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Research Article

Construction of Dominating Sets of Certain Graphs

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Let G be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. We denote the family of dominating sets of a graph G with cardinality i by $\mathscr{D}(G, i)$. In this paper we introduce graphs with specific constructions, which are denoted by $G(m)$. We construct the dominating sets of $G(m)$ by dominating sets of graphs $G(m-1)$, $G(m-2)$, and $G(m-3)$. As an example of $G(m)$, we consider $\mathscr{D}(P_n, i)$. As a consequence, we obtain the recursive formula for the number of dominating sets of $G(m)$.

1. Introduction

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V uv \in E\}$, and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G, and the family of γ-sets is denoted by $\Gamma(G)$. For a detailed treatment of this parameter, the reader is referred to [1]. Let $\mathscr{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i, and let $d(G, i) = |\mathscr{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G.

The domination polynomial of a graph has been introduced by Alikhani in his Ph.D. thesis [2]. More recently it has been investigated with respect to special graphs, zeros, and application in network reliability; see [2–9].

Obviously study of the dominating sets of graphs is a method for finding the coefficients of the domination polynomial of graphs. Authors studied the construction of dominating sets of some families of graphs to study their domination polynomials; see [10–13]. In this paper we would like to study some further results of this kind.

In the next section we introduce graphs with specific construction which is denoted by $G(m)$. As examples of these graphs, in Section 3 we study the dominating sets of paths and some other graphs. As a consequence, we give a recurrence relation for $|\mathscr{D}(G(m), i)|$ and $D(G(m), x)$.

As usual we use $\lceil x \rceil$, $\lfloor x \rfloor$ for the smallest integer greater than or equal to x and the largest integer less than or equal to x, respectively. In this paper we denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$.

2. Dominating Sets of Graphs $G(m)$

A *path* is a connected graph in which two vertices have degree 1, and the remaining vertices have degree 2. Let P_n be the path with $V(P_n) = [n]$ and $E(P_n) = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$; see Figure 1(a).

In this section we introduce graphs with specific construction and study their dominating sets. Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \ldots, y_m$, for $m \geq 3$. Let $G(m)$ be a graph obtained from G by identifying a vertex of G with an end vertex y_0 of P_{m+1}. For example, if G is a path P_2, then $G(m) = P_2(m)$ is a path P_{m+2}.

We need some lemmas and theorems to obtain main results in this section.

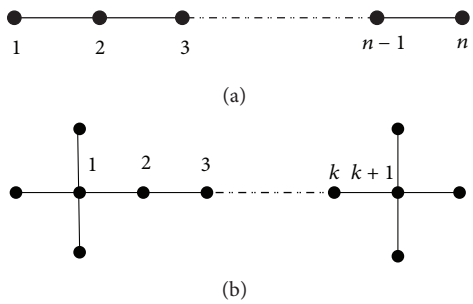

Figure 1: (a) The path P_n; (b) graph G contains a simple path of length k.

Lemma 1. *$\mathscr{D}(G(m), i) = \emptyset$ if and only if $i > |V(G(m))|$ or $i < \gamma(G(m))$.*

Proof. It follows from the definition of dominating set of graph. □

We recall the following theorem.

Theorem 2 (see [14]). *If $e \in E(G)$, then $\gamma(G-e) - 1 \leq \gamma(G) \leq \gamma(G-e)$.*

The following lemma follows from Theorem 2.

Lemma 3. *For any $m \in N$, $\gamma(G(m-1)) \leq \gamma(G(m)) \leq \gamma(G(m-1)) + 1$.*

Lemma 4 (see [15, page 371]). *The domination number of path is $\gamma(P_n) = \lceil n/3 \rceil$.*

A *simple path* is a path where all its internal vertices have degree two. In Figure 1(b), we have shown a graph G which contains a simple path of length k with vertices labeled $1, \ldots, k, k+1$. The following lemma follows from Lemma 4.

Lemma 5. *If a graph G contains a simple path of length $3k-1$, then every dominating set of G must contain at least k vertices of the path.*

Lemma 6. *If $Y \in \mathscr{D}(G(m-4), i-1)$, and there exists $x \in \{y_1, y_2, \ldots, y_m\}$, such that $Y \cup \{x\} \in \mathscr{D}(G(m, i))$, then $Y \in \mathscr{D}(G(m-3), i-1)$.*

Proof. Suppose that $Y \notin \mathscr{D}(G(m-3), i-1)$. Since $Y \in \mathscr{D}(G(m-4), i-1)$, Y contains at least one vertex labeled y_{m-5} or y_{m-4}. If $y_{m-4} \in Y$, then $Y \in \mathscr{D}(G(m-3), i-1)$, a contradiction. Hence, $y_{m-5} \in Y$. But then in this case, $Y \cup \{x\} \notin \mathscr{D}(G(m), i)$, for any $x \in \{y_1, y_2, \ldots, y_m\}$, also a contradiction. □

Lemma 7. (i) *If $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$, then $\mathscr{D}(G(m-2), i-1) = \emptyset$.*

(ii) *If $\mathscr{D}(G(m-1), i-1) \neq \emptyset, \mathscr{D}(G(m-3), i-1) \neq \emptyset$ then, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$.*

(iii) *If $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$, then $\mathscr{D}(G(m), i) = \emptyset$.*

Proof. (i) Since $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$, by Lemmas 1 and 3, $i-1 > |V(G(m-1))|$ or $i-1 < \gamma(G(m-3))$. In either case we have $\mathscr{D}(G(m-2), i-1) = \emptyset$.

(ii) Suppose that $\mathscr{D}(G(m-2), i-1) = \emptyset$, so by Lemma 1, we have $i-1 > |V(G(m-2))|$ or $i-1 < \gamma(G(m-2))$. If $i-1 > |V(G(m-2))|$, then $i-1 > |V(G(m-3))|$, and hence, $\mathscr{D}(G(m-3), i-1) = \emptyset$, a contradiction. So we have $i-1 < \gamma(G(m-2), i-1)$, and hence, $\mathscr{D}(G(m-1), i-1) = \emptyset$, also a contradiction.

