Research Article

A Study on Fuzzy Ideals of \(N\)-Groups

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Received 10 February 2013; Accepted 19 June 2013

Academic Editor: Antonio M. Cegarra

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Using the idea of the new sort of fuzzy subnear-ring of a near-ring, fuzzy subgroups, and their generalizations defined by various researchers, we try to introduce the notion of \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideals of \(N\)-groups. These fuzzy ideals are characterized by their level ideals, and some other related properties are investigated.

1. Introduction and Basic Definitions

The concept of a fuzzy set was introduced by Zadeh [1] in 1965, utilizing what Rosenfeld [2] defined as fuzzy subgroups. This was studied further in detail by different researchers in various algebraic systems. The concept of a fuzzy ideal of a ring was introduced by Liu [3]. The notion of fuzzy subnear-ring and fuzzy ideals was introduced by Abou-Zaid [4]. Then in many papers, fuzzy ideals of near-rings were discussed for example, see [5–11]. In [12], the idea of fuzzy point and its belongingness to and quasi coincidence with a fuzzy set were used to define \((\alpha, \beta)\)-fuzzy subgroup, where \(\alpha, \beta\) take one of the values from \(\{e, q, e \land q, e \lor q\}\), \(\alpha \neq e \land q\). A fuzzy subgroup in the sense of Rosenfeld is in fact an \((e, e)\)-fuzzy subgroup. Thus, the concept of \((e, e \lor q)\)-fuzzy subgroup was introduced and discussed thoroughly in [7]. Bhakat and Das [13] introduced the concept of \((e, e \lor q)\)-fuzzy subrings and ideals of a ring. Davvaz [14, 15], Narayanan and Manikantan [16], and Zhan and Davvaz [17] studied a new sort of fuzzy subnear-ring (ideal and prime ideal) called \((e, e \lor q)\)-fuzzy subnear-ring (ideal and prime ideal) and gave characterizations in terms of the level ideals. In [18, 19], the idea of fuzzy ideals of \(N\)-groups was defined, and various properties such as fundamental theorem of fuzzy ideals and fuzzy congruence were studied, respectively. In the present paper, we extend the idea of \((e, e \lor q)\)-fuzzy ideals of near-rings to the case of \(N\)-groups and introduce the idea of fuzzy cosets with some results.

We first recall some basic concepts for the sake of completeness.

By a near-ring we mean a nonempty set \(N\) with two binary operations “+” and “⋅” satisfying the following axioms:

(i) \((N, +)\) is a group,

(ii) \((N, \cdot)\) is a semigroup,

(iii) \((x + y) \cdot z = x \cdot z + y \cdot z\) for all \(x, y, z \in N\).

It is in fact a right near-ring because it satisfies the right distributive law. We will use the word “near-ring” to mean “right near-ring.” \(N\) is said to be zero symmetric if \(0 \cdot x = x \cdot 0 = 0\) for all \(x \in N\). We denote \(x \cdot y\) by \(xy\).

Note that the missing left distributive law, \(x \cdot (y + z) = x \cdot y + x \cdot z\), has to do with linearity if \(x\) is considered as a function.

Example 1. Let \(\mathcal{G}\) be a group, and let \(M(\mathcal{G})\) be the set of all mappings from \(\mathcal{G}\) into \(\mathcal{G}\). We define + and \(\cdot\) on \(M(\mathcal{G})\) by

\[
(f + g)(x) := f(x) + g(x),
\]

\[
(f \cdot g)(x) := f(g(x)).
\]

Then, \((M(\mathcal{G})), +, \cdot)\) is a near-ring.

Just in the same way as \(R\)-modules or vector spaces are used in ring theory, \(N\)-groups are used in near-ring theory.
By an $N$-group we mean a nonempty set $G$ together with a map $Φ : N \times G \to G$ written as $Φ(n, g) = ng$ satisfying the following conditions:

(i) $(G, +)$ is a group (not necessarily abelian),

(ii) $(n_1 + n_2)g = n_1g + n_2g$,

(iii) $(n_1n_2)g = n_1(n_2g)$ for all $n_1, n_2 \in N, g \in G$.

Example 2. Let $N$ be a subnear-ring of $M(\mathcal{S})$. Then, $\mathcal{S}$ is an $N$-group via function application as operation.

Example 3. The additive group $(N, +)$ of a near-ring $(N, +, \cdot)$ is an $N$-group via the near-ring multiplication.

