Research Article

Freely Solvable Graphs in Peg Solitaire

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In a 2011 paper, the game of peg solitaire is generalized to arbitrary boards, which are treated as graphs in the combinatorial sense. Of particular interest are graphs that are freely solvable, that is, graphs that can be solved from any starting position. In this paper we give several examples of freely solvable graphs including all such trees with ten vertices or less, numerous cycles with a subdivided chord, meshes, and generalizations of the wheel, helm, and web.

1. Introduction

Peg solitaire is a table game which traditionally begins with “pegs” in every space except for one which is left empty (i.e., a “hole”). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z . The peg in y is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved [1, 2].

In [3], this notion is generalized to graphs. A graph, $G = (V, E)$, is a set of vertices, V , and a set of edges, E . Because of the restrictions of peg solitaire, we will assume that all graphs are finite, undirected, and connected graphs with no loops or multiple edges. For all undefined graph theory terminology, refer to West [4]. If there are pegs in vertices x and y and a hole in z , then we allow the peg in x to jump over the peg in y into the hole in z provided that $xy, yz \in E$. The peg in y is then removed. This jump is denoted by $x \cdot \vec{y} \cdot z$.

A graph G is *solvable* if there exists some vertex s so that, starting with a hole in s , there exists an associated terminal state consisting of a single peg. A graph G is *freely solvable* if this is true for all vertices. A graph G is *distance 2-solvable* if there exists some vertex s so that, starting with a hole in s , there exists an associated terminal state consisting of two pegs that are distance 2 apart. In Figure 2, the left graph is not solvable. The middle graph in Figure 2 is solvable but not freely solvable. The right graph in this figure is freely solvable.

In [5], it is shown that, of the 996 nonisomorphic connected graphs on seven vertices or less, only 54 are not freely solvable. However, determining whether a specific graph is freely solvable is NP-hard. Also [6] shows that, if G is freely solvable and G' is obtained by appending a pendent vertex to any vertex of G , then G' is (at worst) solvable. We are motivated by the above comments to give examples of freely solvable graphs in this paper. Known examples of freely solvable graphs include the Petersen graph, the platonic solids, the Archimedean solids, the complete graph, and the complete k -partite graph. Other examples are given in [3, 5, 6].

Several useful results are included in [3]. The following theorem allows the completion of the game in reverse by exchanging the roles of pegs and holes.

Theorem 1 (see [3]). *Suppose that S is a starting state of G with associated terminal state T . Let S' and T' be the duals of S and T , respectively. It follows that T' is a starting state of G with associated terminal state S' .*

The following observation is also useful.

Observation 1 (see [3]). (i) If G can be k -solved with the initial hole in vertex s and a jump is possible, then there is a first jump; say $s'' \cdot \vec{s'} \cdot s$. Hence, if there are holes in s'' and s' and pegs elsewhere, then G can be k -solved from this configuration.

FIGURE 1: A typical jump in peg solitaire, $x \cdot \vec{y} \cdot z$.

FIGURE 2: An example of an unsolvable, a solvable, and a freely solvable graph.

(ii) If G is a k -solvable spanning subgraph of H , then H is (at worst) k -solvable.

Because of this observation, it is of particular interest to determine which trees are freely solvable. In Figure 3, we list all nonisomorphic freely solvable trees with ten vertices or less. The list of trees comes from the appendix to Harary [7]. An exhaustive computer search [8] was used to determine solvability.

2. The Cycle with a Subdivided Chord

The cycle with a subdivided chord, denoted by $\text{CSC}(n, m)$, is formed from a cycle on n vertices, labeled $0, 1, \dots, n-1$ in the usual way, by adding an edge from vertex 0 to vertex m to form a chord. This edge is then subdivided. The resulting vertex on the chord is labeled c . Figure 4 shows a cycle with a subdivided chord. As usual, P_n will denote the path on n vertices with vertices labeled $0, 1, \dots, n-1$ in the usual way.

