Research Article

Construction of Dominating Sets of Certain Graphs

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Let G be a simple graph. A set S ⊆ V is a dominating set of G, if every vertex in V \ S is adjacent to at least one vertex in S. We denote the family of dominating sets of a graph G with cardinality i by \( \mathcal{D}(G, i) \). In this paper we introduce graphs with specific constructions, which are denoted by \( G(m) \). We construct the dominating sets of \( G(m) \) by dominating sets of graphs \( G(m-1) \), \( G(m-2) \), and \( G(m-3) \). As an example of \( G(m) \), we consider \( G(P_{m}, i) \). As a consequence, we obtain the recursive formula for the number of dominating sets of \( G(m) \).

1. Introduction

Let \( G = (V, E) \) be a graph of order \( |V| = n \). For any vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N(v) = \{ u \in V \mid uv \in E \} \), and the closed neighborhood is the set \( N[v] = N(v) \cup \{ v \} \). For a set \( S \subseteq V \), the open neighborhood is \( N(S) = \bigcup_{v \in S} N(v) \), and the closed neighborhood is \( N[S] = N(S) \cup S \). A set \( S \subseteq V \) is a dominating set if \( N[S] = V \), or equivalently, every vertex in \( V \setminus S \) is adjacent to at least one vertex in \( S \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set in \( G \), and the family of \( \gamma \)-sets is denoted by \( \Gamma(G) \). For a detailed treatment of this parameter, the reader is referred to [1]. Let \( \mathcal{D}(G, i) \) be the family of dominating sets of a graph \( G \) with cardinality \( i \), and let \( d(G, i) = |\mathcal{D}(G, i)| \). The domination polynomial \( D(G, x) \) of \( G \) is defined as \( D(G, x) = \sum_{i \in \Gamma(G)} d(G, i) x^i \), where \( \gamma(G) \) is the domination number of \( G \).

The domination polynomial of a graph has been introduced by Alikhani in his Ph.D. thesis [2]. More recently it has been investigated with respect to special graphs, zeros, and application in network reliability; see [2–9].

Obviously study of the dominating sets of graphs is a method for finding the coefficients of the domination polynomial of graphs. Authors studied the construction of dominating sets of some families of graphs to study their domination polynomials; see [10–13]. In this paper we would like to study some further results of this kind.

In the next section we introduce graphs with specific construction which is denoted by \( G(m) \). As examples of these graphs, in Section 3 we study the dominating sets of paths and some other graphs. As a consequence, we give a recurrence relation for \( |\mathcal{D}(G(m), i)| \) and \( D(G(m), x) \).

As usual we use \( \lceil x \rceil \) and \( \lfloor x \rfloor \) for the smallest integer greater than or equal to \( x \) and the largest integer less than or equal to \( x \), respectively. In this paper we denote the set \( \{1, 2, \ldots, n\} \) simply by \( [n] \).

2. Dominating Sets of Graphs \( G(m) \)

A path is a connected graph in which two vertices have degree 1, and the remaining vertices have degree 2. Let \( P_n \) be the path with \( V(P_n) = [n] \) and \( E(P_n) = \{(1, 2), (2, 3), \ldots, (n-1, n)\} \); see Figure I(a).

In this section we introduce graphs with specific construction and study their dominating sets. Let \( P_{m+1} \) be a path with vertices labeled by \( y_0, y_1, \ldots, y_m \), for \( m \geq 3 \). Let \( G(m) \) be a graph obtained from \( G \) by identifying a vertex of \( G \) with an end vertex \( y_0 \) of \( P_{m+1} \). For example, if \( G \) is a path \( P_5 \), then \( G(m) = P_5(m) \) is a path \( P_{m+2} \).

We need some lemmas and theorems to obtain main results in this section.
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Lemma 1. \( \mathcal{D}(G, i) = \emptyset \) if and only if \( i > |V(G)| \) or \( i < \gamma(G) \).

Proof. It follows from the definition of dominating set of graph. \( \Box \)

We recall the following theorem.

Theorem 2 (see [14]). If \( e \in E(G) \), then \( \gamma(G - e) \leq \gamma(G) \).

The following lemma follows from Theorem 2.

Lemma 3. For any \( m \in N \), \( \gamma(G(m - 1)) \leq \gamma(G(m)) \leq \gamma(G(m - 1)) + 1 \).

Lemma 4 (see [15, page 371]). The domination number of path \( n \) is \( \gamma(P_n) = \lceil n/3 \rceil \).