An ideal $I$ of $N$-group $G$ is an additive normal subgroup of $G$ such that $NI \subseteq I$ and $n(g + h) - ng \in I$ for all $h \in I, g \in G, n \in N$. A mapping between two $N$-groups $G$ and $G'$ is called an $N$-homomorphism if $f(g + h) = f(g) + f(h)$ and $f(n g) = n f(g)$ for all $g, h \in G, n \in N$.

Throughout this study, we use $N$ to denote a zero-symmetric near-ring and $G$ to denote an $N$-group.

For any fuzzy subset $A$ of $G$, $ImA = \{A(x) \mid x \in G\}$ denotes the image of $A$. For any subset $I$ of $G$, $\chi_I$ denotes the characteristic function of $I$.

Definition 4 (see [2]). A fuzzy subset $A$ of a group $G$ is called a fuzzy subgroup of $G$ if it satisfies the following conditions:

(i) $A(x + y) \geq \min\{A(x), A(y)\}$,

(ii) $A(-x) \geq A(x)$,

for all $x, y \in G$.

Definition 5. For a fuzzy subset $A$ of $G$, $t \in (0, 1]$, the subset $A_t = \{x \in G \mid A(x) \geq t\}$ is called a level subset of $G$ determined by $A$ and $t$.

The set $\{x \in G \mid A(x) > 0\}$ is called the support of $A$ and is denoted by $SuppA$. A fuzzy subset $A$ of $G$ of the form

$$A(y) = \begin{cases} t & (t \neq 0) \\ 0 & (t \neq 0) \end{cases}$$

is said to be a fuzzy point denoted by $x_t$. Here $x$ is called the support point, and $t$ is called its value. A fuzzy point $x_t$ is said to belong to (resp., quasi coincide with) a fuzzy set $A$ written as $x_t \vDash A$ (resp., $x_t \vDash A$) if $A(x) \geq t$ (resp., $A(x) > t > 1$). If $x_t \in A$ or $x_t \vDash A$, then we write $x_t \vDash \vDash A$. The symbols $x_t \vDash A, x_t \vDash A, x_t \vDash \vDash A \vDash A$ mean that $x_t \in A, x_t \vDash A, x_t \vDash \vDash A$ do not hold, respectively.

Definition 6 (see [7, 12]). A fuzzy subset of a group $G$ is said to be an $(\epsilon, \epsilon \lor \epsilon)$-fuzzy subgroup of $G$ if it satisfies the following conditions:

(i) $x_t, y_t \in A \Rightarrow (x + y)_{min(t, r)} \vDash \vDash A$,

(ii) $x_t \in A \Rightarrow (-x)_{t} \vDash \vDash A$.

Remark 7 (see [7]). The conditions (i) and (ii) of Definition 6 are respectively equivalent to

(i) $A(x + y) \geq \min\{A(x), A(y), 0.5\}$,

(ii) $A(-x) \geq \min\{A(x), 0.5\}$,

for all $x, y \in G$.

Remark 8. For any $(\epsilon, \epsilon \lor \epsilon)$-fuzzy subgroup $A$ of $G$ such that $A(x) \geq 0.5$ for some $x \in G$, then $A(0) \geq 0.5$ and if $A(0) < 0.5$, then $A(x) < 0.5$ for all $x \in G$. So, $A$ is just the usual fuzzy subgroup in the sense of Rosenfeld.

Remark 9. It is noted that if $A$ is a fuzzy subgroup then it is an $(\epsilon, \epsilon \lor \epsilon)$-fuzzy subgroup of $G$. However the converse may not be true.

Here onwards we assume that $A$ is an $(\epsilon, \epsilon \lor \epsilon)$-fuzzy subgroup in the nontrivial sense for which case we have $A(0) \geq 0.5$.

Definition 10 (see [7]). An $(\epsilon, \epsilon \lor \epsilon)$-fuzzy subgroup of a group $G$ is said to be an $(\epsilon, \epsilon \lor \epsilon)$-fuzzy normal subgroup if for any $x, y \in G$ and $t \in (0, 1]$,

$$x_t \vDash A \Rightarrow (x + y - x)_{t} \vDash \vDash A$$

Remark 11. If $\chi$ is a fuzzy subset of a group $G$, then $\chi_S$ denotes the commutator of $x, y \in G$.

In the light of this fact, the condition of Definition 10 can be replaced by any one of the above conditions in Remark 8.

Definition 12 (see [18]). Let $A$ be a fuzzy subset of an $N$-group $G$. It is called a fuzzy $N$-subgroup of $G$ if it satisfies the following conditions:

(i) $A(x + y) \geq \min\{A(x), A(y)\}$,

(ii) $A(nx) \geq A(x)$,

for all $x, y \in G, n \in N$.