In the following result, we show that $\text{CSC}(n, m)$ is solvable with the initial hole in several vertices. However, these graphs may be solvable from additional vertices as well. In [3], it was shown that P_{2n} is solvable with the initial hole in vertex 1 and the final peg in $(2n-2)$, where $n \geq 2$. If $n \geq 5$ is odd, then P_n is distance 2-solvable with the initial hole in 1 and the final two pegs in $(n-3)$ and $(n-1)$.

Lemma 2. *The graph $\text{CSC}(2k, m)$ is solvable with the initial hole in vertex $0, 1, 2, 3, m-3, m-2, m-1, m, m+1, m+2, m+3, 2k-3, 2k-2, 2k-1$, or c . The graph $\text{CSC}(2k+1, m)$ is solvable with the initial hole in vertex $0, 2, m-2, m, m+2$, or $2k-1$.*

Proof. Note that the vertices i and $i+m \pmod{n}$ are symmetric.

For $\text{CSC}(2k, m)$, suppose the initial hole is in vertex 0 ; jump $m \cdot \vec{c} \cdot 0$. An even path subgraph is formed by $m-1, m, \dots, 2k-1, 0, \dots, m-2$ with a hole in m . This subgraph is solvable with the final peg in $m-3$ by [3]. Alternately, an even path is formed by $m+1, m, m-1, \dots, 0, 2k-1, \dots, m+2$ with a hole in m . Solve this path with the final peg in $m+3$. By Theorem 1, the graph may also be solved with the initial hole in $m-3$ or $m+3$. A similar argument holds if the initial hole is in $m, 3$, or $2k-3$.

FIGURE 3: Freely solvable trees with $n(T) \leq 10$. FIGURE 4: The cycle with a subdivided chord, $\text{CSC}(6, 2)$.

Suppose that the initial hole is in 1. Jump $c \cdot \vec{0} \cdot 1$. Solve the even cycle formed by the remaining pegs with a hole in 0. A similar argument holds when the initial hole is in $m-1, m+1$, or $2k-1$.

Suppose that the initial hole is in 2. Jump $0 \cdot \vec{1} \cdot 2$. Solve the even path formed by the remaining pegs with a hole in 0. A similar argument holds for the case when the initial hole is in $m-2, m+2$, or $2k-2$.

Suppose that the initial hole is in c . Jump 1 over 0 into c . Solve the even path formed by the remaining pegs with a hole in 0.

For $\text{CSC}(2k+1, m)$, suppose the initial hole is in vertex 0 . An even path is formed by $c, 0, 1, \dots, 2k$ with a hole in 0 . Solve this path with the final peg in $2k-1$. Alternately, an even path is formed by $c, 0, 2k, 2k-1, \dots, 1$ with a hole in 0 . Solve this path with the final peg in 2. By Theorem 1, the graph may also be solved with the initial hole in $2k-1$ or 2. A similar argument holds when the initial hole is in $m, m-2$, or $m+2$. $\square$

We now give some freely solvable examples of this type of graph.

Theorem 3. *The graph $\text{CSC}(2k, 4)$ is freely solvable.*

Proof. If the initial hole is in vertex $\{2k-3, 2k-1, 0, 1, \dots, 7, c\}$, then the graph is solvable by Lemma 2.

Suppose the initial hole is in i , where i is even and $8 \leq i \leq 2k-4$. An even path is formed by $i-1, i, \dots, 0, c, 4$ with a hole in i . Solve this path, ending in c . Then jump $2 \cdot \vec{1} \cdot 0, 0 \cdot \vec{c} \cdot 4$, and

$3 \cdot \vec{4} \cdot c$. Finally, solve the even path with a hole in 4, formed by $c, 4, 5, \dots, i-2$.

Suppose the initial hole is in j , where j is odd and $9 \leq j \leq 2k-5$. An even path is formed by $j-1, j, \dots, 2k-1$ with a hole in j . Solve this path, ending in $2k-2$. Then jump $c \cdot \vec{0} \cdot 2k-1, 2k-2 \cdot 2k-1 \cdot 0, 1 \cdot \vec{0} \cdot c, 4 \cdot \vec{c} \cdot 0$, and $2 \cdot \vec{3} \cdot 4$. Finally, solve the even path with a hole in c , formed by $0, c, 4, 5, \dots, j-2$. $\square$

Theorem 4. *The graph $CSC(2k, 1)$ is freely solvable.*

Proof. If the initial hole is in vertex $\{2k-3, 2k-1, 0, 1, 2, 4, c\}$, then the graph is solvable by Lemma 2.