A simple path is a path where all its internal vertices have degree two. In Figure 1(b), we have shown a graph \( G \) which contains a simple path of length \( k \) with vertices labeled 1, \ldots, \( k \), \( k + 1 \). The following lemma follows from Lemma 4.

Lemma 5. If a graph \( G \) contains a simple path of length \( 3k - 1 \), then every dominating set of \( G \) must contain at least \( k \) vertices of the path.

Lemma 6. If \( Y \in \mathcal{D}(G(m - 4), i - 1) \), and there exists \( x \in \{y_1, y_2, \ldots, y_m\} \), such that \( Y \cup \{x\} \in \mathcal{D}(G(m, i)) \), then \( Y \in \mathcal{D}(G(m - 3), i - 1) \).

Proof. Suppose that \( Y \notin \mathcal{D}(G(m - 3), i - 1) \). Since \( Y \in \mathcal{D}(G(m - 4), i - 1) \), \( Y \) contains at least one vertex labeled \( y_{m-5} \) or \( y_{m-4} \). If \( y_{m-4} \notin Y \), then \( Y \in \mathcal{D}(G(m - 3), i - 1) \), a contradiction. Hence, \( y_{m-5} \notin Y \). But then in this case, \( Y \notin \mathcal{D}(G(m, i)) \), for any \( x \in \{y_1, y_2, \ldots, y_m\} \), also a contradiction. \( \Box \)

Lemma 7. (i) If \( \mathcal{D}(G(m - 1), i - 1) = \mathcal{D}(G(m - 3), i - 1) = \emptyset \), by Lemmas 1 and 3, \( i - 1 \geq |V(G(m - 1))| \) or \( i - 1 < \gamma(G(m - 3)) \). In either case we have \( \mathcal{D}(G(m - 2), i - 1) = \emptyset \).

(ii) Suppose that \( \mathcal{D}(G(m - 2), i - 1) = \emptyset \), so by Lemma 1, we have \( i - 1 > |V(G(m - 2))| \) or \( i - 1 < \gamma(G(m - 2)) \). If \( i - 1 > |V(G(m - 2))| \), then \( i - 1 > |V(G(m - 3))| \), and hence, \( \mathcal{D}(G(m - 3), i - 1) = \emptyset \), a contradiction. So we have \( i - 1 < \gamma(G(m - 2), i - 1) \), and hence, \( \mathcal{D}(G(m - 1), i - 1) = \emptyset \), also a contradiction.

(iii) Suppose that \( \mathcal{D}(G(m), i) \neq \emptyset \). Let \( Y \in \mathcal{D}(G(m), i) \), then at least one vertex labeled \( y_m \) or \( y_{m-1} \) is in \( Y \). If \( y_m \notin Y \), then by Lemma 5, at least one vertex labeled \( y_{m-2} \) or \( y_{m-3} \) is in \( Y \). If \( y_{m-1} \notin Y \), then \( Y \notin \mathcal{D}(G(m-2), i-1) \), a contradiction. Now suppose that \( y_m \notin Y \), then by Lemma 5, at least one vertex labeled \( y_{m-2} \) or \( y_m \) is in \( Y \). If \( y_{m-2} \notin Y \) or \( y_{m-3} \notin Y \), then \( Y \notin \mathcal{D}(G(m-2), i-1) \), a contradiction. If \( y_{m-4} \notin Y \), then \( Y \notin \mathcal{D}(G(m-3), i-1) \), a contradiction. Therefore \( \mathcal{D}(G(m), i) = \emptyset \). \( \Box \)

Lemma 8. Suppose that \( \mathcal{D}(G(m), i) \neq \emptyset \), then

(i) \( \mathcal{D}(G(m - 1), i - 1) = \mathcal{D}(G(m - 2), i - 1) = \emptyset \) and \( \mathcal{D}(G(m - 3), i - 1) \neq \emptyset \) if and only if \( \gamma(G(m - 3)) + 1 \leq i \leq \gamma(G(m - 2)) + 1 \).

(ii) \( \mathcal{D}(G(m), i) = \emptyset \) if and only if \( i = |V(G(m))| \).

(iii) \( \mathcal{D}(G(m), i) \neq \emptyset \) if and only if \( i = |V(G(m))| \).