(iii) Suppose that $\mathscr{D}(G(m), i) \neq \emptyset$. Let $Y \in \mathscr{D}(G(m), i)$, then at least one vertex labeled y_m or y_{m-1} is in Y. If $y_m \in Y$, then by Lemma 5, at least one vertex labeled y_{m-1}, y_{m-2} or y_{m-3} is in Y. If $y_{m-1} \in Y$ or $y_{m-2} \in Y$, then $Y - \{y_m\} \in \mathscr{D}(G(m-1), i-1)$, a contradiction. If $y_{m-3} \in Y$, then $Y - \{y_m\} \in \mathscr{D}(G(m-2), i-1)$, a contradiction. Now suppose that $y_{m-1} \in Y$, then by Lemma 5, at least one vertex labeled y_{m-2}, y_{m-3} or y_{m-4} is in Y. If $y_{m-2} \in Y$ or $y_{m-3} \in Y$, then $Y - \{y_{m-1}\} \in \mathscr{D}(G(m-2), i-1)$, a contradiction. If $y_{m-4} \in Y$, then $Y - \{y_{m-1}\} \in \mathscr{D}(G(m-3), i-1)$, a contradiction. Therefore $\mathscr{D}(G(m), i) = \emptyset$. □

Lemma 8. *Suppose that $\mathscr{D}(G(m), i) \neq \emptyset$, then*

(i) *$\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-2), i-1) = \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$ if and only if $\gamma(G(m-3)) + 1 \leq i < \gamma(G(m-2)) + 1$,*

(ii) *$\mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$ and $\mathscr{D}(G(m-1), i-1) \neq \emptyset$ if and only if $i = |V(G(m))|$,*

(iii) *$\mathscr{D}(G(m-1), i-1) = \emptyset, \mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$ if and only if $\gamma(G(m-2)) + 1 \leq i < \gamma(G(m-1)) + 1$,*

(iv) *$\mathscr{D}(G(m-1), i-1) \neq \emptyset, \mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) = \emptyset$ if and only if $i = |V(G(m))| - 1$,*

(v) *$\mathscr{D}(G(m-1), i-1) \neq \emptyset, \mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$ if and only if $\gamma(G(m-1)) + 1 \leq i \leq |V(G(m))| - 2$.*

Proof. (i) ($\Rightarrow$) Since $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-2), i-1) = \emptyset$, by Lemmas 1 and 3, we have $i-1 > |V(G(m))| - 1$ or $i-1 < \gamma(G(m-2))$. If $i-1 > |V(G(m))| - 1$, then $i > |V(G(m))|$, and by Lemma 1, $\mathscr{D}(G(m), i) = \emptyset$, a contradiction. So we have $i-1 < \gamma(G(m-2))$, and since $\mathscr{D}(G(m-3), i-1) \neq \emptyset$, together we have $\gamma(G(m-3)) \leq i-1 < \gamma(G(m-2))$. So $\gamma(G(m-3)) + 1 \leq i < \gamma(G(m-2)) + 1$.

($\Leftarrow$) If $\gamma(G(m-3)) + 1 \leq i < \gamma(G(m-2)) + 1$, then by Lemma 1, we have $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-2), i-1) = \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$.

(ii) ($\Rightarrow$) Since $\mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$, by Lemmas 1 and 3, $i-1 > |V(G(m))| - 2$ or $i-1 < \gamma(G(m-3))$. If $i-1 < \gamma(G(m-3))$, then $i-1 < \gamma(G(m-1))$, and hence $\mathscr{D}(G(m-1), i-1) = \emptyset$, a contradiction. So we must have $i-1 > |V(G(m))| - 2$. Also since $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, we have $i-1 \leq |V(G(m))| - 1$. Therefore we have $i = |V(G(m))|$.

($\Leftarrow$) If $i = |V(G(m))|$, then by Lemma 1, we have $\mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$ and $\mathscr{D}(G(m-1), i-1) \neq \emptyset$.

(iii) ($\Rightarrow$) Since $\mathscr{D}(G(m-1), i-1) = \emptyset$, by Lemma 1, $i-1 > |V(G(m))| - 1$ or $i-1 < \gamma(G(m-1))$. If $i-1 > |V(G(m))| - 1$, then $i-1 > |V(G(m))| - 2$, and by Lemma 1,

$\mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$, a contradiction. So we must have $i - 1 < \gamma(G(m-1))$. But we also have $i - 1 \geq \gamma(G(m-2))$ because $\mathscr{D}(G(m-2), i-1) \neq \emptyset$. Hence, we have $\gamma(G(m-2)) + 1 \leq i < \gamma(G(m-1)) + 1$.

($\Leftarrow$) If $\gamma(G(m-2)) + 1 \leq i < \gamma(G(m-1)) + 1$, then by Lemma 1, $\mathscr{D}(G(m-1), i-1) = \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$.

(iv) ($\Rightarrow$) Since $\mathscr{D}(G(m-3), i-1) = \emptyset$, by Lemma 1, we have $i - 1 > |V(G(m))| - 3$ or $i - 1 < \gamma(G(m-3))$. Since $\mathscr{D}(G(m-2)), i-1) \neq \emptyset$, by Lemma 1, we have $\gamma(G(m-2)) \leq i - 1 \leq |V(G(m))| - 2$. Therefore $i - 1 < \gamma(G(m-3))$ is not possible. Hence we must have $i - 1 > |V(G(m))| - 3$. Thus $i = |V(G(m))| - 1$ or $|V(G(m))|$. But $i \neq |V(G(m))|$ because $\mathscr{D}(G(m-3), i-1) = \emptyset$. So we have $i = |V(G(m))| - 1$.

($\Leftarrow$) If $i = |V(G(m))| - 1$, then by Lemma 1, $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) = \emptyset$.

(v) ($\Rightarrow$) Since $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$, and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$, then by applying Lemma 1, we have $\gamma(G(m-1)) \leq i - 1 \leq |V(G(m))| - 1$, $\gamma(G(m-2)) \leq i - 1 \leq |V(G(m))| - 2$, and $\gamma(G(m-3)) \leq i - 1 \leq |V(G(m))| - 3$. So, by Lemma 3, $\gamma(G(m-1)) \leq i - 1 \leq |V(G(m))| - 3$, and hence $\gamma(G(m-1)) + 1 \leq i \leq |V(G(m))| - 2$.