Suppose the initial hole is in i , where i is even and $6 \leq i \leq 2k-4$. An even path is formed by $i+1, i, \dots, 1, c$ with a hole in i . Solve this path, ending in 1. Then jump $1 \cdot \vec{0} \cdot c$. Finally, solve the even path with a hole in 1, formed by $c, 1, \dots, i+2$.

Suppose the initial hole is in j , where j is odd and $3 \leq j \leq 2k-5$. An even path is formed by $j-1, j, \dots, 2k-1, 0, c$ with a hole in j . Solve this path, ending in 0. Now jump $1 \cdot \vec{0} \cdot c$. Finally, solve the even path with a hole in 1, formed by $c, 1, \dots, j-2$. $\square$

Note that $CSC(2k, 1)$ is isomorphic to the graph formed from C_{2k+1} by adding a chord between vertices 0 and 2. Thus, this provides partial progress on the open question from [5] as to whether all chorded odd cycles are freely solvable.

Theorem 5. *The graph $CSC(2k+1, 3)$ is freely solvable.*

Proof. If the initial hole is in vertex $0, 1, 2, 3, 5$, or $2k-1$, then the graph is solvable by Lemma 2.

Suppose the initial hole is in i , where i is even and $4 \leq i \leq 2k$. An even path is formed by $i-1, i, \dots, 2k, 0, 1$ with a hole in i . Solve this path, ending in 0. Then jump $3 \cdot \vec{2} \cdot 1$ and $0 \cdot \vec{c} \cdot 3$. Finally, solve the even path with a hole in 2, formed by $1, 2, \dots, i-2$.

Suppose the initial hole is in j , where j is odd and $7 \leq j \leq 2k-3$. An even path is formed by $j-1, j, \dots, 2k, 0$ with a hole in j . Solve this path, ending in $2k$. Then jump $3 \cdot \vec{c} \cdot 0, 1 \cdot \vec{2} \cdot 3, 2k \cdot \vec{0} \cdot c$, and $c \cdot \vec{3} \cdot 2$. Finally, solve the even path with a hole in 3, formed by $2, 3, \dots, j-2$.

Suppose the initial hole is in c . Jump $2 \cdot \vec{3} \cdot c$ and $0 \cdot \vec{c} \cdot 3$. Solve the even path formed by the remaining pegs with a hole in 2. $\square$

It is currently unknown if $CSC(n, m)$ is freely solvable for other values of n and m .

3. The Mesh and Related Graphs

In [3], it is shown that, if G is solvable or distance 2-solvable and H is solvable or distance 2-solvable, then the cartesian product $G \square H$ is solvable. We now consider a specific case of when such graphs are freely solvable, namely, meshes $P_n \square P_m$. If n or m is even, then $P_n \square P_m$ is Hamiltonian and has an even number of vertices. Hence, it is freely solvable by [3]. Thus, it suffices to consider the case where both n and m are odd. Label the vertex of $G \square H$ induced by $g \in V(G)$ and $h \in V(H)$ by (g, h) .

Lemma 6. *For $m \geq 3$, there exists a peg solitaire solution to the mesh $P_m \square P_3$ such that (i) the initial jump is $(m-1, 2) \cdot (m-1, 1) \cdot (m-1, 0)$ and (ii) the final peg is in vertex $(m-2, 0)$.*

Proof. Proceed by induction on m . An exhaustive computer search [8] shows that the claim holds for $m = 3$. Assume the claim holds for $P_m \square P_3$, and consider $P_{m+1} \square P_3$. With the initial hole in vertex $(m, 0)$, jump $(m, 2) \cdot (m, 1) \cdot (m, 0), (m-1, 2) \cdot (m-1, 1) \cdot (m, 1)$, and $(m-2, 2) \cdot (m-2, 1) \cdot (m-1, 1)$. The $P_{m-1} \square P_3$ subgraph is solvable by hypothesis, ending in $(m-3, 0)$. Now jump $(m, 0) \cdot (m-1, 0) \cdot (m-2, 0), (m-3, 0) \cdot (m-2, 0) \cdot (m-1, 0), (m-1, 1) \cdot (m-1, 0) \cdot (m, 0)$, and $(m, 1) \cdot (m, 0) \cdot (m-1, 0)$ to solve the graph. $\square$

