Research Article

The Generalization of Prime Modules

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Piecewise prime (PWP) module M_R is defined in terms of a set of triangulating idempotents in R . The class of PWP modules properly contains the class of prime modules. Some properties of these modules are investigated here.

1. Introduction

All rings are associative, and R denotes a ring with unity 1. The word ideal without the adjective right or left means two-sided ideal. The right annihilator of ideals of R is denoted by $\text{r.ann}_R(I)$. A ring R is *quasi-Baer* (Baer) if the right annihilator of every right ideal (nonempty subset) of R is generated as a right ideal by an idempotent. We now recall a few definitions and results from [1] which motivated our study and serve as the background material for the present work. An idempotent $e \in R$ is a *left semicentral* idempotent if $exe = xe$, for all $x \in R$. Similarly right semicentral idempotent can be defined. The set of all left (right) semicentral idempotents of R is denoted by $S_l(R)$ ($S_r(R)$). An idempotent $e \in R$ is *semicentral reduced* if $S_l(eRe) = \{0, e\}$. If 1 is semicentral reduced, then R is called semicentral reduced. An ordered set $\{e_1, \dots, e_n\}$ of nonzero distinct idempotents of R is called a set of *left triangulating idempotents* of R if all the following hold:

- (i) $e_1 + \dots + e_n = 1$,
- (ii) $e_1 \in S_l(R)$,
- (iii) $e_{k+1} \in S_l(c_k R c_k)$, where $c_k = 1 - (e_1 + \dots + e_k)$ for $1 \leq k \leq n$.

From part (iii) of the previous definition, it can be seen that a set of left triangulating idempotents is a set of pairwise orthogonal idempotents. A set $E = \{e_1, \dots, e_n\}$ of left triangulating idempotents of R is *complete*, if each e_i is semicentral reduced. A (complete) set of right triangulating idempotents is defined similarly. The cardinalities of complete sets of left triangulating idempotents of R are the same and are denoted by $\tau \dim(R)$ [1, Theorem 2.10]. A ring R is called *piecewise*

prime if there exists a complete set of left triangulating idempotents $E = \{e_1, \dots, e_n\}$ of R , such that $xRy = 0$ implies $x = 0$ or $y = 0$ where $x \in e_i R e_j$ and $y \in e_j R e_k$ for $1 \leq i, j, k \leq n$. In view of this definition we say a proper ideal I in R is a PWP ideal if there is a complete set of left triangulating idempotents $E = \{e_1, \dots, e_n\}$, such that $xRy \subseteq I$ implies $x \in I$ or $y \in I$, where $x \in e_i R e_j$ and $y \in e_j R e_k$ for $1 \leq i, j, k \leq n$. If R is PWP, then it is PWP with respect to any complete set of left triangulating idempotents of R ; furthermore for a ring R with finite $\tau \dim(R)$, R is PWP if and only if R is quasi-Baer [1, Theorem 4.11].

A nonzero right R -module M is called a prime module if for any nonzero submodule N of M , $\text{r.ann}_R(N) = \text{r.ann}_R(M)$, and a proper submodule P of M is a prime submodule of M if the quotient module M/P is a prime module. The notion of prime submodule was first introduced in [2, 3]; see also [4, 5]. It is easy to see that M is a prime R -module if and only if for any $m \in M$, and $b \in R$ if $mRb = 0$, then $m = 0$ or $Mb = 0$.

In this work the concept of prime modules is developed to *piecewise prime* modules as it is done for rings in [1]. Throughout this work it is considered that $\tau \dim(R)$ is finite.

2. Main Results

Definition 1. Let M be an R -module and $S = \text{End}_R(M)$.

- (1) M is a *piecewise prime* (PWP) R -module with respect to a complete set of left triangulating idempotents $E = \{e_1, \dots, e_n\}$ of R , if for any $m \in M$, $e_i \in E$, and $b \in R$,

$$me_i Re_i b = 0 \implies me_i = 0 \quad \text{or} \quad Me_i b = 0. \quad (1)$$

- (2) Let N be a submodule of M . Then N is a *piecewise prime submodule* of M with respect to E if M/N is a PWP module with respect to E .
- (3) M is *piecewise endoprime* (PWE) with respect to a complete set of left triangulating idempotents $F = \{b_1, \dots, b_m\}$ of S , such that for each nonzero submodule $N \subseteq M$, $f \in S$, and $b_i \in F$, if $fb_iN = 0$, then $fb_i = 0$.

By Definition 1, N is a piecewise prime submodule of M with respect to a set of left triangulating idempotents E if for any $m \in M$, $e_i \in E$, and $b \in R$,

$$me_iRe_ib \subseteq N \implies me_i \in N \quad \text{or} \quad Me_ib \subseteq N. \quad (2)$$

Example 2. Let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R .