($\Leftarrow$) If $\gamma(G(m-1)) + 1 \leq i \leq |V(G(m))| - 2$, then by Lemma 1 we have the result. □

By Lemma 6, for the construction of $\mathscr{D}(G(m), i)$, suffices to consider $\mathscr{D}(G(m-1), i-1)$, $\mathscr{D}(G(m-2), i-1)$, and $\mathscr{D}(G(m-3), i-1)$. By Lemma 7, we need only to consider the five cases in the following theorem.

Theorem 9. *For every $i \geq \gamma(G(m))$,*

(i) *if $\mathscr{D}(G(m-1), i-1) = \mathscr{D}(G(m-2), i-1) = \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$, then $\mathscr{D}(G(m), i) = \{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1)\}$;*

(ii) *if $\mathscr{D}(G(m-2), i-1) = \mathscr{D}(G(m-3), i-1) = \emptyset$ and $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, then $\mathscr{D}(G(m), i) = \mathscr{D}(G(m), |V(G(m))|) = \{\{y_m\} \cup X \mid X \in \mathscr{D}(G(m-1), i-1)\}$;*

(iii) *if $\mathscr{D}(G(m-1), i-1) = \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$, then $\mathscr{D}(G(m), i) = \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-2), i-1), X_2 \in \mathscr{D}(G(m-3), i-1)\}$;*

(iv) *if $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) = \emptyset$, then $\mathscr{D}(G(m), i) = \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-1), i-1), X_2 \in \mathscr{D}(G(m-2), i-1)\}$;*

(v) *if $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$ are nonempty, then $\mathscr{D}(G(m), i) = \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-1), i-1), X_2 \in \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \setminus \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_m\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \cap \mathscr{D}(G(m-2), i-1)\}$.*

Proof. (i) Obviously $\{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1)\} \subseteq \mathscr{D}(G(m), i)$. Now suppose that $Y \in \mathscr{D}(G(m), i)$, then at least one of the vertices y_m or y_{m-1} is in Y. If $y_m \in Y$ then by Lemma 5 at least one of the vertices y_{m-1}, y_{m-2}, or y_{m-3} is in Y. If $y_{m-1} \in Y$ or $y_{m-2} \in Y$, then $Y - \{y_m\} \in \mathscr{D}(G(m-1), i-1)$, a contradiction. If $y_{m-3} \in Y$, then $Y - \{y_m\} \in \mathscr{D}(G(m-2), i-1)$, a contradiction. Now suppose that $y_{m-1} \in Y$, then by Lemma 5 at least one of the vertices y_{m-2}, y_{m-3}, or y_{m-4} is in Y. If $y_{m-2} \in Y$ or $y_{m-3} \in Y$, then $Y - \{y_{m-1}\} \in \mathscr{D}(G(m-2), i-1)$, a contradiction. If $y_{m-4} \in Y$, then $Y - \{y_{m-1}\} \in \mathscr{D}(G(m-3), i-1)$. So $\mathscr{D}(G(m), i) \subseteq \{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1)\}$.

(ii) By Lemma 8(ii), $i = |V(G(m))|$. If we suppose that $G(m)$ has labeled with numbers in $[|V(G(m))|]$, then $\mathscr{D}(G(m), i) = \mathscr{D}(G(m), |V(G)|) = \{[|V(G)|]\} = \{\{y_m\} \cup X \mid X \in \mathscr{D}(G(m-1), i-1)\}$.

(iii) It is obvious that $\{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-2), i-1), X_2 \in \mathscr{D}(G(m-3), i-1)\} \subseteq \mathscr{D}(G(m), i)$. Now let $Y \in \mathscr{D}(G(m), i)$. Then y_m or y_{m-1} is in Y. If $y_m \in Y$, then by Lemma 5, at least one vertices y_{m-1}, y_{m-2}, or y_{m-3} is in Y. If y_{m-1} or y_{m-2} is in Y, then $Y - \{y_m\} \in \mathscr{D}(G(m-1), i-1)$, a contradiction, because $\mathscr{D}(G(m-1), i-1) = \emptyset$. Hence, $y_{m-3} \in Y$, $y_{m-2} \notin Y$, and $y_{m-1} \notin Y$. Therefore $Y = X \cup \{y_m\}$ for some $X \in \mathscr{D}(G(m-2), i-1)$. Now suppose that $y_{m-1} \in Y$ and $y_m \notin Y$. By Lemma 5, at least one vertex labeled y_{m-2}, y_{m-3}, or y_{m-4} is in Y. By Lemma 8(iii), $\gamma(G(m-2)) \leq i - 1 < \gamma(G(m-1)) \leq \gamma(G(m))$, and on the other hand since $\mathscr{D}(G(m), i) \neq \emptyset$, then $i \geq \gamma(G(m))$. So together we have $i = \gamma(G(m))$. Therefore $\mathscr{D}(G(m), i) = \Gamma(G(m))$, and so all these sets in $\Gamma(G(m))$ have minimum cardinality. Therefore, since $Y \in \mathscr{D}(G(m), i)$ and $y_{m-1} \in Y$, then $y_{m-2} \notin Y$. Hence y_{m-3} or y_{m-4} is in Y, but $y_{m-2} \notin Y$. Thus $Y = X \cup \{y_{m-1}\}$ for some $X \in \mathscr{D}(G(m-3), i-1)$. So $\mathscr{D}(G(m), i) \subseteq \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-2), i-1), X_2 \in \mathscr{D}(G(m-3), i-1)\}$.

(iv) By Lemma 8, $i = |V(G(m))| - 1$. If we suppose that $G(m)$ has labeled with numbers in $[|V(G(m))|]$, then $\mathscr{D}(G(m), i) = \{[|V(G(m))|] - \{x\} \mid x \in [|V(G(m))|]\} = \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-1), i-1), X_2 \in \mathscr{D}(G(m-2), i-1)\}$.