Remark 13. If $G$ is a unitary $N$-group, the above conditions are equivalent to conditions $A(x - y) \geq \min\{A(x), A(y)\}$ and $A(nx) \geq A(x)$ for all $x, y \in G, n \in N$.

Definition 14 (see [18, 19]). A nonempty fuzzy subset $A$ of an $N$-group $G$ is called a fuzzy ideal if it satisfies the following conditions:

(i) $A(x - y) \geq \min\{A(x), A(y)\}$,

(ii) $A(nx) \geq A(x)$,

(iii) $A(y + x - y) \geq A(x)$,

(iv) $A(n(x + y) - nx) \geq A(y)$,

for all $x, y \in G, n \in N$. 
Definition 15 (see [14]). A fuzzy set $A$ of a near-ring $N$ is called an $(\varepsilon, \varepsilon \vee q)$-fuzzy subnear-ring of $N$ if for all $t, r \in (0, 1]$, and $x, y \in N$

(i) $x_t, y_t \in A \Rightarrow (x + y)_{\min(t, r)} \in \mathcal{V}A$, 
(ii) $x_t, y_t \in A \Rightarrow (xy)_{\min(t, r)} \in \mathcal{V}A$.

A is called an $(\varepsilon, \varepsilon \vee q)$-fuzzy subnear-ring of $N$ if it is an $(\varepsilon, \varepsilon \vee q)$-fuzzy near-ring of $N$ and

(iii) $x_t \in A \Rightarrow (y + x - y) \in \mathcal{V}A$,
(iv) $y_t \in A, x \in N \Rightarrow (yx)_t \in \mathcal{V}A$,
(v) $q_t \in A \Rightarrow (y(x + a) - xy)_t \in \mathcal{V}A$, for all $x, y, a \in N$.

2. Generalized Fuzzy Ideals

In this section, we give the definition of $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup and ideal of an $N$-group $G$ based on Definitions 14 and 15.

Definition 16. A fuzzy subset $A$ of an $N$-group $G$ is said to be an $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup of $G$ if $x, y \in G, n \in N, t, r \in (0, 1]$, 

(i) $x_t, y_t \in A \Rightarrow (x + y)_{\min(t, r)} \in \mathcal{V}A$,
(ii) $x_t \in A \Rightarrow (-x)_t \in \mathcal{V}A$,
(iii) $x_t, y_t \in A \Rightarrow (xy)_t \in \mathcal{V}A$.

Lemma 17. Let $A$ be a fuzzy subset of $G$ and $t, r \in (0, 1]$. Then,

(i) $x_t, y_t \in A \Rightarrow (x + y)_{\min(t, r)} \in \mathcal{V}A \Leftrightarrow A(x + y) \geq \min\{A(x), A(y), 0.5\}$, 
(ii) $x_t \in A \Rightarrow (-x)_t \in \mathcal{V}A \Leftrightarrow A(-x) \geq \min\{A(x), 0.5\}$, 
for all $x, n \in G$.

Proof. (i) Let $x, y \in G$. Consider the case (a): $\min\{A(x), A(y)\} < 0.5$.

Assume that $A(x + y) < \min\{A(x), A(y), 0.5\} = \min\{A(x), A(y)\}$. Choose $t$ such that $A(x + y) < t < \min\{A(x), A(y)\}$ which implies that $x_t \in A, y_t \in A$ but $(x + y)_t \in \mathcal{V}A$ [as $A(x + y) + t < 1$ and $A(x + y) < t$]. Consider the case (b): $\min\{A(x), A(y)\} \geq 0.5$. Assume that $A(x + y) < \min\{A(x), A(y), 0.5\} = 0.5$. Choose $t$ such that $A(x + y) < t < 0.5$ so that $x_t, y_t \in A$ but $(x + y)_t \in \mathcal{V}A$.

Conversely, let $x_t, y_t \in A$ and $A(x + y) \geq r$. Then, $A(x + y) \geq \min\{A(x), A(y), 0.5\} \geq \min\{A(x), A(y), r, 0.5\}$. Thus $A(x + y) \geq \min\{t, r\}$ if either $t$ or $r \leq 0.5$ and $A(x + y) \geq 0.5$ if both $t$ and $r > 0.5$ which means $(x + y)_{\min(t, r)} \in \mathcal{V}A$.