Proof. (i) Since \( \mathcal{D}(G(m - 1), i - 1) = \mathcal{D}(G(m - 3), i - 1) = \emptyset \), by Lemmas 1 and 3, \( i - 1 > |V(G(m - 1))| \) or \( i - 1 < \gamma(G(m - 3)) \). In either case we have \( \mathcal{D}(G(m - 2), i - 1) = \emptyset \).

(ii) Suppose that \( \mathcal{D}(G(m - 2), i - 1) = \emptyset \), so by Lemma 1, we have \( i - 1 > |V(G(m - 2))| \) or \( i - 1 < \gamma(G(m - 2)) \). If \( i - 1 > |V(G(m - 2))| \), then \( i - 1 > |V(G(m - 3))| \), and hence, \( \mathcal{D}(G(m - 3), i - 1) = \emptyset \), a contradiction. So we have \( i - 1 < \gamma(G(m - 2), i - 1) \), and hence, \( \mathcal{D}(G(m - 1), i - 1) = \emptyset \), also a contradiction.

(iii) Suppose that \( \mathcal{D}(G(m), i) \neq \emptyset \). Let \( Y \in \mathcal{D}(G(m), i) \), then at least one vertex labeled \( y_m \) or \( y_{m-1} \) is in \( Y \). If \( y_m \notin Y \), then by Lemma 5, at least one vertex labeled \( y_{m-2} \) or \( y_{m-3} \) is in \( Y \). If \( y_{m-1} \notin Y \), then \( Y \notin \mathcal{D}(G(m-2), i-1) \), a contradiction. Now suppose that \( y_m \notin Y \), then by Lemma 5, at least one vertex labeled \( y_{m-2} \) or \( y_m \) is in \( Y \). If \( y_{m-2} \notin Y \) or \( y_{m-3} \notin Y \), then \( Y \notin \mathcal{D}(G(m-2), i-1) \), a contradiction. If \( y_{m-4} \notin Y \), then \( Y \notin \mathcal{D}(G(m-3), i-1) \), a contradiction. Therefore \( \mathcal{D}(G(m), i) = \emptyset \). \( \Box \)
\(\mathcal{D}(G(m-2), i-1) = \mathcal{D}(G(m-3), i-1) = 0\), a contradiction. So we must have \(i-1 < \gamma(G(m-1))\). But we also have \(i-1 \geq \gamma(G(m-2))\) because \(\mathcal{D}(G(m-2), i-1) \neq 0\). Hence, we have \(\gamma(G(m-2)) + 1 \leq i < \gamma(G(m-1)) + 1\).

(\(\Rightarrow\)) If \(\gamma(G(m-2)) + 1 \leq i < \gamma(G(m-1)) + 1\), then by Lemma 1, \(\mathcal{D}(G(m-1), i-1) = 0\) and \(\mathcal{D}(G(m-2), i-1) \neq 0\) and \(\mathcal{D}(G(m-3), i-1) \neq 0\).

(\(\Leftarrow\)) Since \(\mathcal{D}(G(m-3), i-1) = 0\), by Lemma 1, we have \(i-1 > [V(G(m))] - 3\) or \(i-1 < \gamma(G(m-3))\). Since \(\mathcal{D}(G(m-2), i-1) \neq 0\), \(\mathcal{D}(G(m-3), i-1) \neq 0\), then by applying Lemma 1, we have \(\gamma(G(m-1)) \leq i-1 \leq [V(G(m))] - 3\), \(\gamma(G(m-2)) \leq i-1 \leq \gamma(G(m-3))\), and \(\gamma(G(m-3)) \leq i-1 \leq [V(G(m))] - 3\). So, by Lemma 3, \(\gamma(G(m-1)) \leq i-1 \leq \gamma(G(m-2))\), and hence \(\gamma(G(m-1)) + 1 \leq i \leq \gamma(G(m-2))\).

(ii) By Lemma 8(iii), we have that \(\mathcal{D}(G(m), i) \neq \emptyset\). If \(\mathcal{D}(G(m), i) \neq \emptyset\), then by applying Lemma 1, we have \(\gamma(G(m)) \leq i-1 \leq [V(G(m))] - 3\). Thus \(i = [V(G(m))] - 1\) or \([V(G(m))]\). But \(i \neq [V(G(m))]\) because \(\gamma(G(m-1)) \leq i-1 \leq [V(G(m))] - 3\).