(v) $\mathscr{D}(G(m-1), i-1) \neq \emptyset$, $\mathscr{D}(G(m-2), i-1) \neq \emptyset$ and $\mathscr{D}(G(m-3), i-1) \neq \emptyset$. Let $X \in \mathscr{D}(G(m-1), i-1)$, so at least one vertex labeled y_{m-1} or y_{m-2} is in X. If y_{m-1} or $y_{m-2} \in X$, then $X \cup \{y_m\} \in \mathscr{D}(G(m), i)$. Now let $X \in \mathscr{D}(G(m-2), i-1)$, then y_{m-2} or y_{m-3} is in X. If y_{m-2} or $y_{m-3} \in X$, then $X \cup \{y_{m-1}\} \in \mathscr{D}(G(m), i)$. Now let $X \in \mathscr{D}(G(m-3), i-1)$, then y_{m-3} or y_{m-4} is in X. If $y_{m-3} \in X$, then $X \cup \{x\} \in \mathscr{D}(G(m), i)$, for $x \in \{y_m, y_{m-1}\}$. If $y_{m-4} \in X$, then $X \cup \{y_{m-1}\} \in \mathscr{D}(G(m), i)$. Therefore we have $\{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-1), i-1), X_2 \in \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \setminus \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_m\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \cap \mathscr{D}(G(m-2), i-1)\} \subseteq \mathscr{D}(G(m), i)$. Now, let $Y \in \mathscr{D}(G(m), i)$, then $y_m \in Y$ or $y_{m-1} \in Y$. If $y_m \in Y$, then by Lemma 5, at least one vertex labeled y_{m-1}, y_{m-2}, or y_{m-3} is in Y. If $y_{m-1} \in Y$ or $y_{m-2} \in Y$, then $Y = X \cup \{y_m\}$ for some $X \in \mathscr{D}(G(m-1), i-1)$. If $y_{m-3} \in Y$, $y_{m-2} \notin Y$, and $y_{m-1} \notin Y$, then $Y = X \cup \{y_m\}$ for some $X \in \mathscr{D}(G(m-2), i-1) \cap \mathscr{D}(G(m-3), i-1)$. Now suppose that $y_{m-1} \in Y$ and $y_m \notin Y$, then by Lemma 5, at least one vertex labeled y_{m-2}, y_{m-3}, or y_{m-4} is in Y. If $y_{m-2} \in Y$ or $y_{m-3} \in Y$, then $Y = X \cup \{y_{m-1}\}$ for some $X \in \mathscr{D}(G(m-2), i-1)$. If $y_{m-4} \in Y$, $y_{m-3} \notin Y$, and $y_{m-2} \notin Y$, then $Y = X \cup \{y_{m-1}\}$ for some $X \in \mathscr{D}(G(m-3), i-1) \setminus \mathscr{D}(G(m-2), i-1)$. So $\mathscr{D}(G(m), i) \subseteq \{\{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathscr{D}(G(m-$

$1), i-1)$, $X_2 \in \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_{m-1}\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \setminus \mathscr{D}(G(m-2), i-1)\} \cup \{\{y_m\} \cup X \mid X \in \mathscr{D}(G(m-3), i-1) \cap \mathscr{D}(G(m-2), i-1)\}$. □

Theorem 10. *For any $m \geq 3$ and $i \geq \gamma(G(m))$,*

$$|\mathscr{D}(G(m), i)| = |\mathscr{D}(G(m-1), i-1)| + |\mathscr{D}(G(m-2), i-1)| + |\mathscr{D}(G(m-3), i-1)|. \tag{1}$$

Proof. It follows from Theorem 9. □

Theorem 11. *For any $m \geq 3$,*

$$D(G(m), x) = x(D(G(m-1), x) + D(G(m-2), x) + D(G(m-3), x)). \tag{2}$$

Proof. We have the result by definition of domination polynomial and Theorem 10. □

3. Dominating Sets of Paths and Some Other Graphs

First we investigate the dominating sets of paths. Since $P_n = P_1(n-1)$, we use the results for graph $G(m)$ in the previous section to obtain properties of dominating sets of path. We suppose that P_n is labeled as in Figure 1(a) and denote $\mathscr{D}(P_n, i)$ simply by P_i^n.

Lemma 12. *Suppose that $\mathscr{P}_i^n \neq \emptyset$, then one has*

(i) *$\mathscr{P}_{i-1}^{n-1} = \mathscr{P}_{i-1}^{n-2} = \emptyset$ and $\mathscr{P}_{i-1}^{n-3} \neq \emptyset$ if and only if $n = 3k$ and $i = k$ for some $k \in N$,*

(ii) *$\mathscr{P}_{i-1}^{n-2} = \mathscr{P}_{i-1}^{n-3} = \emptyset$ and $\mathscr{P}_{i-1}^{n-1} \neq \emptyset$ if and only if $i = n$,*

(iii) *$\mathscr{P}_{i-1}^{n-1} = \emptyset, \mathscr{P}_{i-1}^{n-2} \neq \emptyset$, and $\mathscr{P}_{i-1}^{n-3} \neq \emptyset$ if and only if $n = 3k+2$ and $i = \lceil (3k+2)/3 \rceil$ for some $k \in N$,*

(iv) *$\mathscr{P}_{i-1}^{n-1} \neq \emptyset, \mathscr{P}_{i-1}^{n-2} \neq \emptyset$, and $\mathscr{P}_{i-1}^{n-3} = \emptyset$ if and only if $i = n-1$,*

(v) *$\mathscr{P}_{i-1}^{n-1} \neq \emptyset, \mathscr{P}_{i-1}^{n-2} \neq \emptyset$, and $\mathscr{P}_{i-1}^{n-3} \neq \emptyset$ if and only if $\lceil (n-1)/3 \rceil + 1 \leq i \leq n-2$.*

Proof. (i) ($\Rightarrow$) By Lemmas 4 and 8(i), we have $\lceil n/3 \rceil \leq i < \lceil (n-2)/3 \rceil + 1$, which give us $n = 3k$ and $i = k$ for some $k \in N$.

($\Leftarrow$) It follows from Lemmas 4 and 8(i).

(ii) It follows from Lemma 8(ii).

(iii) ($\Rightarrow$) By Lemmas 4 and 8(iii), we have $\lceil (n-2)/3 \rceil + 1 \leq i < \lceil (n-1)/3 \rceil + 1$, which give us $n = 3k+2$ and $i = k+1 = \lceil (3k+2)/3 \rceil$ for some $k \in N$.