(ii) Let $x \in G$, $\min\{A(x), 0.5\} \leq 0.5$. Suppose $A(-x) < \min\{A(x), 0.5\} \leq 0.5$. Choose $r$ such that $A(-x) < r < \min\{A(x), 0.5\} \leq 0.5$. Then, $x_t \in A$ but $(-x)_t \in \mathcal{V}A$ which contradicts the hypothesis. So, $A(-x) \geq \min\{A(x), 0.5\}$ for all $x \in G$.

Conversely, let $x_t \in A$. Then, $A(x) \geq t$. But we have $A(-x) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} \Rightarrow A(-x) \geq t$ or $A(-x) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5 \Rightarrow (-x)_t \in \mathcal{V}A$.

(iii) Let $x \in G$ and $\min\{A(x), 0.5\} \leq 0.5$. Suppose $A(nx) < \min\{A(x), 0.5\} \leq 0.5$. Choose $r$ such that $A(nx) < r < \min\{A(x), 0.5\} \leq 0.5$. Then, $A(nx) > r$ that is, $x_t \in A$, but $(nx)_t \in \mathcal{V}A$ as $A(nx) < t$ and $(nx) + t \leq 1$.

Conversely let $x_t \in A, n \in N$; then $A(x) \geq t$. But $A(nx) \geq \min\{A(x), 0.5\} \geq \min\{r, 0.5\} \Rightarrow A(nx) \geq t$ or $A(nx) \geq 0.5$ according as $t \leq 0.5$ or $t > 0.5 \Rightarrow A(nx) \geq t$ or $A(nx) + t > 1 \Rightarrow (nx)_t \in \mathcal{V}A$.

Theorem 18. Let $A$ be a fuzzy subset of $G$. Then, $A$ is an $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup of $G$ if and only if the following conditions are satisfied:

(i) $A(x + y) \geq \min\{A(x), A(y), 0.5\}$,
(ii) $A(-x) \geq \min\{A(x), 0.5\}$,
(iii) $A(nx) \geq \min\{A(x), 0.5\}$,
for all $x, y \in G, n \in N$.

Proof. It follows from the previous lemma.

Definition 19. A fuzzy subset $A$ of an $N$-group $G$ is said to be $(\varepsilon, \varepsilon \vee q)$-fuzzy ideal of $G$ if it is an $(\varepsilon, \varepsilon \vee q)$-fuzzy subgroup and satisfies the following conditions:

(i) $x_t \in A \Rightarrow (y + x - y)_t \in \mathcal{V}A$,
(ii) $a_t \in A \Rightarrow (nx + a - nx)_t \in \mathcal{V}A$,
for any $n \in N, x, a \in G$.

Lemma 20. Let $A$ be a fuzzy subset of $G$ and $t, r \in (0, 1]$. Then,

(i) $x_t \in A \Rightarrow (y + x - y)_t \in \mathcal{V}A \Leftrightarrow A(y + x - y) \geq \min\{A(x), 0.5\}$,
(ii) $a_t \in A \Rightarrow (nx + a - nx)_t \in \mathcal{V}A \Leftrightarrow A(nx + a - nx) \geq \min\{A(x), 0.5\}$.

Proof. (i) Assume that $A(y + x - y) < \min\{A(x), 0.5\}$. Choose $t$ such that $A(y + x - y) < t < \min\{A(x), 0.5\}$. But $\min\{A(x), 0.5\} \leq A(x) < 0.5$ or $A(x) \geq 0.5$ and $A(x) > 0.5$, respectively, which contradicts the hypothesis.

Conversely, assume that $x_t \in A$, then $A(x) \geq t$. For any $y \in G$, we have $A(y + x - y) \geq \min\{A(x), 0.5\} \geq \min\{t, 0.5\} \Rightarrow A(y + x - y) \geq t$ or $0.5$ according as $t \leq 0.5$ or $t > 0.5 \Rightarrow (y + x - y)_t \in \mathcal{V}A$ or $(y + x - y)_{0.5} \in \mathcal{V}A$, respectively, which is in contradiction. $A(x) \geq 0.5$ or for some $n \in N, x, a \in G$. According $A(a) < 0.5$ or $A(a) > 0.5$. Choose $t \in (0, 1]$ such that $A(nx + a - nx) < t < \min\{A(a), 0.5\} \leq 0.5$. In either case, $A(nx + a - nx) < t$ or $A(nx + a - nx) + t > 1$. So, $(nx + a - nx)_t \in \mathcal{V}A$, which is a contradiction.