(iii) By Lemma 8(iv), we have that \(\mathcal{D}(G(m), i) \neq \emptyset\). If \(\mathcal{D}(G(m), i) \neq \emptyset\), then by applying Lemma 1, we have \(\gamma(G(m)) \leq i-1 \leq [V(G(m))] - 3\). Therefore \(i = [V(G(m))] - 1\) or \([V(G(m))]\). But \(i \neq [V(G(m))]\) because \(\gamma(G(m-1)) \leq i-1 \leq [V(G(m))] - 3\). Thus \(i = [V(G(m))] - 1\) or \([V(G(m))]\)

\[\mathcal{D}(G(m), i) \subseteq \{\{y_{m}\} \cup X | X \in \mathcal{D}(G(m-2), i-1)\} \cup \{\{y_{m-1}\} \cup X | X \in \mathcal{D}(G(m-3), i-1)\} \cap \mathcal{D}(G(m-2), i-1)\].

\[\mathcal{D}(G(m), i) \subseteq \{\{y_{m}\} \cup X | X \in \mathcal{D}(G(m-3), i-1) \cap \mathcal{D}(G(m-2), i-1)\}.\]

**Proof.** (i) Obviously \(\{y_{m-1}\} \cup X | X \in \mathcal{D}(G(m-3), i-1)\) \(\subseteq \mathcal{D}(G(m), i)\). Now suppose that \(Y \in \mathcal{D}(G(m), i)\), then at least one of the vertices \(y_{m}\) or \(y_{m-1}\) is in \(Y\). If \(y_{m}\) \(\in Y\) then by Lemma 5 at least one of the vertices \(y_{m-1}, y_{m-2}, \cdots, y_{m-3}\) is in \(Y\). If \(y_{m-1} \in Y\) or \(y_{m-2} \in Y\), then \(Y \\neq \{y_{m}\} \in \mathcal{D}(G(m-1), i-1)\), a contradiction. If \(y_{m-3} \in Y\), then \(Y \setminus \{y_{m}\} \in \mathcal{D}(G(m-2), i-1)\), a contradiction. Now suppose that \(y_{m-1} \in Y\), then by Lemma 5 at least one of the vertices \(y_{m-2}, y_{m-3}, \text{ or } y_{m-4}\) is in \(Y\). If \(y_{m-2} \in Y\) or \(y_{m-3} \in Y\), then \(Y \setminus \{y_{m-1}\} \in \mathcal{D}(G(m-2), i-1)\), a contradiction. If \(y_{m-4} \in Y\), then \(Y \setminus \{y_{m-1}\} \in \mathcal{D}(G(m-3), i-1)\). So \(\mathcal{D}(G(m), i) \subseteq \{\{y_{m}\} \cup X | X \in \mathcal{D}(G(m-3), i-1)\} \cap \mathcal{D}(G(m-2), i-1)\).
1), \(i - 1\), \(X_2 \in D(G(m - 2), i - 1)\) \(\cup \{y_{m-1}\} \cup X \mid X \in D(G(m - 3), i - 1) \setminus D(G(m - 2), i - 1)\) \(\cup \{y_m\} \cup X \mid X \in D(G(m - 3), i - 1) \cap D(G(m - 2), i - 1)\). □

Theorem 10. For any \(m \geq 3\) and \(i \geq \gamma(G(m))\),

\[
|D(G(m), i)| = |D(G(m - 1), i - 1)| + |D(G(m - 2), i - 1)| + |D(G(m - 3), i - 1)|. \tag{1}
\]

Proof. It follows from Theorem 9. □

Theorem 11. For any \(m \geq 3\),

\[
D(G(m), x) = x(D(G(m - 1), x) + D(G(m - 2), x) + D(G(m - 3), x)). \tag{2}
\]

Proof. We have the result by definition of domination polynomial and Theorem 10. □

3. Dominating Sets of Paths and Some Other Graphs

First we investigate the dominating sets of paths. Since \(P_i = P_i(n - 1)\), we use the results for graph \(G(m)\) in the previous section to obtain properties of dominating sets of path. We suppose that \(P_i\) is labeled as in Figure 1(a) and denote \(D(P_n, i)\) simply by \(P^n_i\).