($\Leftarrow$) It follows from Lemmas 4 and 8(iii).

(iv) It follows from Lemma 8(iv).

(v) It follows from Lemmas 4 and 8(v). □

The following theorem specify $\mathscr{P}_i^n$.

Theorem 13. *For every $n \geq 4$ and $i \geq \lceil n/3 \rceil$,*

$$\mathscr{P}_i^n = \begin{cases} \{\{2, 5, \ldots, n-4, n-1\}\}, & \text{if } \mathscr{P}_{i-1}^{n-1} = \mathscr{P}_{i-1}^{n-2} = \emptyset, \\ \{[n]\}, & \text{if } \mathscr{P}_{i-1}^{n-2} = \mathscr{P}_{i-1}^{n-3} = \emptyset, \\ \{\{2, 5, \ldots, n-3, n\}\} \cup \left\{X \cup \{n-1\} \mid X \in \mathscr{P}_{i-1}^{n-3}\right\}, & \text{if } \mathscr{P}_{i-1}^{n-1} = \emptyset, \\ \{[n] - \{x\} \mid x \in [n]\}, & \text{if } \mathscr{P}_{i-1}^{n-3} = \emptyset, \\ \left\{\{n\} \cup X_1, \{n-1\} \cup X_2 \mid X_1 \in \mathscr{P}_{i-1}^{n-1}, X_2 \in \mathscr{P}_{i-1}^{n-2}\right\} & \\ \quad \cup \left\{\{n-1\} \cup X \mid X \in \mathscr{P}_{i-1}^{n-3} \setminus \mathscr{P}_{i-1}^{n-2}\right\} & \\ \quad \cup \left\{\{n\} \cup X \mid X \in \mathscr{P}_{i-1}^{n-3} \cap \mathscr{P}_{i-1}^{n-2}\right\}, & \text{if } \mathscr{P}_{i-1}^{n-1} \neq \emptyset, \mathscr{P}_{i-1}^{n-2} \neq \emptyset, \mathscr{P}_{i-1}^{n-3} \neq \emptyset. \end{cases} \tag{3}$$

Proof.

Case 1. $\mathscr{P}_{i-1}^{n-1} = \mathscr{P}_{i-1}^{n-2} = \emptyset$ and $\mathscr{P}_{i-1}^{n-3} \neq \emptyset$. By Lemma 12(i), $n = 3k$ and $i = k$ for some $k \in N$. Therefore $\mathscr{P}_i^n = \mathscr{P}_{n/3}^n = \{\{2, 5, \ldots, n-4, n-1\}\}$.

Case 2. It follows from Theorem 9(ii).

Case 3. By Theorem 9(iii), $\mathscr{P}_i^n = \{\{n\} \cup X_1, \{n-1\} \cup X_2 \mid X_1 \in \mathscr{P}_{i-1}^{n-2}, X_2 \in \mathscr{P}_{i-1}^{n-3}\}$. By Lemma 12(iii), $n = 3k+2$ and $i = k+1$ for some $k \in N$. Since $\mathscr{P}_{i-1}^{n-2} = \mathscr{P}_k^{3k} = \{\{2, 5, \ldots, 3k-1\}\}$, then $\{X \cup \{3k+2\} \mid X \in \mathscr{P}_k^{3k}\} = \{\{2, 5, \ldots, 3k-1, 3k+2\}\}$. Therefore $\mathscr{P}_i^n = \{\{2, 5, \ldots, n-3, n\}\} \cup \{X \cup \{n-1\} \mid X \in \mathscr{P}_{i-1}^{n-3}\}$.

Cases 4 and 5 follow from Theorem 9(iv) and (v), respectively. □

Example 14. We use Theorem 13 to construct $\mathscr{P}_i^6$ for $2 \leq i \leq 6$. Since $\mathscr{P}_1^5 = \mathscr{P}_1^4 = \emptyset$ and $\mathscr{P}_1^3 = \{\{2\}\}$, by Theorem 13, $\mathscr{P}_2^6 = \{\{2, 5\}\}$. Since $\mathscr{P}_5^5 = \{[5]\}, \mathscr{P}_5^4 = \emptyset$, and $\mathscr{P}_5^3 = \emptyset$,

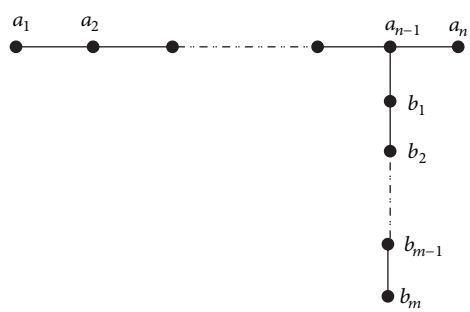

FIGURE 2: The tree $T_n(m)$.

we get $\mathscr{P}_6^6 = \{[6]\}$. Since $\mathscr{P}_4^5 = \{\{1,2,3,4\}, \{1,2,3,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,4,5\}\}$, $\mathscr{P}_4^4 = \{[4]\}$, and $\mathscr{P}_4^3 = \emptyset$, then by Theorem 13, $\mathscr{P}_5^6 = \{[6] - \{x\} \mid x \in [6]\} = \{\{1,2,3,4,6\}, \{1,2,3,5,6\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{1,2,4,5,6\}, \{1,2,3,4,5\}\}$. And, for construction $\mathscr{P}_3^6 = \{\{1,3,5\}, \{1,3,6\}, \{2,3,6\}, \{2,3,5\}, \{1,4,6\}, \{1,4,5\}, \{2,5,6\}, \{2,4,6\}, \{2,4,5\}, \{1,2,5\}\}$, by Theorem 13, $\mathscr{P}_3^6 = \{X_1 \cup \{6\}, X_2 \cup \{5\} \mid X_1 \in \mathscr{P}_2^5, X_2 \in \mathscr{P}_2^4\} \cup \{\{1,2\} \cup \{5\}, \{1,3\} \cup \{6\}, \{2,3\} \cup \{6\}\}$. Finally, since $\mathscr{P}_3^5 = \{\{1,3,5\}, \{1,2,4\}, \{2,4,5\}, \{2,3,4\}, \{2,3,5\}, \{1,4,5\}, \{1,3,4\}, \{1,2,5\}\}$, $\mathscr{P}_3^4 = \{\{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}\}$, and $\mathscr{P}_3^3 = \{[3]\}$, then $\mathscr{P}_4^6 = \{X_1 \cup \{6\}, X_2 \cup \{5\} \mid X_1 \in \mathscr{P}_3^5, X_2 \in \mathscr{P}_3^4\} \cup \{X \cup \{6\} \mid X \in \mathscr{P}_3^3\} = \{\{1,2,3,5\}, \{1,2,3,6\}, \{1,2,4,6\}, \{1,3,5,6\}, \{1,3,4,6\}, \{1,3,4,5\}, \{1,2,5,6\}, \{1,2,4,5\}, \{2,3,4,6\}, \{1,4,5,6\}, \{2,3,4,5\}, \{2,4,5,6\}, \{2,3,5,6\}\}$.

