3 + \cdots + 2nb_{2n} z^{2n} + \cdots \right) + \cdots + (2n - 1) 2nb_{2n} z^{2n} + \cdots \right]\times \left(1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots + p_{2n} z^{2n} + \cdots \right).
\]

On equating the coefficients, we obtain

\[
2(1 + \alpha) a_2 = p_1 + (1 + \alpha) b_2, \tag{54}
\]

\[
3(1 + 2\alpha) a_3 = p_2 + (1 + \alpha) b_2 p_1 + (1 + 2\alpha) b_3, \tag{55}
\]

\[
4(1 + 3\alpha) a_4 = p_3 + (1 + \alpha) b_2 p_2 + (1 + 2\alpha) b_3 p_1 + (1 + 3\alpha) b_4, \tag{56}
\]

and so

\[
2n(1 + (2n - 1)\alpha) a_{2n} = p_{2n-1} + (1 + \alpha) b_2 p_{2n-2} + (1 + 2\alpha) b_3 p_{2n-3} + \cdots + (1 + (2n - 2)\alpha) b_{2n-1} p_1 + (1 + (2n - 1)\alpha) b_{2n} p_{2n-1} + \cdots \tag{57}
\]

\[
(2n + 1)(1 + 2n\alpha) a_{2n+1} = p_{2n} + (1 + \alpha) b_2 p_{2n-1} + (1 + 2\alpha) b_3 p_{2n-2} + \cdots + (1 + (2n - 1)\alpha) b_{2n} p_{2n-1} + (1 + 2na_{2n} p_{2n+1} + \cdots \tag{58}
\]

By using Lemma 3, (48), and (54), we have

\[
|a_2| \leq \frac{(C - D) + (A - B)}{2 \cdot (1 + \alpha)}, \tag{59}
\]

\[
|a_3| \leq \frac{(A - B + 1) [(A - B) + 2(C - D)]}{3 \cdot 2 \cdot (1 + 2\alpha)}.
\]
Again, by applying Lemma 3 and using (48)–(50), we obtain from (55) and (56)
\[
|a_n| \leq \frac{(A - B + 1)(A - B + 3)(C - D)}{4 \cdot 6 \cdot (1 + 3\alpha)}.
\]
(57)

\[
|a_n| \leq (A - B + 1)(A - B + 3)
\times [(A - B + 4(C - D))(5 \cdot 24 \cdot (1 + 4\alpha))^{-1}.
\]

It follows that (45) and (46) hold for \(n = 1, 2\). We now prove (45) by induction.
Equation (57) together with Lemma 3 yield
\[
|a_n| \leq \frac{1}{2n(1 + (2n - 1) \alpha)}
\times \left\{ (C - D) \left[ 1 + \sum_{k=1}^{n-1} (1 + (2k - 1) \alpha) |b_{2k}| \right.ight.
\left. \left. + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right] + (1 + (2n - 1) \alpha) |b_{2n}| \right\}.
\]
(58)

Again, by using Lemma 3 in (51), we have
\[
|b_{2n}| \leq \frac{(A - B)}{(2n - 1)(1 + (2n - 1) \alpha)}
\times \left[ 1 + \sum_{k=1}^{n-1} (1 + (2k - 1) \alpha) |b_{2k}| \right.
\left. + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right].
\]
(59)

Using (60) in (58), we obtain
\[
|a_n| \leq \frac{1}{2n(1 + (2n - 1) \alpha)}
\times \left\{ (C - D) + \frac{(A - B)}{2n - 1} \right\}
\times \left\{ 1 + \sum_{k=1}^{n-1} (1 + (2k - 1) \alpha) |b_{2k}| \right.
\left. + \sum_{k=1}^{n-1} (1 + 2k\alpha) |b_{2k+1}| \right\}.
\]
(61)

Using Lemma 5 in (63), we get
\[
|a_{2n}| \leq \frac{1}{2n(1 + (2n - 1) \alpha)}
\times \left\{ (C - D) + \frac{(A - B)}{2n - 1} \right\}
\times \left\{ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-1} (A - B + j)
\right.
\left. + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right\}.
\]
(62)

In order to prove (45), it is sufficient to show that
\[
\frac{1}{2m(1 + (2m - 1) \alpha)}
\times \left\{ (C - D) + \frac{(A - B)}{2m - 1} \right\}
\times \left\{ 1 + \sum_{k=1}^{m-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-1} (A - B + j)
\right.
\left. + \sum_{k=1}^{m-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right\}.
\]
(63)