Lemma 7. *The mesh $P_{2k+1} \square P_3$ is freely solvable with the final peg in vertex $(2k-1, 0)$.*

Proof. Proceed by induction on k . If $k = 1$, then use Lemma 6. Assume the claim holds for $P_{2k+1} \square P_3$, and consider $P_{2k+3} \square P_3$. Lemma 6 shows that the claim holds with the initial hole in vertex $(2k+2, 0)$.

Case 1. Suppose the initial hole is in (i, j) , where $0 \leq i \leq 2k$ and $0 \leq j \leq 2$. The $P_{2k+1} \square P_3$ subgraph is solvable by hypothesis, ending in $(2k-1, 0)$. Now jump $(2k+2, 0) \cdot (2k+1, 0) \cdot (2k, 0), (2k-1, 0) \cdot (2k, 0) \cdot (2k+1, 0), (2k+2, 2) \cdot (2k+2, 1) \cdot (2k+2, 0), (2k+1, 2) \cdot (2k+1, 1) \cdot (2k+2, 1), (2k+2, 0) \cdot (2k+1, 0) \cdot (2k+1, 1)$, and $(2k+2, 1) \cdot (2k+1, 1) \cdot (2k+1, 0)$ to solve the graph.

Case 2. Suppose the initial hole is in $(2k+1, 0)$. Jump $(2k, 1) \cdot (2k+1, 1) \cdot (2k+1, 0)$ and $(2k, 2) \cdot (2k+1, 2) \cdot (2k+1, 1)$. The $P_{2k+1} \square P_3$ subgraph is solvable by hypothesis, ending in $(2k-1, 0)$. Finally, jump $(2k+2, 0) \cdot (2k+1, 0) \cdot (2k, 0), (2k-1, 0) \cdot (2k, 0) \cdot (2k+1, 0), (2k+2, 2) \cdot (2k, 0) \cdot (2k-1, 0), (2k+2, 2) \cdot (2k+2, 1) \cdot (2k+2, 0), (2k+2, 0) \cdot (2k+1, 0) \cdot (2k+1, 1)$, and $(2k+1, 2) \cdot (2k+1, 1) \cdot (2k+1, 0)$ to solve the graph.

Case 3. Suppose the initial hole is in $(2k+1, 1)$. Jump $(2k, 2) \cdot (2k, 1) \cdot (2k+1, 1)$. The $P_{2k+1} \square P_3$ subgraph is solvable by Lemma 6, ending in $(2k-1, 0)$. This reduces the graph to Case 1. Suppose the initial hole is in $(2k+1, 2)$. Jump $(2k, 1) \cdot (2k, 2) \cdot (2k+1, 2)$. This reduces the graph to the above case.

Case 4. Suppose the initial hole is in $(2k+2, 1)$. Jump $(2k+1, 2) \cdot (2k+1, 1) \cdot (2k+2, 1)$ and $(2k, 1) \cdot (2k, 2) \cdot (2k+1, 2)$. The $P_{2k+1} \square P_3$ subgraph is solvable by Lemma 6, ending in $(2k-1, 0)$. This reduces the graph to Case 2. Suppose the initial hole is in $(2k+2, 2)$. Jump $(2k+1, 1) \cdot (2k+1, 2) \cdot (2k+2, 2)$ and $(2k, 1) \cdot (2k, 2) \cdot (2k+1, 2)$. This reduces the graph to Case 2. The case where the initial hole is in $(2k+2, 0)$ is analogous. $\square$

Theorem 8. *The mesh $P_{2k+1} \square P_{2n+1}$ is freely solvable.*

Proof. We prove the stronger statement that $P_{2k+1} \square P_{2n+1}$ is freely solvable with the final peg in vertex $(2k-1, 2n)$, which is symmetric to vertex $(2k-1, 0)$. Proceed by induction on n .