Conversely, assume that $A(nx + a - nx) \geq \min\{A(a), 0.5\}$ for all $a, x \in G, n \in N$. Let $a_t \in A$. Then, $A(a) \geq t$. So, $A(nx + a - nx) \geq \min\{t, 0.5\} \leq 0.5$ or $0.5$ according as $t \leq 0.5$ or $t > 0.5$. So, $A(nx + a - nx)_t \in \mathcal{V}A$. □
Theorem 21. Let $A$ be an $(e, \varepsilon \lor q)$ fuzzy subgroup of $G$. Then, $A$ is an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$ if and only if

(i) $A(y + x - y) \geq \min[A(x), 0.5]$, for all $x, y \in G$,

(ii) $A(n(x + a) - nx) \geq \min[A(a), 0.5]$, for all $n \in N$, $x, a \in G$.

Proof. It is immediate from Lemma 20. $\square$

By definition, a fuzzy ideal of $G$ is an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$. But the converse is not true in general as shown by the following example.

Example 22. Consider $G = S_3 = \{i, r_1, r_2, r_3\}$ (written additively) to be a $\mathcal{Z}$-group. Define a fuzzy subset $A$ of $G$ as $A(i) = 1$, $A(r_1) = A(r_2) = A(r_3) = 0.6$, $A(r_1) = 0.8$ which is not a fuzzy ideal as $A[2(r_1 + r_2) - 2r_3] = A(r_1) = 0.6 < A(r_1)$; it contradicts the condition (iv) of Definition 14. As $A(x - y), A(nx), A(y + x - y)$ and $A(n(x + a) - nx)$ are $0.6$ or $0.8 \geq \min(0.5, 0.6 \lor 0.8) = 0.5$, thus, the notion of $(e, \varepsilon \lor q)$-fuzzy ideal is a successful generalization of fuzzy ideals of $G$ as introduced in [18].

Theorem 23. Let $\{A_i, i \in I\}$ be any family of $(e, \varepsilon \lor q)$-fuzzy ideals of $G$. Then, $A = \bigcap_{i \in I} A_i$ is an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$.

Proof. It is straightforward. $\square$

Theorem 24. A nonempty subset $I$ of $G$ is an ideal of $G$ if and only if $\chi_I$ is an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$.

Proof. If $I$ is an ideal of $G$, it is clear from [18, Proposition 2.1] that $\chi_I$ is fuzzy ideal of $G$. Since every fuzzy ideal is an $(e, \varepsilon \lor q)$-fuzzy ideal, $\chi_I$ is an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$.

Conversely, let $\chi_I$ be an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$. Let $x, y \in I, \chi_I(x - y) \geq \min[\chi_I(x), \chi_I(y), 0.5] = 0.5$. So, $\chi_I(x - y) = 1 \Rightarrow x - y \in I$. Let $n \in N, x \in I, \chi_I(nx) \geq \min[\chi_I(x), 0.5] = 0.5 \Rightarrow \chi_I(nx) = 1 \Rightarrow nx \in n'I \in G, x \in I, \chi_I(y + x - y) \geq \min[\chi_I(x), 0.5] = 0.5 \Rightarrow \chi_I(x + y - x - y) = 1 \Rightarrow y + x - y \in 1n'I \in G, x \in I, y \in G, \chi_I(n(y + x - ny)) = 0.5 \Rightarrow \chi_I(n(y + x - ny)) = 1 \Rightarrow n(y + x - ny) \in I$. Then, $I$ is an ideal of $G$. $\square$

Theorem 25. A fuzzy subset $A$ of $G$ is an $(e, \varepsilon \lor q)$-fuzzy (subgroup) ideal of $G$ if and only if the level subset $A_i$ is a (subgroup) ideal for $0 < t \leq 0.5$.

Proof. We prove the result for $(e, \varepsilon \lor q)$-fuzzy ideal $S$. Let $A$ be an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$. Let $t \leq 0.5, x, y, i \in A_i, n \in N$.

(i) $A(x - y) \geq \min[A(x), A(y), 0.5] \geq \min[t, 0.5] = t \Rightarrow x - y \in A_i$,

(ii) $A(nx) \geq \min[A(a), 0.5] \geq \min[t, 0.5] = t \Rightarrow nx \in A_i$,

(iii) $A(y + x - y) \geq \min[A(x), 0.5] \geq \min[t, 0.5] = t \Rightarrow y + x - y \in A_i, y \in G$

(iv) $A(n(y + x) - ny) \geq \min[A(x), 0.5] \geq \min[t, 0.5] = t \Rightarrow n(y + x) - ny \in A_i, y \in G, n \in N$.