Lemma 12. Suppose that \(P^n_i \neq \emptyset\), then one has

(i) \(P^n_{i-1} = P^n_{i-2} = \emptyset\) and \(P^n_{i-3} \neq \emptyset\) if and only if \(n = 3k\) and \(i = k\) for some \(k \in N\),

(ii) \(P^n_{i-1} = P^n_{i-2} = \emptyset\) and \(P^n_{i-3} \neq \emptyset\) if and only if \(i = n\),

(iii) \(P^n_{i-1} = \emptyset, P^n_{i-2} \neq \emptyset\), and \(P^n_{i-3} \neq \emptyset\) if and only if \(n = 3k + 2\) and \(i = \lceil (3k + 2)/3\rceil\) for some \(k \in N\),

(iv) \(P^n_{i-1} \neq \emptyset, P^n_{i-2} \neq \emptyset\), and \(P^n_{i-3} = \emptyset\) if and only if \(i = n - 1\),

(v) \(P^n_{i-1} \neq \emptyset, P^n_{i-2} = \emptyset\), and \(P^n_{i-3} \neq \emptyset\) if and only if \([n - 1)/3\]+1 \leq i \leq n - 2\).

Proof. (i) \((\Rightarrow)\) By Lemmas 4 and 8(ii), we have \([n/3] \leq i < \lceil (n - 2)/3\rceil + 1\), which gives us \(n = 3k\) and \(i = k\) for some \(k \in N\).

(\(\Leftarrow\)) It follows from Lemmas 4 and 8(i).

(ii) It follows from Lemma 8(ii).

(iii) \((\Rightarrow)\) By Lemmas 4 and 8(iii), we have \([n - 2)/3\] + 1 \leq i < \lceil (n - 1)/3\rceil + 1\), which gives us \(n = 3k + 2\) and \(i = k + 1 = \lceil (3k + 2)/3\rceil\) for some \(k \in N\).

(\(\Leftarrow\)) It follows from Lemmas 4 and 8(iii).

(iv) It follows from Lemma 8(iv).

(v) It follows from Lemmas 4 and 8(v). □

The following theorem specify \(P^n_i\).

Theorem 13. For every \(n \geq 4\) and \(i \geq \lceil n/3 \rceil\),

\[
P^n_i = \begin{cases} \{2, 5, \ldots, n - 4, n - 1\}, & \text{if } P^n_{i-1} = P^n_{i-2} = \emptyset, \\ \{n\}, & \text{if } P^n_{i-1} = P^n_{i-2} = \emptyset, \\ \{2, 5, \ldots, n - 3, n\} \cup \{X \mid X \in P^n_{i-3}\}, & \text{if } P^n_{i-1} = \emptyset, \\ \{n\} \cup X_1, \{n - 1\} \cup X_2 | X_1 \in P^n_{i-1}, X_2 \in P^n_{i-2}, & \text{if } P^n_{i-3} = \emptyset, \\ \{n\} \cup X | X \in P^n_{i-3} \cap P^n_{i-2}, & \text{if } P^n_{i-1} \neq \emptyset, P^n_{i-2} \neq \emptyset, P^n_{i-3} \neq \emptyset, \end{cases}
\]

for some \(k \in N\). Since \(P^n_{i-3} = P^n_{i-2} = \emptyset\), then \(\{X \mid X \in P^n_{i-1}\} = \{2, 5, \ldots, n - 3, n\}\). Therefore \(P^n_i = \{2, 5, \ldots, n - 3, n\} \cup \{X \mid X \in P^n_{i-3}\}\).

Cases 4 and 5 follow from Theorem 9(iv) and (v), respectively. □

Example 14. We use Theorem 13 to construct \(P^n_i\) for \(2 \leq i \leq 6\). Since \(P^n_2 = P^n_3 = \emptyset\), by Theorem 13, \(P^n_2 = \{[2, 5]\}\). Since \(P^n_3 = [5]\), \(P^n_4 = \emptyset\), and \(P^n_5 = \emptyset\),
we get $\mathcal{P}^6_0 = \{[6]\}$. Since $\mathcal{P}^6_4 = \{[1,2,3,4], [1,2,3,5], [1,3,4,5], [1,3,4,5,6], [2,3,4,5,6], [1,2,3,4,5,6]\}$, $\mathcal{P}^6_5 = \{[4]\}$, and $\mathcal{P}^6_6 = \emptyset$, then by Theorem 13, $\mathcal{P}^6_k = \{[6] - x \mid x \in [6]\} = \{[1,2,3,4,6], [1,2,3,5,6], [1,3,4,5,6], [2,3,4,5,6], [1,2,3,4,5,6]\}$. And, for construction $\mathcal{P}^6_3 = \{[1,2,3,5,6], [1,2,3,4,5,6]\}$, we get $\gamma(T_n(m)) = \{[6]\}$, the result is true for $k = 0$. Now suppose that the result is true for all numbers less than or equal $m - 1$, and we prove it for $m$. By applying Theorem 2 for $e = a_{n-1}b_1$, we have the following inequalities:

$$\left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-1}{3} \right\rceil - 1 \leq \gamma(T_n(m)) \leq \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{m}{3} \right\rfloor. \quad (4)$$