If $n = 1$, then the claim holds by Lemma 7. Assume the claim holds for $P_{2k+1} \square P_{2n+1}$, and consider $P_{2k+1} \square P_{2n+3}$.

Case 1. Suppose the initial hole is in (i, j) , where $j \leq 2n$. The subgraph $P_{2k+1} \square P_{2n+1}$ is solvable by hypothesis, ending in $(2k-1, 2n)$. Now jump $(2k-1, 2n+1) \cdot (2k-2, 2n+1) \cdot (2k-2, 2n)$ and $(2k-1, 2n) \cdot (2k-2, 2n) \cdot (2k-2, 2n+1)$. The remaining pegs form an even path with a hole in $(2k-1, 2n+1)$. Solve this path with the final peg in $(2k-1, 2n+2)$.

Case 2. Suppose the initial hole is in $(i, 2n+2)$. Jump $(i, 2n) \cdot (i, 2n+1) \cdot (i, 2n+2)$. A hole is now in $(i, 2n)$. Solve the $P_{2k+1} \square P_{2n+1}$ subgraph by hypothesis, ending in $(2k-1, 2n)$.

If $i = 2k-1$, then jump $(2k-3, 2n+1) \cdot (2k-2, 2n+1) \cdot (2k-2, 2n)$, $(2k-2, 2n) \cdot (2k-1, 2n) \cdot (2k-1, 2n+1)$, and $(2k, 2n+1) \cdot (2k-1, 2n+1) \cdot (2k-2, 2n+1)$. The remaining pegs form an even path with a hole in $(2k-3, 2n+1)$. Solve this path with the final peg in $(2k-1, 2n+2)$.

If $i = 2k-2$, then jump $(2k-1, 2n) \cdot (2k-1, 2n+1) \cdot (2k-2, 2n+1)$. If $i = 2k$, then jump $(2k-1, 2n) \cdot (2k-1, 2n+1) \cdot (2k, 2n+1)$. In either case, this reduces the graph to Case 1.

If $i \notin \{2k-1, 2k-2\}$, then jump $(2k-1, 2n+1) \cdot (2k, 2n+1) \cdot (2k, 2n)$ and $(2k, 2n) \cdot (2k-1, 2n) \cdot (2k-2, 2n)$.

If $i = 0$, then, for $j = 1, \dots, 2k-3$, jump $(2k-j-1, 2n+1) \cdot (2k-j-1, 2n) \cdot (2k-j-2, 2n)$. Now jump $(1, 2n+2) \cdot (0, 2n+2) \cdot (0, 2n+1)$, $(0, 2n+1) \cdot (1, 2n+1) \cdot (2, 2n+1)$, $(2, 2n+1) \cdot (2, 2n+1) \cdot (1, 2n+1)$, and $(1, 2n) \cdot (1, 2n+1) \cdot (1, 2n+2)$. The remaining pegs form an even path with a hole in $(2, 2n+2)$. Solve this path with the final peg in $(2k-1, 2n+2)$.

If $i \neq 0$, then, for $j = 1, \dots, 2k-i-2$, jump $(2k-j-1, 2n+1) \cdot (2k-j-1, 2n) \cdot (2k-j-2, 2n)$. If i is odd, then the remaining pegs form an even path with a hole in $(i, 2n+1)$. Solve this path with the final peg in $(2k-1, 2n+2)$. If i is even, then $(0, 2n+1)$, $(1, 2n+1)$, $\dots$, $(i, 2n+1)$, $(i, 2n)$ form an even path with a hole in $(i, 2n+1)$. Solve this path, ending in $(1, 2n+1)$. Now jump $(1, 2n+2) \cdot (0, 2n+2) \cdot (0, 2n+1)$, $(0, 2n+1) \cdot (1, 2n+1) \cdot (2, 2n+1)$, and $(2, 2n+1) \cdot (2, 2n+1) \cdot (1, 2n+2)$. The remaining pegs form an even path with a hole in $(2, 2n+2)$. Solve this path with the final peg in $(2k-1, 2n+2)$.