Thus, (65) is valid for \(m = 3\).
Let us assume that (65) is true for all \(m, 3 < m \leq (n - 1)\).
Then, from (64), we have
\[
\frac{1}{2n(1 + (2n - 1) \alpha)}
\times \left\{ \sum_{k=1}^{n-1} \left\{ (C - D) + \frac{(A - B)}{2n - 1} \right\}
\times \left\{ 1 + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k - 1)!} \prod_{j=1}^{2k-1} (A - B + j)
\right.
\left. + \sum_{k=1}^{n-1} \frac{(A - B)}{(2k)!} \prod_{j=1}^{2k-1} (A - B + j) \right\} \right|.
\]
(64)
\[
= \frac{n-1}{n} \left\{ \frac{r-2}{2} \frac{1}{n-1(1 + (2n-1)\alpha)} \right. \\
\times \left[ (C - D) + \frac{(A - B)}{2n-1} \right] \\
\times \left[ 1 + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k-1)!} \prod_{j=1}^{k} (A - B + j) \right] \\
\left. + \sum_{k=1}^{n-2} \frac{(A - B)}{(2k)!} \prod_{j=1}^{k} (A - B + j) \right\} \\
+ \frac{1}{2n(1 + (2n-1)\alpha)} \left[ (C - D) + \frac{(A - B)}{2n-1} \right] \\
\times \left[ \frac{(A - B)}{(2(n-1) - 1)} \prod_{j=1}^{2n-4} (A - B + j) \right] \\
\times \frac{1}{(2(n-1) - 2)!} \prod_{j=1}^{2n-3} (A - B + j) \right) \\
+ \frac{1}{2n(1 + (2n-1)\alpha)} \left[ (C - D) + \frac{(A - B)}{2n-1} \right] \\
\times \left[ \frac{(A - B)}{(2(n-1) - 1)} \prod_{j=1}^{2n-4} (A - B + j) \right] \\
\times \frac{1}{(2(n-1) - 2)!} \prod_{j=1}^{2n-3} (A - B + j) \right) \right) \\
+ \frac{1}{2n(1 + (2n-1)\alpha)} \left[ (C - D) + \frac{(A - B)}{2n-1} \right] \\
\times \left[ \frac{(A - B)}{(2(n-1) - 1)} \prod_{j=1}^{2n-4} (A - B + j) \right] \\
\times \frac{1}{(2(n-1) - 2)!} \prod_{j=1}^{2n-3} (A - B + j) \right) \right) \\
+ \frac{1}{2n(1 + (2n-1)\alpha)} \left[ (C - D) + \frac{(A - B)}{2n-1} \right] \\
\times \left[ \frac{(A - B)}{(2(n-1) - 1)} \prod_{j=1}^{2n-4} (A - B + j) \right] \\
\times \frac{1}{(2(n-1) - 2)!} \prod_{j=1}^{2n-3} (A - B + j) \right) \right) \\
\left. \right\}.
\] (66)

Thus, (65) holds for \( m = n \), and hence, (45) follows. Similarly, we can prove (46).

By taking \( \alpha = 0 \) and \( \alpha = 1 \) in Theorem 11, respectively, we can easily derive the following corollaries.

**Corollary 12.** Let \( f \in K_c(A, B; C, D) \), then, for \( n \geq 1 \),

\[
|\alpha_{2n}| \leq \frac{1}{2n} \left\{ (C - D) + \frac{(A - B)}{2n-1} \right. \\
\times \left[ \frac{1}{(2n-2)!} \prod_{j=1}^{2n-2} (A - B + j) \right] \right\},
\]

\[
|\alpha_{2n+1}| \leq \frac{1}{2n + 1} \left\{ (C - D) + \frac{(A - B)}{2n} \right. \\
\times \left[ \frac{1}{(2n-1)!} \prod_{j=1}^{2n-1} (A - B + j) \right] \right\}.
\] (67)
Corollary 13. Let \( f \in K^*_1(A, B; C, D) \), then, for \( n \geq 1 \),

\[
\begin{align*}
|a_{2n}| & \leq \frac{1}{(2n)^2} \left\{ (C - D) + \frac{(A - B)}{2n - 1} \right. \\
& \quad \times \left. \left[ \frac{1}{(2n - 2)!} \prod_{j=1}^{2n-2} (A - B + j) \right] \right\}, \\
\left| a_{2n+1} \right| & \leq \frac{1}{(2n + 1)^2} \left\{ (C - D) + \frac{(A - B)}{2n} \right. \\
& \quad \times \left. \left[ \frac{1}{(2n - 1)!} \prod_{j=1}^{2n-1} (A - B + j) \right] \right\}.
\end{align*}
\]

(68)

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