- (1) Let k_1 and k_2 be two fields and $R = k_1 \times k_2$. Then $M = R/k_1 \oplus R/k_2$ is not a prime module, but it is piecewise prime with respect to $\{(1, 0), (0, 1)\}$.
- (2) If M is a prime R -module, then it is piecewise prime with respect to any set of left triangulating idempotents of R .
- (3) Homomorphic image of M_R needs to be PWP with respect to E . For example, $\mathbb{Z}_{\mathbb{Z}}$ is a PWP module with respect to $\{0, 1\}$, but $\mathbb{Z}_4$ is not PWP because $r.\text{ann}_r(2) \neq r.\text{ann}_r(\mathbb{Z}_4)$.

Corollary 3. If M is a PWP R -module with respect to E , then any submodule of M is PWP with respect to E .

Proof. It can be seen by Definition 1. $\square$

Proposition 4. Let R be a ring with finite triangulating dimension.

- (1) I is a PWP ideal of R if and only if R/I is a PWP R -module.
- (2) R is a PWP ring if and only if R_R is PWP.

Proof. The part one is obtained by Definition 1, and for second let $I = 0$ in part one. $\square$

Proposition 5. Let M be an R -module, and let $E = \{e_1, \dots, e_n\}$ be a set of left triangulating idempotents of R . Then the following statements are equivalent:

- (1) M is PWP with respect to E ;
- (2) for each $N \subseteq M$, ideal I in R , and $e_i \in E$ if $Ne_iI = 0$ then $Ne_i = 0$ or $Me_iI = 0$;
- (3) for each $(m) \subseteq M$, ideal (a) in R , and $e_i \in E$ if $(m)e_i(a) = 0$ then $(m)e_i = 0$ or $Me_i(a) = 0$.

Proof. (1) $\implies$ (2) If $Ne_i \neq 0$, then there exists $n \in N$, such that $ne_i \neq 0$, and for any $b \in I$, $ne_iRe_ib = 0$. By Definition 1, for each $b \in I$, $Me_ib = 0$. This implies that $Me_iI = 0$.

(2) $\implies$ (3) In (2), let $N = (m)$ and $I = (a)$.

(3) $\implies$ (1) Let $me_iRe_ib = 0$ where $m \in M$, $e_i \in E$, and $b \in R$. Thus $me_iRe_ibR = 0$ or $(me_i)e_i(e_ib) = 0$. By (3),

$(me_i)e_i = 0$ or $Me_i(e_ib) = 0$. This implies that $me_i = 0$ or $Me_ib = 0$. $\square$

Proposition 6. Let M be an R -module, $S = \text{End}_R(M)$, let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R , and let $F = \{b_1, \dots, b_m\}$ be a complete set of left triangulating idempotents of S .

- (1) M is a PWP R -module with respect to E if and only if for each $N \subseteq M$ with $Ne_i \neq 0$, $\text{ann}_r(Ne_i) = \text{ann}_r(Me_i)$.
- (2) If M_R is PWP R -module with respect to E , then $\text{ann}_r(M)$ is a PWP ideal of R with respect to E .
- (3) If M_R is PWE with respect to F and retractable, then $\text{ann}_r(M)$ is a PWP ideal of R with respect to E .

Proof. (1) If $b \in \text{ann}_r(Ne_i)$, then there exists $n \in N$, such that $ne_iRe_i \neq 0$ and $ne_iRe_ib = 0$. Since M is PWP R -module with respect to E by Definition 1, $Me_ib = 0$. Hence $\text{ann}_r(Ne_i) = \text{ann}_r(Me_i)$. Conversely let $me_iRe_ib = 0$ where $e_i \in E$, $m \in M$, $b \in R$, and $me_i \neq 0$. Thus $b \in \text{ann}_r((me_iR)e_i)$ which means $b \in \text{ann}_r(Me_i)$ or $Me_ib = 0$.

(2) Let $Ie_iJ \subseteq \text{ann}_r(M)$ and $Ie_i \not\subseteq \text{ann}_r(M)$. Since $(MIe_i)e_iJ = 0$, and M is a PWP R -module with respect to E , by Proposition 5, $Me_iJ = 0$. Thus $e_iJ \subseteq \text{ann}_r(M)$. This implies that $\text{ann}_r(M)$ is a PWP ideal of R with respect to E .