Hence, $A_i$ is an ideal of $G$. Again, let $A_i$ be an ideal of $G$ for all $t \leq 0.5$. If possible, let there exist $x, y \in G$ such that $A(x - y) < t < \min[A(x), A(y), 0.5]$. Let $t$ be such that $A(x - y) < t < \min[A(x), A(y), 0.5] \Rightarrow x, y \in A_i, x - y \notin A_i$, a contradiction. So, $A(x - y) \geq \min[A(x), A(y), 0.5]$, for all $x, y \in G$. For $n \in N, x \in G$ let $A(nx) < \min[A(x), 0.5]$. If possible let $t$ be such that $A(nx) < t < \min[A(x), 0.5]$. This implies $x \notin A_i, n \in A_i$, a contradiction. Similarly, we can prove that $A(y + x - y) \geq \min[A(x), 0.5], A(n(y + x) - ny) \geq \min[A(x), 0.5], x, y \in G, n \in N$. $\square$

Remark 26. For $t \in (0.5, 1)$, $A$ may be an $(e, \varepsilon \lor q)$-fuzzy ideal of $G$, but $A_i$ may not be an ideal of $G$. Let $t = 0.8$ in Example 22. Then, $A_i = \{i, r_1\}$. $A_i$ is not an ideal of $S_3$ as it is not a normal subgroup of $S_3$.

We are looking for a corresponding result when $A_i$ is an ideal of $G$ for all $t \in (0.5, 1]$.

Theorem 27. Let $A$ be a fuzzy subset of an $N$-group $G$. Then, $A_i \neq \emptyset$ is an ideal of $G$ for all $t \in (0.5, 1]$ if and only if $A$ satisfies the following conditions:

(i) $\max[A(x - y), 0.5] \geq \min[A(x), A(y)]$,

(ii) $\max[A(nx), 0.5] \geq A(x),$

(iii) $\max[A(y + x - y), 0.5] \geq A(x),$

(iv) $\max[A(n(y + x) - ny), 0.5] \geq A(x),$

for all $x, y \in G, n \in N$.

Proof. Suppose that $A_i \neq \emptyset$ is an ideal of $G$ for all $t \in (0.5, 1]$. In order to prove (i), suppose that for some $x, y \in G, \max[A(x - y), 0.5] < \min[A(x), A(y)]$. Let $t = \min[A(x), A(y)]$. So, $x, y \in A_i$, and $t \in (0.5, 1]$. Since $A_i$ is an ideal, $x - y \notin A_i$. So, $A(x - y) \geq \max[A(x - y), 0.5], 0.5$, a contradiction. In order to prove (ii), suppose that $x \in G, n \in N$ and $\max[A(nx), 0.5] < A(x) = t$ (say). Then, $x \in A_i \Rightarrow nx \in A_i \Rightarrow A(nx) \geq t > \max[A(nx), 0.5]$, a contradiction. Similarly, we can prove (iii) and (iv).

Conversely, suppose that conditions (i) to (iv) hold. We show that $A_i$ is an ideal of $G$ for all $t \in (0.5, 1]$. Let $x, y \in A_i$. Then, $0.5 < t \leq \min[A(x), A(y)] \leq \max[A(x - y), 0.5] = A(x - y)$. So, $x - y \in A_i$. Let $n \in N, x \in A_i$. Then, $0.5 < t \leq \min[A(x), 0.5] = A(x) \Rightarrow nx \in A_i$. For $x \in A_i, y \in G, 0.5 < t \leq \min[A(x + y - x), 0.5] = A(x + y - x) \Rightarrow y + x - y \in A_i$. Also, if $n \in N, x \in A_i, y \in G, 0.5 < t \leq \min[A(n(y + x) - ny), 0.5] = A(n(y + x) - ny)$.

Hence, $n(y + x) - ny \in A_i$. Then, $A_i$ is an ideal of $G$. $\square$

A definition for the previous kind of fuzzy subset was given for the case of near-rings in [17]. Now, we give the definition for $N$-groups.

