Research Article

Note on a Class of Subsets of $AG(3, q)$ with Intersection Numbers 1, q and n with respect to the Planes

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We give a new and correct proof of a result of O. Ferri and S. Ferri (1995) on q^2 -caps of $AG(3, q)$ in this paper; moreover we prove that sets of $AG(3, q)$ of type $(1, q, q+1)$ with respect to the planes of $AG(3, q)$ have size at most q^2 with equality if and only if $\mathcal{K}$ is a cap.

1. Introduction

Let $\mathbb{G}$ denote either a finite projective space or a finite affine space, and let $\{m_0, \dots, m_s\}$ be a set of nonnegative integers with $m_0 < m_1 < \dots < m_s$. A subset $\mathcal{K}$ of $\mathbb{G}$ is of class $[m_0, \dots, m_s]_d$ with respect to the subspaces of dimension d of $\mathbb{G}$ if any subspaces intersect $\mathcal{K}$ either in $m_0, \dots, m_{s-1}$ or m_s points, and $\mathcal{K}$ is of type $(m_0, \dots, m_s)_d$ with respect to the subspaces of dimension d if for every integer m_i , $i \in \{0, \dots, s\}$ there exists a d -subspace H meeting $\mathcal{K}$ in exactly m_i points. The numbers m_i are called *intersection numbers* of $\mathcal{K}$. As usual, by a k -set we mean a set of size k .

In the literature one can find many papers devoted to the study of k -sets of given type, not only in affine and projective geometries (cf., e.g., [1–15]), and most of these results are characterizations of classical geometric objects. Recently, characterizations of Hermitian varieties and quadrics of $PG(r, q)$ as k -sets with given intersection numbers with respect to more than one family of subspaces (e.g., with respect to planes and solids) have been considered [16, 17].

A *cap* of an affine or projective space of dimension ≥ 3 is a subset of points no three of which are collinear.

In 1995, O. Ferri and S. Ferri [18] gave a characterization of q^2 -caps of $AG(3, q)$ in terms of sets with three given intersection numbers with respect to the planes. Their result reads as follows.

Theorem 1 (see [18]). *Let K be a subset of $AG(3, q)$ with $|K| = q^2$ points and of type $(1, q, n)_2$. Then, $n = q+1$ and K is a cap of $AG(3, q)$.*

Unfortunately, a step of the proof of that theorem is not correct; in fact it contains a counting argument which does not give the contradiction they want (see [18] page 71 line +7). However, the statement of the result is true as we are going to prove in Lemma 4.

In this paper we will prove the following slight extension of the O. Ferri and S. Ferri result.

Theorem 2. *Let $\mathcal{K}$ be a set of $AG(3, q)$ of size $k \geq q^2$ and with three intersection numbers 1, q , and n . Then $n \geq q+1$, and $n = q+1$ if and only if $k = q^2$. Moreover, if $k = q^2$ the set $\mathcal{K}$ is a cap.*

Thus, it follows that the sets of type $(1, q, q+1)_2$ of $AG(3, q)$ have size at most q^2 and that equality holds if and only if they are caps.

2. Proof of Theorem 1

In this section, first, we briefly recall the basic equations for a k -set of $AG(3, q)$ with three intersection numbers, and then we will assume that $k = q^2$ and we will give the proof of Theorem 1.

2.1. The Basic Equations for k -Sets with Intersection Numbers l , q , and n . Let t_i , ($i = 1, q, n$), denote the number of planes intersecting K in exactly i -points (such numbers are called *characters of $\mathcal{K}$*).

Double counting gives

$$\begin{aligned} t_1 + t_q + t_n &= q^3 + q^2 + q, \\ t_1 + qt_q + nt_n &= k(q^2 + q + 1), \\ q(q-1)t_q + n(n-1)t_n &= k(k-1)(q+1). \end{aligned} \quad (1)$$

From (1) it follows that

$$\begin{aligned} t_1 &= (q^2n(q^2 + q + 1) - k(q^2 + q + 1)) \\ &\quad \times (n + q - 1) + k(k-1)(q+1) \\ &\quad \times ((n-1)(q-1))^{-1}, \end{aligned} \quad (2)$$

$$t_q = \frac{n(q^2 + q + 1)(k - q) - k(k-1)(q+1)}{(q-1)(n-q)}, \quad (3)$$

$$t_n = \frac{k(k-1)(q+1) - (k-q)q(q^2 + q + 1)}{(n-1)(n-q)}. \quad (4)$$

2.2. q^2 -Sets of $AG(3, q)$ with Three Intersection Numbers. From now on, $\mathcal{K}$ is a k -set of $AG(3, q)$ with $k \geq q^2$ and with intersection numbers l , q , and n with respect to the planes.

Since $k \geq 2$, there are at least two distinct points in K . Let x and y be two points of K , and let ℓ be the line connecting them. Put $s := |\ell \cap K|$. Namely, $s \geq 2$.

Let x denote the number of planes on ℓ intersecting K in n points. Counting k via the planes passing through ℓ , we have

$$k = q^2 - (s-1)q + x(n-q). \quad (5)$$

Being $k \geq q^2$ and $s \geq 2$ it follows that $n > q$; that is, $n \geq q+1$.