Similarly by applying Theorem 2 for $e = b_1b_2, b_{m-3}b_{m-2}$ and induction hypothesis, we have the following inequalities; respectively,

$$\left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil \leq \gamma(T_n(m)) \leq \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{m-2}{3} \right\rfloor + 1, \quad (5)$$

$$\left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-4}{3} \right\rceil \leq \gamma(T_n(m)) \leq \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{m-4}{3} \right\rfloor + 1. \quad (6)$$

Now if $m = 3k$ for some $k \in \mathbb{N}$, then by (4) and (5) we have $\gamma(T_n(m)) = \left\lceil \frac{n}{3} \right\rceil + k = \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{(m-1)}{3} \right\rfloor$.

If $m = 3k + 1$ for some $k \in \mathbb{N}$, then by (4) and (6) we have $\gamma(T_n(m)) = \left\lceil \frac{n}{3} \right\rceil + k = \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{(m-1)}{3} \right\rfloor$.

Now if $m = 3k + 2$ for some $k \in \mathbb{N}$, we will consider the following cases.

(i) One has $n = 3k$ for some $k' \in \mathbb{N}$. By applying Theorem 2 for $a_{n-1}a_n$, we have the following inequalities:

$$\left\lceil \frac{n + m - 1}{3} \right\rceil \leq \gamma(T_n(m)) \leq \left\lceil \frac{n + m - 1}{3} \right\rceil + 1. \quad (7)$$

Now, by (4) and (7), we have $\gamma(T_{3k'}(3k + 2)) = k' + 1 = \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{(m-1)}{3} \right\rfloor$.

(ii) One has $n = 3k' + 2$ for some $k' \in \mathbb{N}$. By Theorem 2 for $e = a_1a_2$ we have

$$\left\lceil \frac{n - 1}{3} \right\rceil + \left\lceil \frac{m - 1}{3} \right\rceil \leq \gamma(T_n(m)) \leq \left\lceil \frac{n - 1}{3} \right\rceil + \left\lfloor \frac{m - 1}{3} \right\rfloor + 1. \quad (8)$$

By (4) and (8), we have $\gamma(T_{3k'+2}(3k + 2)) = k' + 2 = \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{(m-1)}{3} \right\rfloor$.

(iii) One has $n = 3k' + 1$ for some $k' \in \mathbb{N}$. We will prove that $\gamma(T_{3k'+2}(3k + 2)) = k' + 2$. We do it by induction on $k$. If $k = 0$, then by Lemma 17, $\gamma(T_{3k'+2}(2)) = \gamma(T_{3k'+1}(2)) = 2 + k'$. So the result is true for $k = 0$. Now suppose that the result is true for all number less than $k$, and we prove it for $k$. Figure 2: The tree $T_n(m)$. Theorem 18. For every $n \geq 3$ and $m \geq 0$, $\gamma(T_n(m)) = \left\lceil \frac{n}{3} \right\rceil + \left\lfloor \frac{(m-1)}{3} \right\rfloor$. Proof. By induction on $m$. If $m = 0$, then $\gamma(T_n(0)) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Now suppose that the theorem is true for all numbers less than or equal $m - 1$, and we prove it for $m$. By applying Theorem 2 for $e = a_{n-1}b_1$, we have the following inequalities:
By Lemma 17 and the induction hypothesis, we have the following equalities for \( k' - 1 < k \):

\[
y(T_{3k'+1} (3k + 2)) = y(T_{3k'+1} (3k' - 1)) = y(T_{3k'+1} (3k' + 2)) = k' - 1 + k + 1 + 2 = k' + k + 2.
\]

If \( k' - 1 > k \), then for some \( t > 0, k' = k + 1 + t \). Again by Lemma 17,

\[
y(T_{3k'+1} (3k + 2)) = y(T_{3k'+3t+4} (3k + 2)) = y(T_{3k'+3t+1} (3k + 2)) = k + t + k + 1 + 2 = k' + k + 2.
\]

Finally, for \( k' - 1 = k \), we have \( y(T_{3k'+1} (3k + 2)) = y(T_{3k'+4} (3k + 2)) = y(T_{3(k+1)+1} (3k + 2)) = 2k + 3 = k' + k + 2 \). Therefore in all cases \( y(T_n(m)) = \lceil n/3 \rceil + \lfloor (m - 1)/3 \rfloor \).

\[\square\]

Let \( D(T_n(m), i) \) or simply \( D_i \) be the family of dominating set of \( T_n(m) \) with cardinality \( i \). By Lemma 8 and Theorem 9 we prove the following lemma, which is some properties of \( D_i \).