(3) Let $Ie_iJ \subseteq \text{ann}_r(M)$ where $Ie_i, e_iJ \not\subseteq \text{ann}_r(M)$. Since M is retractable, then there exists a nonzero homomorphism $f : M \rightarrow MIe_i$. There exists $b_j \in F$, such that $fb_j \neq 0$. Since $Ie_iJ \subseteq \text{ann}_r(M)$, $fb_jMe_iJ = 0$. By assumption M is PWE with respect to F . This implies that $fb_j = 0$ which is a contradiction. Hence $\text{ann}_r(M)$ is a PWP ideal of R with respect to E . $\square$

A module M_R is called *retractable* if for any nonzero submodule N of M , $\text{Hom}_R(M, N) \neq 0$.

Theorem 7. Let M be an R -module, $S = \text{End}_R(M)$, and let $F = \{b_1, \dots, b_m\}$ be a complete set of left triangulating idempotents of S .

- (1) If ${}_S M$ is a PWP module with respect to F , then S is a PWP ring. The converse is true when M_R is retractable.
- (2) ${}_S M$ is a PWP module with respect to F , if and only if M_R is PWE with respect to F .

Proof. (1) Let $fb_iSb_i g = 0$ where $f, g \in S$, $b_i \in F$ and $b_i g \neq 0$. Thus there exists $m \in M$, such that $b_i gm \neq 0$ and $fb_iSb_i gm = 0$. Since ${}_S M$ is PWP with respect to F , $fb_iM = 0$ which means $fb_i = 0$. Conversely let $fb_iSb_i m = 0$ and $b_i m \neq 0$. Since M_R is retractable, there exists a nonzero homomorphism $b_i g \in \text{Hom}_R(M, b_i mR)$. Thus $fb_iSb_i g = 0$. Since S is PWP, $fb_i = 0$.

(2) Assume M is a PWP S -module with respect to F . Let $N \subseteq M$ and $fb_iN = 0$ where $f \in S$ and $b_i \in F$. Since ${}_S M$ is PWP, by Proposition 6(1), $fb_iM = 0$. Thus $fb_i = 0$. Conversely assume M_R be PWE with respect to F . Let $fb_iSb_i m = 0$ where $f \in S$, $b_i \in F$, $m \in M$, and $b_i m \neq 0$. If $N = Sb_i m$, then $fb_iN = 0$. This implies that $fb_i = 0$ or $fb_iM = 0$. Hence ${}_S M$ is PWP with respect to F . $\square$

Let M be a right R -module with $S = \text{End}_R(M)$. Then M_R is called a *quasi-Baer* module, if for any $N \subseteq_S M$, $\text{l.ann}_S(N) = Se$, where $e = e^2 \in S$ [6].

Corollary 8. *Let M be a retractable R -module, $S = \text{End}_R(M)$, and let $F = \{b_1, \dots, b_m\}$ be a complete set of left triangulating idempotents of S . Then the following statements are equivalent:*

- (1) M_R is a PWEP module with respect to F ;
- (2) ${}_S M$ is a PWP module with respect to F ;
- (3) M_R is quasi-Baer.

Proof. (1) $\Leftrightarrow$ (2) This is evident by Theorem 7(2).

(2) $\Leftrightarrow$ (3) By [6, Proposition 4.7], M_R is quasi-Baer if and only if S is quasi-Baer. By [1, Theorem 4.11], S is PWP with respect to F if and only if S is quasi-Baer. The result is obtained by Theorem 7(1). $\square$

Proposition 9. *Let Λ be an index set, and let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R .*

- (1) *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$. M is PWP with respect to E if and only if for each $\lambda \in \Lambda$, M_λ is PWP with respect to E .*
- (2) *Let $M = \prod_{\lambda \in \Lambda} M_\lambda$. M is PWP with respect to E if and only if for each $\lambda \in \Lambda$, M_λ is PWP with respect to E .*

Proof. (1) Assume M is PWP with respect to E . If $m_\lambda e_i Re_i b = 0$, where $m_\lambda \in M_\lambda$, $e_i \in E$, and $b \in R$ then $(0, \dots, m_\lambda, 0, \dots, 0)e_i Re_i b = 0$. Since M is PWP, $(0, \dots, m_\lambda, 0, \dots, 0)e_i = 0$ or $Me_i b = 0$. This implies that $m_\lambda e_i = 0$ or $M_\lambda e_i b = 0$ which means for each $\lambda \in \Lambda$, M_λ is PWP with respect to E . Conversely assume that for each $\lambda \in \Lambda$, M_λ is PWP with respect to E , and $(m_1, \dots, m_n, 0, \dots)e_i Re_i b = 0$. This implies that $m_\lambda e_i Re_i b = 0$. Since M_λ is PWP with respect to E , $m_\lambda e_i = 0$ or $Me_i b = 0$. Hence $(m_1, \dots, m_n, 0, \dots)e_i = 0$ or $Me_i b = 0$. Thus M is PWP with respect to E .