Theorem 15. *If $\mathscr{P}_i^n$ is the family of dominating set of P_n with cardinality i, then $|\mathscr{P}_i^n| = |\mathscr{P}_{i-1}^{n-1}| + |\mathscr{P}_{i-1}^{n-2}| + |\mathscr{P}_{i-1}^{n-3}|$.*

Proof. It follows from Theorem 10. □

Remark 16. We have obtained [12, Lemma 12 and Theorem 5] with a different approach.

As other examples of graph $G(m)$, we study the dominating sets of some other graphs. By Theorem 10, the recursive formula in this theorem is true for every graphs which contain a gluing path with at least 3 vertices. Also, if we have the domination number of graphs, then we can study the properties of their dominating sets by Theorem 9 and Lemma 8. Here, we consider a special case of graph $P_n(m)$ which is a tree. We denote this tree by $T_n(m)$. In other words $T_n(m)$ is a tree, such that $T_n(0) = P_n$, and for $m \geq 1$, $T_n(m) = (A, B, E)$ where $A \cup B$ is its vertex set, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$, and the edge set $E = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq m-1\} \cup \{a_{n-1} b_1\}$; see Figure 2. For obtaining the domination number of $T_n(m)$, we need the following lemma which follows from observation.

Lemma 17. *For every $n \geq 0$ and $m \geq 3$, $\gamma(T_n(m)) = \gamma(T_{m-2}(n+2))$.*

Theorem 18. *For every $n \geq 3$ and $m \geq 0$, $\gamma(T_n(m)) = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil$.*

Proof. By induction on m. If $m = 0$, then $\gamma(T_n(0)) = \gamma(P_n) = \lceil n/3 \rceil$. Now suppose that the theorem is true for all numbers less than or equal $m-1$, and we prove it for m. By applying Theorem 2 for $e = a_{n-1}b_1$, we have the following inequalities:

$$\left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil - 1 \leq \gamma(T_n(m)) \leq \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m}{3} \right\rceil. \quad (4)$$

Similarly by applying Theorem 2 for $e = b_2 b_3, b_{m-3} b_{m-2}$ and induction hypothesis, we have the following inequalities; respectively,

$$\begin{aligned} \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil &\leq \gamma(T_n(m)) \\ &\leq \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-2}{3} \right\rceil + 1, \end{aligned} \quad (5)$$

$$\begin{aligned} \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-4}{3} \right\rceil &\leq \gamma(T_n(m)) \\ &\leq \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{m-4}{3} \right\rceil + 1. \end{aligned} \quad (6)$$

Now if $m = 3k$ for some $k \in N$, then by (4) and (5) we have $\gamma(T_n(m)) = \lceil n/3 \rceil + k = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil$.

If $m = 3k+1$ for some $k \in N$, then by (4) and (6) we have $\gamma(T_n(m)) = \lceil n/3 \rceil + k = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil$.

Now if $m = 3k+2$ for some $k \in N$, we will consider the following cases.

(i) One has $n = 3k'$ for some $k' \in N$. By applying Theorem 2 for $a_{n-1}a_n$, we have the following inequalities:

$$\left\lceil \frac{n+m-1}{3} \right\rceil \leq \gamma(T_n(m)) \leq \left\lceil \frac{n+m-1}{3} \right\rceil + 1. \quad (7)$$

Now, by (4) and (7), we have $\gamma(T_{3k'}(3k+2)) = k + k' + 1 = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil$.

(ii) One has $n = 3k' + 2$ for some $k' \in N$. By Theorem 2 for $e = a_1 a_2$ we have

$$\begin{aligned} \left\lceil \frac{n-1}{3} \right\rceil + \left\lceil \frac{m-1}{3} \right\rceil &\leq \gamma(T_n(m)) \\ &\leq \left\lceil \frac{n-1}{3} \right\rceil + \left\lceil \frac{m-1}{3} \right\rceil + 1. \end{aligned} \quad (8)$$

By (4) and (8), we have $\gamma(T_{3k'+2}(3k+2)) = k + k' + 2 = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil$.

(iii) One has $n = 3k' + 1$ for some $k' \in N$. We will prove that $\gamma(T_{3k'+1}(3k+2)) = k + k' + 2$. We do it by induction on k. If $k = 0$, then by Lemma 17, $\gamma(T_{3k'+1}(2)) = \gamma(T_4(3k'-1)) = 2 + k'$. So the result is true for $k = 0$. Now suppose that the result is true for all number less than k, and we prove it for k.

By Lemma 17 and the induction hypothesis, we have the following equalities for $k' - 1 < k$:

$$\begin{aligned}
&\gamma\left(T_{3k'+1}(3k+2)\right)\\
&\quad = \gamma\left(T_{3k+4}\left(3k'-1\right)\right)\\
&\quad = \gamma\left(T_{3(k+1)+1}\left(3\left(k'-1\right)+2\right)\right)\\
&\quad = k'-1+k+1+2 = k'+k+2.
\end{aligned} \tag{9}$$

If $k' - 1 > k$, then for some $t > 0$, $k' = k + 1 + t$. Again by Lemma 17,

$$\begin{aligned}
&\gamma\left(T_{3k'+1}(3k+2)\right)\\
&\quad = \gamma\left(T_{3k+3t+4}(3k+2)\right)\\
&\quad = \gamma\left(T_{3k+4}(3(k+t)+2)\right)\\
&\quad = k+t+k+1+2 = k'+k+2.
\end{aligned} \tag{10}$$

Finally, for $k' - 1 = k$, we have $\gamma(T_{3k'+1}(3k + 2)) = \gamma(T_{3k+4}(3k + 2)) = \gamma(T_{3(k+1)+1}(3k + 2) = 2k + 3 = k + k' + 2$. Therefore in all cases $\gamma(T_n(m)) = \lceil n/3\rceil + \lceil (m-1)/3\rceil$.