**Lemma 19.** Suppose that \( D_i \neq \emptyset \), then

\[\text{(i)} \quad D_{i-1} = D_{i-2} = \emptyset \text{ and } D_{i-3} \neq \emptyset \text{ if and only if } m = 3k + 1, i = \lfloor n/3 \rfloor + k, \text{ for some } k \in N,\]

\[\text{(ii)} \quad D_{i-1} = D_{i-3} = \emptyset, \text{ and } D_{i-1} \neq \emptyset \text{ if and only if } i = n + m,\]

\[\text{(iii)} \quad D_{i-1} = D_{i-2} = \emptyset, \text{ and } D_{i-3} \neq \emptyset \text{ if and only if } m = 3k, i = \lfloor n/3 \rfloor + k, \text{ for some } k \in N,\]

\[\text{(iv)} \quad D_{i-1} \neq \emptyset, D_{i-2} = \emptyset, \text{ and } D_{i-3} = \emptyset \text{ if and only if } i = m + n - 1,\]

\[\text{(v)} \quad D_{i-1} \neq \emptyset, D_{i-2} \neq \emptyset, \text{ and } D_{i-3} \neq \emptyset \text{ if and only if } \lfloor n/3 \rfloor + \lfloor (m - 2)/3 \rfloor + 1 \leq i \leq m + n - 2.\]

**Proof.**

\[\text{(i)} \quad \text{By Lemma 8(i) and Theorem 18, } \lfloor n/3 \rfloor + \lfloor (m-4)/3 \rfloor + 1 \leq i < \lfloor n/3 \rfloor + \lfloor (m-3)/3 \rfloor + 1, \text{ which give us } m = 3k + 1, i = \lfloor n/3 \rfloor + k, \text{ for some } k \in N.\]

\[\text{(ii)} \quad \text{By Lemma 8(ii), } i = |V(T_n(m))| = m + n.\]

\[\text{(iii)} \quad \text{By Lemma 8(iii) and Theorem 18, } \lfloor n/3 \rfloor + \lfloor (m-3)/3 \rfloor + 1 \leq i < \lfloor n/3 \rfloor + \lfloor (m-2)/3 \rfloor + 1. \text{ Therefore } m = 3k \text{ and } i = \lfloor n/3 \rfloor + k \text{ for some } k \in N.\]

\[\text{(iv)} \quad \text{By Lemma 8(iv), } i = |V(T_n(m))| - 1 = n + m - 1.\]

\[\text{(v)} \quad \text{By Lemma 8(v) and Theorem 18, } \lfloor n/3 \rfloor + \lfloor (m-2)/3 \rfloor + 1 \leq i \leq m + n - 2.\]

Here, we suppose that the tree \( T_n(m) \) labeled as shown in Figure 3. The following theorem constructs \( D_i \) from \( D_i \), \( D_{i-1} \), and \( D_{i-2} \).

**Theorem 20.**

\[\text{(i)} \quad \text{If } D_i = D_{i-1} = D_{i-2} = \emptyset \text{ and } D_{i-3} \neq \emptyset, \text{ then } D_i = \left\{ \{m+n-1\} \cup X \mid X \in D_{i-3} \right\}. \]

\[\text{(ii)} \quad \text{If } D_{i-1} = D_{i-2} = D_{i-3} = \emptyset, \text{ and } D_{i-1} \neq \emptyset, \text{ then } D_i = \left\{ \{m+n\} \cup X \mid X \in D_{i-1} \right\}. \]

\[\text{(iii)} \quad \text{If } D_{i-1} = D_{i-2} = \emptyset, \text{ and } D_{i-3} \neq \emptyset, \text{ then } D_i = \left\{ \{m+n\} \cup X \mid X \in D_{i-3} \right\}. \]

\[\text{(iv)} \quad \text{If } D_{i-1} = D_{i-2} = \emptyset, \text{ and } D_{i-1} \neq \emptyset, \text{ then } D_i = \left\{ \{m+n\} \cup X \left| X \in D_{i-1} \right. \right\}. \]

\[\text{(v)} \quad \text{If } D_{i-1} = D_{i-2} = \emptyset, \text{ and } D_{i-3} \neq \emptyset, \text{ and } D_{i-1} \neq \emptyset, \text{ then } D_i = \left\{ \{m+n\} \cup X \mid X \in D_{i-3} \cup D_{i-1} \right\}. \]