(2) It can be seen by similar method as in part (1). $\square$

Corollary 10. *Let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R , let M be an R -module, and let F be a free R -module.*

- (1) *R is quasi-Baer if and only if F is a PWP module with respect to E .*
- (2) *M is PWP with respect to E if and only if $F \otimes_R M$ is PWP with respect to E .*

Proof. It follows by [1, Theorem 4.11] and Proposition 9. $\square$

Proposition 11. *Let M be an R -module, and $S = \text{End}_R(M)$. Then ${}_S M$ is prime if and only if $\tau \dim(S) = 1$, and M_R is quasi-Baer.*

Proof. ($\Rightarrow$) Since M is a prime S -module, then for each $N \subseteq M$, $\text{l.ann}_S(N) = \text{l.ann}_S(M) = 0$. This implies that M_R is quasi-Baer. If $e^2 = e \in S$, then $M = eM \oplus (1-e)M$. Since ${}_S M$ is prime, $\text{l.ann}_S(eM) = \text{l.ann}_S((1-e)M) = \text{l.ann}_S(M)$. This implies that $e = 1$ or $e = 0$. Thus $\tau \dim(S) = 1$.

($\Leftarrow$) Let N be any submodule of ${}_S M$. Since M_R is quasi-Baer, $\text{l.ann}_S(N) = Se$, where $e \in S_r(S)$. Since $\tau \dim(S) = 1$, $e \in \{0, 1\}$. If $e = 1$, then $N = 0$. Thus $e = 0$. This implies that for each nonzero submodule $N \subseteq_S M$, $\text{l.ann}_S(N) = \text{l.ann}_S(M) = 0$. This means ${}_S M$ is prime. $\square$

It is folklore that prime radical plays an important role in the study of rings [7]. Following this concept is developed for modules of course by using a complete set of left triangulating idempotents of R .

Definition 12. Let M be an R -module, let N be a proper submodule of M , and let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R .

- (1) The *piecewise prime radical* of N in M with respect to E is denoted by $\text{PRad}(N)$ and is defined to be the intersection of all piecewise prime submodules of M with respect to E containing N .
- (2) $\text{PRad}(M)$ means the intersection of all piecewise prime submodules of M with respect to E . If M has no piecewise prime submodule with respect to E , then $\text{PRad}(M) = M$.

Proposition 13. *Let N be a submodule of R -module M .*

- (1) *If N is a submodule of R -module M , then $\text{PRad}(N) \subseteq \text{PRad}(M)$.*
- (2) *If $\text{PRad}(M) = K$, then $\text{PRad}(M/K) = 0$.*
- (3) *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of submodules M_i , then*

$$\text{PRad}(M) = \bigoplus_{i \in I} \text{PRad}(M_i). \quad (3)$$

Proof. Let $E = \{e_1, \dots, e_n\}$ be a complete set of left triangulating idempotents of R .

- (1) Let K be any piecewise prime submodule of M with respect to E . If $N \subseteq K$, then $\text{PRad}(N) \subseteq K$. If $N \not\subseteq K$, then by the definition it is easy to see that $N \cap K$ is a piecewise prime submodule of N with respect to E . Thus $\text{PRad}(N) \subseteq (N \cap K) \subseteq K$. Hence $\text{PRad}(N) \subseteq \text{PRad}(M)$.
- (2) Let P/K be a piecewise prime submodules of M/K with respect to E . By definition $(M/K)/(P/K)$ is a piecewise prime module with respect to E . Thus M/P is a piecewise prime module with respect to E . This implies that P is a piecewise prime submodules of M with respect to E . Hence $\text{PRad}(M/K) = 0$.
- (3) By (1) for each $i \in I$, $\text{PRad}(M_i) \subseteq \text{PRad}(M)$. This implies that

$$\bigoplus_{i \in I} \text{PRad}(M_i) \subseteq \text{PRad}(M). \quad (4)$$

Let $(m_i)_{i \in I} \in M \setminus \bigoplus_{i \in I} \text{PRad}(M_i)$. Then there exists $i \in I$, such that $m_i \notin \text{PRad}(M_i)$. By the definition there exists a piecewise prime submodule $N_i \subseteq M_i$ with respect to E , such that $m_i \notin N_i$. If $K = N_i \oplus (\bigoplus_{j \neq i} M_j)$, then K is a piecewise

prime submodule of M with respect to E , and $m \notin K$. Thus $m \notin \text{PRad}(M)$. It means that

$$\text{PRad}(M) = \bigoplus_{i \in I} \text{PRad}(M_i). \quad (5)$$

□

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