Case 3. Suppose the initial hole is in $(i, 2n+1)$. Jump $(i+1, 2n) \cdot (i+1, 2n+1) \cdot (i, 2n+1)$ or $(i-1, 2n) \cdot (i-1, 2n+1) \cdot (i, 2n+1)$. In either case, this reduces the graph to Case 2. $\square$

It follows from Theorem 8 that the cylinder $C_n \square P_t$ is freely solvable. The *generalized web* is obtained from $C_n \square P_t$ by appending a pendant vertex z_i to $(i, t-1)$ for $i = 0, \dots, n-1$. Similarly, the *generalized wheel* is obtained from $C_n \square P_t$ by adding a new vertex c such that c is adjacent to $(i, 0)$ for $i = 0, \dots, t-1$. Finally, the *generalized helm* is obtained from the generalized wheel by appending a pendant vertex z_i to $(i, t-1)$ for $i = 0, \dots, n-1$.

Corollary 9. *The generalized web, generalized wheel, and the generalized helm are freely solvable for $n \geq 3$ and $t \geq 2$.*

Proof. We first show that a mesh of even order may be solved with the final peg in an outermost vertex. For $P_{2k+1} \square P_{2n}$, suppose the initial hole is in vertex (i, j) . Without loss of generality, assume $j \leq 2n-2$. Solve the $P_{2k+1} \square P_{2n-1}$ subgraph ending in $(2k-1, 2n-2)$. Jump $(2k-1, 2n-1) \cdot (2k, 2n-1) \cdot (2k, 2n-2)$, $(2k, 2n-2) \cdot (2k-1, 2n-2) \cdot (2k-2, 2n-2)$, and $(2k-2, 2n-2) \cdot (2k-2, 2n-1) \cdot (2k-1, 2n-1)$. The remaining pegs form an even path with a hole in $(2k-2, 2n-1)$. Solve this path with the final peg in $(1, 2n-1)$.

For $P_{2k} \square P_{2n}$, suppose the initial hole is in vertex (i, j) . Without loss of generality, assume that $i \leq 2k-2$. Solve the $P_{2k-1} \square P_{2n}$ subgraph, ending in $(2k-2, 2n-2)$. The remaining pegs form an even path with a hole in $(2k-3, 2n-1)$. Solve this path, ending in $(1, 2n-1)$.

For the generalized web, if the hole is in (i, j) , then solve the mesh subgraph with the final peg in $(0, t-1)$. Now, jump $z_k \cdot (k, t-1) \cdot (k+1, t-1)$ for $k = 0, \dots, n-1$. If the hole is in z_0 , then jump $(0, t-2) \cdot (0, t-1) \cdot z_0$. Next, jump $z_i \cdot (i, t-1) \cdot (i-1, t-1)$ for $i = 1, \dots, n-1$. Finally, jump $z_0 \cdot (0, t-1) \cdot (0, t-2)$. Ignoring the pendants, the graph is a mesh after the first jump.

For the generalized wheel, if the hole is in (i, j) , then solve the mesh subgraph with the final peg in $(0, 0)$. Now jump $c \cdot (0, 0) \cdot (1, 0)$. If the hole is in c , then jump $(0, 0) \cdot (1, 0) \cdot c$. This reduces the graph to the case of the mesh after the first jump.

For the generalized helm, if the hole is in (i, j) or z_i , then solve the generalized web subgraph with the final peg in $(0, 0)$. Now jump $c \cdot (0, 0) \cdot (1, 0)$. If the initial hole is in c , then solve the wheel subgraph, ending in $(0, t-1)$. Now, jump $z_k \cdot (k, t-1) \cdot (k+1, t-1)$ for $k = 0, \dots, n-1$. $\square$