□

Let $\mathscr{D}(T_n(m), i)$ or simply $\mathscr{D}_i^{n,m}$ be the family of dominating set of $T_n(m)$ with cardinality i. By Lemma 8 and Theorem 9 we prove the following lemma, which is some properties of $\mathscr{D}_i^{n,m}$.

Lemma 19. *Suppose that $\mathscr{D}_i^{n,m} \neq \emptyset$, then*

(i) $\mathscr{D}_{i-1}^{n,m-1} = \mathscr{D}_{i-1}^{n,m-2} = \emptyset$ *and* $\mathscr{D}_{i-1}^{n,m-3} \neq \emptyset$ *if and only if* $m = 3k + 1$, $i = \lceil n/3\rceil + k$, *for some* $k \in N$,

(ii) $\mathscr{D}_{i-1}^{n,m-2} = \mathscr{D}_{i-1}^{n,m-3} = \emptyset$, *and* $\mathscr{D}_{i-1}^{n,m-1} \neq \emptyset$ *if and only if* $i = n + m$,

(iii) $\mathscr{D}_{i-1}^{n,m-1} = \emptyset, \mathscr{D}_{i-1}^{n,m-2} \neq \emptyset$, *and* $\mathscr{D}_{i-1}^{n,m-3} \neq \emptyset$ *if and only if* $m = 3k$, $i = \lceil n/3\rceil + k$, *for some* $k \in N$,

(iv) $\mathscr{D}_{i-1}^{n,m-1} \neq \emptyset, \mathscr{D}_{i-1}^{n,m-2} \neq \emptyset$, *and* $\mathscr{D}_{i-1}^{n,m-3} = \emptyset$, *if and only if* $i = m + n - 1$,

(v) $\mathscr{D}_{i-1}^{n,m-1} \neq \emptyset, \mathscr{D}_{i-1}^{n,m-2} \neq \emptyset$ *and* $\mathscr{D}_{i-1}^{n,m-3} \neq \emptyset$ *if and only if* $\lceil n/3\rceil + \lceil (m-2)/3\rceil + 1 \le i \le m + n - 2$.

Proof. (i) By Lemma 8(i) and Theorem 18, $\lceil n/3\rceil + \lceil (m-4)/3\rceil + 1 \le i < \lceil n/3\rceil + \lceil (m-3)/3\rceil + 1$, which give us $m = 3k + 1$, $i = \lceil n/3\rceil + k$, for some $k \in N$.

(ii) By Lemma 8(ii), $i = |V(T_n(m))| = m + n$.

(iii) By Lemma 8(iii) and Theorem 18, $\lceil n/3\rceil + \lceil (m-3)/3\rceil + 1 \le i < \lceil n/3\rceil + \lceil (m-2)/3\rceil + 1$. Therefore $m = 3k$ and $i = \lceil n/3\rceil + k$ for some $k \in N$.

(iv) By Lemma 8(iv), $i = |V(T_n(m))| - 1 = n + m - 1$.

(v) By Lemma 8(v) and Theorem 18, $\lceil n/3\rceil + \lceil (m-2)/3\rceil + 1 \le i \le m + n - 2$. □

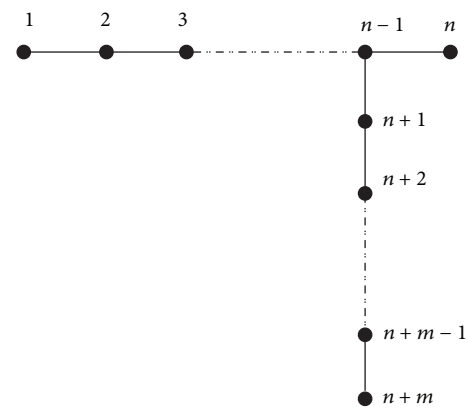

Figure 3: Labeled $T_n(m)$.

Here, we suppose that the tree $T_n(m)$ labeled as shown in Figure 3. The following theorem constructs $\mathscr{D}_i^{n,m}$ from $\mathscr{D}_{i-1}^{n,m-1}, \mathscr{D}_{i-1}^{n,m-2}$, and $\mathscr{D}_{i-1}^{n,m-3}$.

Theorem 20. (i) *If* $\mathscr{D}_{i-1}^{n,m-1} = \mathscr{D}_{i-1}^{n,m-2} = \emptyset$ *and* $\mathscr{D}_{i-1}^{n,m-3} \neq \emptyset$, *then*

$$\mathscr{D}_i^{n,m} = \left\{\{m+n-1\} \cup X \mid X \in \mathscr{D}_{i-1}^{n,m-3}\right\}. \tag{11}$$

(ii) *If* $\mathscr{D}_{i-1}^{n,m-2} = \mathscr{D}_{i-1}^{n,m-3} = \emptyset$ *and* $\mathscr{D}_{i-1}^{n,m-1} \neq \emptyset$, *then*

$$\mathscr{D}_i^{n} = \left\{\{m+n\} \cup X \mid X \in \mathscr{D}_{i-1}^{n,m-1}\right\} = \{[m+n]\}. \tag{12}$$

(iii) *If* $\mathscr{D}_{i-1}^{n,m-1} = \emptyset, \mathscr{D}_{i-1}^{n,m-2} \neq \emptyset$, *and* $\mathscr{D}_{i-1}^{n,m-3} \neq \emptyset$, *then*

$$\begin{aligned}
\mathscr{D}_i^{n,m} = \Big\{ &\{m+n\} \cup X_1,\\
&\{m+n-1\} \cup X_2 \mid X_1 \in \mathscr{D}_{i-1}^{n,m-2},\ X_2 \in \mathscr{D}_{i-1}^{n,m-3}\Big\}.
\end{aligned} \tag{13}$$