**Proof.** It follows from Theorem 9. \[\square\]

Now we obtain the following theorem from Theorem 20 or Theorem 10.
Theorem 21. If $\mathcal{D}_{i}^{n,m}$ is the family of dominating set of $T_{n}(m)$ with cardinality $i$, then $|\mathcal{D}_{i}^{n,m}| = |\mathcal{D}_{i-1}^{n,m-1}| + |\mathcal{D}_{i-1}^{n,m-2}| + |\mathcal{D}_{i-1}^{n,m-3}|$.

Note that the tree $T_{n}(m)$ is obtained by gluing an end vertex of a path $P_{m}$ to the vertex $a_{n-1}$ of $P_{n}$. We now consider another type of related tree $T_{m}(n)$ (or special case of $(T_{n}(m))(k)$), which is obtained from $T_{m}(n)$ by gluing an end vertex of a path $P_{k}$ to the vertex $a_{2}$ of $P_{n}$, see Figure 4.

In other words, let $T_{m,n}(k) = (A, B, C, E)$, $k, n \geq 0$, where $A \cup B \cup C$ is its vertex set, $A = \{a_{1}, \ldots, a_{k}\}$, $B = \{b_{1}, \ldots, b_{n}\}$, $C = \{c_{1}, \ldots, c_{k}\}$, and the edge set $E = \{a_{i}a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_{i}b_{i+1} : 1 \leq i \leq m-1\} \cup \{c_{i}c_{i+1} : 1 \leq i \leq k-1\} \cup \{a_{n-1}b_{1}, a_{2}c_{1}\}$.

Theorem 22. For every $n \geq 3$ and $m, k \geq 0$, $\gamma(T_{m,n}(k)) = \lceil n/3 \rceil + (\lfloor (m-1)/3 \rfloor) + (k-1)/3$.

Proof. Similar to the proof of Theorem 18. □

By Theorem 10, we have the following theorem.

Theorem 23. If $\mathcal{D}_{i}^{n,m,k}$ is the family of dominating set of $T_{m,n}(k)$ with cardinality $i$, then $|\mathcal{D}_{i}^{n,m,k}| = |\mathcal{D}_{i-1}^{n,m,k-1}| + |\mathcal{D}_{i-1}^{n,m,k-2}| + |\mathcal{D}_{i-1}^{n,m,k-3}|$.

As other examples for graphs $G(m)$, we consider graph $C_{n}(m)$ which obtain by gluing a path to one vertex of cycle as shown in Figure 4. By Theorem 10 we have the following theorem for $C_{n}(m)$.

Theorem 24. For every $n, m \geq 3$, $|\mathcal{D}(C_{n}(m), i)| = |\mathcal{D}(C_{n}(m-1), i-1)| + |\mathcal{D}(C_{n}(m-2), i-1)| + |\mathcal{D}(C_{n}(m-3), i-1)|$.

Let sun be a graph which is obtained by gluing end vertices of paths $P_{i_{1}}, P_{i_{2}}, \ldots, P_{i_{n}}$ to the $n$ vertices of $C_{n}$.

We denote this graph by $S(i_{1}, i_{2}, \ldots, i_{n})$. Since sun is the special case of $(\cdots((C_{n}(i_{1}))(i_{2})) \cdots (i_{n}))$, we have the following theorem for sun.

Theorem 25. Suppose that for every $1 \leq j \leq n$, $i_{j} \geq 3$ and $i \geq \gamma(S(i_{1}, i_{2}, \ldots, i_{n}))$, then $|\mathcal{D}(S(i_{1}, i_{2}, \ldots, i_{n}), i)| = |\mathcal{D}(S(i_{1}, i_{2}, \ldots, i_{n}-1), i-1)| + |\mathcal{D}(S(i_{1}, i_{2}, \ldots, i_{n}-2), i-1)| + |\mathcal{D}(S(i_{1}, i_{2}, \ldots, i_{n}-3), i-1)|$.

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References

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