Definition 28. A fuzzy subset of $G$ is called an $(\bar{e}, \bar{\varepsilon} \lor \bar{q})$-fuzzy subgroup of $G$ if for all $t, r \in (0, 1]$ and for all $x, y \in G, n \in N$,

(i) $(x + y)_{\min(t,r)} \bar{e}A$ implies $x_{\bar{e}q}y_{\bar{e}q}A$

(b) $(x + y)_{\min(t,r)} \bar{e}A$ implies $x_{\bar{e}q}y_{\bar{e}q}A$,
Algebra 5

(ii) \((nx), \overline{vA}\) implies \(x, \overline{vqA}\).
Moreover, \(A\) is called an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G\) if \(A\) is \((\varepsilon, \varepsilon \lor q)\)-fuzzy subgroup of \(G\) and

(iii) \((y + x - y), \overline{vA}\) implies \(x, \overline{vqA}\),
(iv) \((n(x - y) - ny), \overline{vA}\) implies \(x, \overline{vqA}\).

**Theorem 29.** A fuzzy subset \(A\) of \(G\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G\) if and only if

\[
\begin{align*}
(1)(a) & \max\{A(x + y), 0.5\} \geq \min\{A(x), A(y)\}, \\
(b) & \max\{A(-x), 0.5\} \geq A(x), \\
(2) & \max\{A(y + x - y), 0.5\} \geq A(x), \\
(3) & \max\{A(nx), 0.5\} \geq A(x), \\
(4) & \max\{A(n(x + y) - ny), 0.5\} \geq A(x).
\end{align*}
\]

**Proof.** (1a) ⇒ (1a). Let \(x, y \in G\) be such that \(\max\{A(x + y), 0.5\} < \min\{A(x), A(y)\}\). Let \(t = \min\{A(x), A(y)\}\); then \(0.5 \leq t \leq 1\). So we must have \(x, \overline{vqA}\) or \(y, \overline{vqA}\).

Conversely, let \((x + y), \min\{t, r\}\) \(\overline{vA}\). Then, \(\min\{A(x), A(y)\}\) \(\leq A(0.5)\). It follows that either \(x, \overline{vqA}\) or \(y, \overline{vqA}\), and thus \(x, \overline{vqA}\).

(1b) ⇔ (1b). Suppose that there exists \(x \in G\) such that \(\max\{A(-x), 0.5\} < A(x)\). If \(A(x) = t\) then \(0.5 < t \leq 1\) and \(A(-x) < t\) so that \(\max\{A(-x), 0.5\} < t\). But then we must have either \(x, \overline{vA}\) or \(y, \overline{vA}\). Also we have \(x, \overline{vqA}\) and \(y, \overline{vqA}\).

Again if \(A(x + y) < \min\{A(x), A(y)\}\), then by (1a)

\[
0.5 \geq \min\{A(x), A(y)\} > A(x + y). \tag{4}
\]

Suppose that \(x, y \in A\) then \(t \leq A(x) \leq 0.5\) or \(r \leq A(y) \leq 0.5\). It follows that either \(x, \overline{vqA}\) or \(y, \overline{vqA}\), and thus \(x, \overline{vqA}\).

(iii) (1a) \(\implies\) (1b): Suppose that there exists \(x \in G\) such that \(\max\{A(-x), 0.5\} < A(x)\). If \(A(x) = t\) then \(0.5 \leq t \leq 1\) and \(A(-x) < t\) so that \(\max\{A(-x), 0.5\} < t\). But then we must have either \(x, \overline{vA}\) or \(y, \overline{vA}\). Also we have \(x, \overline{vqA}\) and \(y, \overline{vqA}\).

Similarly, we can prove the remaining parts. \(\Box\)

**Theorem 29.** A fuzzy subset \(A\) of \(G\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G\) if and only if \(A\) is an ideal of \(G\) for all \(t \in [0.5, 1]\).

3. Fuzzy Cosets and Isomorphism Theorem

In this section, we first study the properties of \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideals under a homomorphism. Then, we introduce the fuzzy cosets and prove the fundamental isomorphism theorem on \(N\)-groups with respect to the structure induced by these fuzzy cosets.

**Theorem 31.** Let \(G\) and \(G'\) be two \(N\)-groups, and let \(f : G \to G'\) be an \(N\)-homomorphism. If \(f\) is surjective and \(A\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G\), then so is \(f(A)\). If \(B\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G'\), then \(f^{-1}(B)\) is a fuzzy ideal of \(G\).