(iv) *If* $\mathscr{D}_{i-1}^{n,m-3} = \emptyset, \mathscr{D}_{i-1}^{n,m-2} \neq \emptyset$, *and* $\mathscr{D}_{i-1}^{n,m-1} \neq \emptyset$, *then*

$$\begin{aligned}
\mathscr{D}_i^{n} = \Big\{ &\{m+n\} \cup X_1,\\
&\{m+n-1\} \cup X_2 \mid X_1 \in \mathscr{D}_{i-1}^{n,m-1},\ X_2 \in \mathscr{D}_{i-1}^{n,m-2}\Big\}.
\end{aligned} \tag{14}$$

(v) *If* $\mathscr{D}_{i-1}^{n,m-1}, \mathscr{D}_{i-1}^{n,m-2}$, *and* $\mathscr{D}_{i-1}^{n,m-3}$ *are nonempty, then*

$$\begin{aligned}
\mathscr{D}_i^{n,m} = &\left\{\{m+n\} \cup X_1, \{m+n-1\} \cup X_2 \mid \right.\\
&\left. X_1 \in \mathscr{D}_{i-1}^{n,m-1},\ X_2 \in \mathscr{D}_{i-1}^{n,m-2}\right\}\\
&\cup \left\{\{m+n-1\} \cup X \mid X \in \mathscr{D}_{i-1}^{n,m-3} \setminus \mathscr{D}_{i-1}^{n,m-2}\right\}\\
&\cup \left\{\{m+n\} \cup X \mid X \in \mathscr{D}_{i-1}^{n,m-3} \cap \mathscr{D}_{i-1}^{n,m-2}\right\}.
\end{aligned} \tag{15}$$

Proof. It follows from Theorem 9. □

Now we obtain the following theorem from Theorem 20 or Theorem 10.

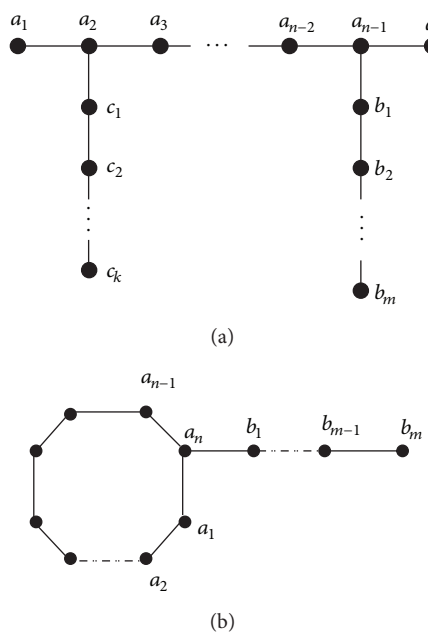

Figure 4: The graphs $T_{m,n}(k)$ and $C_n(m)$, respectively.

Theorem 21. *If $\mathscr{D}_i^{n,m}$ is the family of dominating set of $T_n(m)$ with cardinality i, then $|\mathscr{D}_i^{n,m}| = |\mathscr{D}_{i-1}^{n,m-1}|+|\mathscr{D}_{i-1}^{n,m-2}|+|\mathscr{D}_{i-1}^{n,m-3}|$.*

Note that the tree $T_n(m)$ is obtained by gluing an end vertex of a path P_m to the vertex a_{n-1} of P_n. We now consider another type of related tree $T_{m,n}(k)$ (or special case of $(T_n(m))(k)$), which is obtained from $T_n(m)$ by gluing an end vertex of a path P_k to the vertex a_2 of P_n, see Figure 4. In other words, let $T_{m,n}(k) = (A, B, C, E)$, $k, n \geq 0$, where $A \cup B \cup C$ is its vertex set, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$, $C = \{c_1, \ldots, c_k\}$, and the edge set $E = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{b_i b_{i+1} : 1 \leq i \leq m-1\} \cup \{c_i c_{i+1} : 1 \leq i \leq k-1\} \cup \{a_{n-1} b_1, a_2 c_1\}$.

Theorem 22. *For every $n \geq 3$ and $m, k \geq 0$, $\gamma(T_{m,n}(k)) = \lceil n/3 \rceil + \lceil (m-1)/3 \rceil + \lceil (k-1)/3 \rceil$.*

Proof. Similar to the proof of Theorem 18. □

By Theorem 10, we have the following theorem.

Theorem 23. *If $\mathscr{D}_i^{n,m,k}$ is the family of dominating set of $T_{m,n}(k)$ with cardinality i, then $|\mathscr{D}_i^{n,m,k}| = |\mathscr{D}_{i-1}^{n,m,k-1}| + |\mathscr{D}_{i-1}^{n,m,k-2}| + |\mathscr{D}_{i-1}^{n,m,k-3}|$.*

As other examples for graphs $G(m)$, we consider graph $C_n(m)$ which obtain by gluing a path to one vertex of cycle as shown in Figure 4. By Theorem 10 we have the following theorem for $C_n(m)$.

Theorem 24. *For every $n, m \geq 3$, $|\mathscr{D}(C_n(m), i)| = |\mathscr{D}(C_n(m-1), i-1)| + |\mathscr{D}(C_n(m-2), i-1)| + |\mathscr{D}(C_n(m-3), i-1)|$.*

Let sun be a graph which is obtained by gluing end vertices of paths $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ to the n vertices of C_n. We denote this graph by $S(i_1, i_2, \ldots, i_n)$. Since sun is the special case of $(\cdots((C_n(i_1))(i_2))\cdots(i_n)$, we have the following theorem for sun.

Theorem 25. *Suppose that for every $1 \leq j \leq n$, $i_j \geq 3$ and $i \geq \gamma(S(i_1, i_2, \ldots, i_n))$, then $|\mathscr{D}(S(i_1, i_2, \ldots, i_n), i)| = |\mathscr{D}(S(i_1, i_2, \ldots, i_n - 1), i-1)| + |\mathscr{D}(S(i_1, i_2, \ldots, i_n - 2), i-1)| + |\mathscr{D}(S(i_1, i_2, \ldots, i_n - 3), i-1)|$.*

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