Lemma 3. *If $n = q+1$, then $s = 2$, $x = q$, and K is a q^2 -cap.*

Proof. Let s denote the number of points of a line meeting $\mathcal{K}$ in at least two points. For $n = q+1$ (5) gives $q^2 \leq k = q^2 - (s-1)q + x$. Thus, $s = 2$ and $x \geq q$. If $x = q+1$, then $k = q^2 + 1$, and so there would be no plane intersecting $\mathcal{K}$ in exactly q points, against the assumptions. Thus $s = 2$, $x = q$, $k = q^2$, and $\mathcal{K}$ is a cap. $\square$

Lemma 4 (see S. Ferri and O. Ferri [18]). *If $k = q^2$, then $n = q+1$.*

Proof. From $k = q^2$ it follows that (2) becomes $t_1 = q^3/(n-1)$ and (3) $t_q = q(q^2 + q + 1) - q^3/(n-q)$. Assume that $n \geq q+2$. Let $q = p^h$, with p a prime number, and $h \geq 1$ an integer. We have that p is a divisor of $n-1$ and $n-q$, but this is impossible since the difference of these numbers is $q-1 = p^h - 1$. Thus, $n = q+1$. $\square$

The proof of Theorem 1 follows from Lemmas 3 and 4.

Let us end with the following easy consequence of Theorem 2.

Corollary 5. *A subset $\mathcal{K}$ of points of $AG(3, q)$ of type $(1, q, q+1)_2$ has size $k \leq q^2$, and $k = q^2$ if and only if $\mathcal{K}$ is a cap of $AG(3, q)$.*

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References

- [1] J. Doyen and X. Hubaut, "Finite regular locally projective spaces," *Mathematische Zeitschrift*, vol. 119, pp. 83–88, 1971.
- [2] M. J. de Resmini, "On k -sets of type (m, n) in a steiner $S(2, \ell, v)$," *Finite Geometries and Designs*, London Mathematical Society Lecture Note Series (No. 49), Cambridge University Press, Cambridge, Mass, USA, 1981.
- [3] M. J. de Resmini, "On sets of type (m, n) in BIBDs with $\lambda \geq 2$," in *Combinatorial and Geometric Structures and Their Applications (Trento, 1980)*, vol. 14 of *Ann. Discrete Math.*, pp. 183–206, North-Holland, Amsterdam, The Netherlands, 1982.
- [4] N. Durante, V. Napolitano, and D. Olanda, "On k -sets of class $[1, h]$ in a planar space," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 50, pp. 305–312, 2002.
- [5] S.-M. Kim, "Sets of type $(-1, n)$ in biplanes," *European Journal of Combinatorics*, vol. 25, no. 5, pp. 745–756, 2004.
- [6] V. Napolitano and D. Olanda, "Sets of type $(3, h)_2$ in $PG(3, q)$," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Serie IX. Rendiconti Lincei. Matematica e Applicazioni*, vol. 22, pp. 1–9, 2012.
- [7] V. Napolitano, "On $(q^2 + q + 1)$ -sets of class $[1, m, n]_2$ in $PG(3, q)$," *Electronic Notes in Discrete Math*, vol. 40, pp. 283–287, 2013.
- [8] M. Tallini Scafati and G. Tallini, *Geometria di Galois e teoria dei codici*, C.I.S.U., 1995.
- [9] T. Penttilä and G. F. Royle, "Sets of type (m, n) in the affine and projective planes of order nine," *Designs, Codes and Cryptography*, vol. 6, no. 3, pp. 229–245, 1995.
- [10] G. Tallini, " k -insiemi e blocking sets in $PG(2, q)$ ed $AG(2, q)$," *Quaderni di Geometrie Comb. n. 1 Dipartimento Mat. Univ. l'Aquila*, Dicembre 1982.
- [11] G. Tallini, "Giuseppe On line k -sets of type $(0, n)$ with respect to pencils of lines in $PG(d, q)$," in *Combinatorial and Geometric Structures and Their Applications (Trento, 1980)*, vol. 14 of *Ann. Discrete Math.*, pp. 283–292, North-Holland, Amsterdam, The Netherlands, 1982.
- [12] G. Tallini, "Teoria dei k -insiemi in uno spazio di Galois, Teoria dei codici correttori," *Sem. Geom. Combin. Dip. Mat. Univ. Roma La Sapienza, Quaderno*, vol. 65, 1985.
- [13] G. Tallini, "On sets of given type in a steiner system," in *Finite Geometries (Winnipeg, Man., 1984)*, vol. 103 of *Lecture Notes in Pure and Appl. Math.*, pp. 307–319, Dekker, New York, NY, USA, 1985.
- [14] G. Tallini, "Some new results on sets of type (m, n) in projective planes," *Journal of Geometry*, vol. 29, no. 2, pp. 191–199, 1987.
- [15] M. Tallini Scafati, "The k -sets of type (m, n) of an affine space $A_{r,q}$," *Rendiconti di Matematica. Serie VII*, vol. 1, no. 1, pp. 63–80, 1981.

- [16] J. Schillewaert, “A characterization of quadrics by intersection numbers,” *Designs, Codes and Cryptography*, vol. 47, no. 1–3, pp. 165–175, 2008.
- [17] J. Schillewaert and J. A. Thas, “Characterizations of Hermitian varieties by intersection numbers,” *Designs, Codes and Cryptography*, vol. 50, no. 1, pp. 41–60, 2009.
- [18] O. Ferri and S. Ferri, “Su alcuni q^2 insiemi di $AG(3, q)$ a tre caratteri rispetto ai piani,” *Ratio Mathematica*, vol. 9, pp. 69–72, 1995.