**Proof.** We assume that \(A\) is an \((\varepsilon, \varepsilon \lor q)\)-fuzzy ideal of \(G\). For any \(x, y \in G\); it follows that

\[
f(A)(x + y)
\]

\[
= \sup_{x+y}=f(z)\{A(z)\}
\]

\[
\geq \sup_{f(u)=x, f(v)=y}\{A(u + v)\}
\]

\[
= \min\{\sup_{f(u)=x}\{A(u)\}, \sup_{f(v)=y}\{A(v)\}, 0.5\}
\]

\[
= \min\{f(A)(x), f(A)(y), 0.5\}.
\]

Also,

\[
f(A)(-x)
\]

\[
= \sup_{f(z)=-x}\{A(z)\} = \sup_{f(z)=x}\{A(z)\}
\]

\[
\geq \sup_{f(z)=x}\{\min\{A(u), 0.5\}\}
\]

\[
= \min\{\sup_{f(z)=x}\{A(z)\}, 0.5\}
\]

\[
= \min\{f(A)(x), 0.5\}.
\]

Again,

\[
f(A)(nx)
\]

\[
= \sup_{f(z)=nx}\{A(z)\} \geq \sup_{f(u)=x}\{A(mu)\}
\]

\[
= \sup_{f(u)=x}\{\min\{A(u), 0.5\}\}
\]

\[
= \min\{\sup_{f(u)=x}\{A(u)\}, 0.5\}
\]

\[
= \min\{f(A)(x), 0.5\},
\]

\[
f(A)(y - x)
\]

\[
= \sup_{f(z)=y-x-y}\{A(z)\}
\]

\[
\geq \sup_{f(v)=x,f(u)=y}\{A(u + v - u)\}
\]

\[
= \min\{f(A)(x), f(A)(y), 0.5\}.
\]
\(f(A) \geq \sup_{f(x) = x} \{\min \{A(y), 0.5\}\}\)

\(= \min \left\{ \sup_{f(x) = x} \{A(y)\}, 0.5 \right\}\)

\(= \min \{f(A)(x), 0.5\}\),

\(f(A) (n(y + x) − ny)\)

\(= \sup_{f(x) = y} \{A(n(x + y) − ny)\}\)

\(\geq \sup_{f(x) = x, f(y) = y} \{\min \{A(u), 0.5\}\}\)

\(\geq \sup_{f(x) = x, f(y) = y} \{\min \{A(0), 0.5\}\}\)

\(= \min \{f(A)(x), 0.5\}\).

\(\text{(7)}\)

Therefore, \(f(A)\) is an \((e, e \vee q)\)-fuzzy ideal of \(G\). Similarly, we can show that \(f^{-1}(A)\) is an \((e, e \vee q)\)-fuzzy ideal of \(G\).

**Definition 32.** Let \(A\) be \((e, e \vee q)\)-fuzzy subgroup of \(G\). For any \(x \in G\), let \(A_x\) be defined by \(A_x(g) = \min \{A(g - x), 0.5\}\) for all \(g \in G\). This fuzzy subset \(A_x\) is called the \((e, e \vee q)\)-fuzzy left coset of \(G\) determined by \(A\) and \(x\).

**Remark 33.** Let \(A\) be an \((e, e \vee q)\)-fuzzy subgroup of \(G\). Then, \(A\) is an \((e, e \vee q)\)-fuzzy normal if and only if \(A(x - y) \geq 0.5\) for all \(x, y \in G\). If \(A\) is an \((e, e \vee q)\)-fuzzy ideal, we simply denote fuzzy coset by \(A_x\).

**Lemma 34.** Let \(A\) be an \((e, e \vee q)\)-fuzzy ideal of \(G\). Then, \(A_x = A_y\) if only if \(A(x - y) \geq 0.5\).

**Proof.** Assume that \(A(x - y) \geq 0.5\) and \(y \in G\). \(A_x(g) = \min \{A(g - x), 0.5\}\) for all \(g \in G\). If \(A\) is an \((e, e \vee q)\)-fuzzy ideal, we simply denote fuzzy coset by \(A_x\).

**Proposition 35.** Every fuzzy coset \(A_x\) is constant on every coset of \(G = \{x \in G | A(x) = (0)\}\).

**Proof.** Let \(y + y_0 \in y + G_0\). Now, we have \(A_x(y + y_0) = \min \{A(y + y_0 - x), 0.5\}\) for all \(y, y_0 \in G\). Also, \(A_x(y) = \min \{A(x - y), 0.5\}\) for all \(x \in A\). Therefore, \(A_x(y) = \min \{A(x - y), 0.5\}\) for all \(x \in G_0\).

**Theorem 36.** For any \((e, e \vee q)\)-fuzzy ideal \(A\) of \(G\), \(A_x\) is defined by \(A_x = \{x \in G | A(x) = (0)\}\) for all \(x \in G\). Then, \(A_x = \{x \in G | A(x) = (0)\}\) for all \(x \in G\).

**References**

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