4. The Fat Triangle

We now construct the *fat polygon*, denoted by $\text{Fp}(n; a_0, a_1, \dots, a_{n-1})$. Label the vertices of C_n with $x_0, x_1, \dots, x_{n-1}$. Replace the edge between x_i and $x_{i+1(\text{mod } n)}$ with a_i edges, where $a_i \geq 1$. For $j = 1, \dots, a_i$, place a new vertex, $y_{i,j}$, on the j th edge between x_i and $x_{i+1(\text{mod } n)}$. So $x_i y_{i,j}$ and $y_{i,j} x_{i+1(\text{mod } n)} \in E(\text{Fp}(n; a_0, a_1, \dots, a_{n-1}))$. Further, let $A_i = \{y_{i,j} : j = 1, \dots, a_i\}$. In this section, we show that $\text{Fp}(3; a_0, a_1, a_2)$ (i.e., the *fat triangle*) is freely solvable (see Figure 5). Without loss of generality, we assume $a_0 \geq a_1 \geq a_2 \geq 1$.

Theorem 10. *The fat triangle is freely solvable for all $a_0, a_1, a_2 \geq 1$.*

Proof. For convenience of exposition, we define and solve the following configurations.

Configuration A. Suppose there are pegs in vertices $x_0, y_{0,1}, y_{0,2}, \dots, y_{0,i}$ and a hole in x_1 . If $i = 1$, then jump $x_0 \cdot y_{0,1} \cdot x_1$, ending in x_1 . If $i \geq 2$ is even, say $i = 2t$, then, for $t = 1, \dots, t$, jump $x_0 \cdot y_{0,2t-1} \cdot x_1$ and $x_1 \cdot y_{0,2t} \cdot x_0$, ending in x_0 . If $i \geq 3$ is odd, then make the additional jump $x_0 \cdot y_{0,i} \cdot x_1$, ending in x_1 .

FIGURE 5: Fat triangle $Fp(3; 4, 3, 2)$.

Configuration B. Suppose there are pegs in vertices $x_0, y_{0,1}, y_{0,2}, \dots, y_{0,i}$ and $y_{1,1}, y_{1,2}, \dots, y_{1,i}$ and holes in x_1 and x_2 . If $i = 1$, then jump $x_0 \cdot y_{0,1} \cdot x_1$ and $x_1 \cdot y_{1,1} \cdot x_2$, ending in x_2 . If $i \geq 2$ is even, say $i = 2t$, then, for $t = 1, \dots, t$, jump $x_0 \cdot y_{0,2t-1} \cdot x_1$, $x_1 \cdot y_{1,2t-1} \cdot x_2$, $x_2 \cdot y_{1,2t} \cdot x_1$, and $x_1 \cdot y_{0,2t} \cdot x_0$, ending in x_0 . If $i \geq 3$ is odd, then perform the additional jumps $x_0 \cdot y_{0,i} \cdot x_1$ and $x_1 \cdot y_{1,i} \cdot x_2$, ending in x_2 .

Case 1. If $a_0 = a_1 = a_2 = 1$, then the graph is C_6 , which is freely solvable by [3]. If $a_0 = a_1 = a_2 \geq 2$, then it suffices to show solutions with the initial hole in vertices x_0 and $y_{1,1}$.

Suppose the initial hole is in x_0 . Jump $x_1 \cdot y_{0,1} \cdot x_0, y_{2,1} \cdot \vec{x}_0 \cdot y_{0,1}, x_2 \cdot y_{1,1} \cdot x_1$, and $x_1 \cdot y_{0,1} \cdot x_0$. Now jump $x_0 \cdot y_{0,i} \cdot x_1, x_1 \cdot y_{1,i} \cdot x_2$, and $x_2 \cdot y_{2,i} \cdot x_0$, for $i = 2, \dots, a_0$, to eliminate the $a_0 - 1$ remaining pegs in A_0, A_1 , and A_2 . This solves the graph with the final peg in x_0 . The final jump could also be $y_{2,a_1} \cdot \vec{x}_2 \cdot y_{1,1}$. Thus, by Theorem 1, the graph can also be solved with the initial hole in $y_{1,1}$ and the final peg in x_0 .

Case 2. If $a_0 > a_1 = a_2$, then it suffices to show solutions with the initial hole in vertices $x_0, y_{0,1}, x_2$, and $y_{2,1}$.

Suppose the initial hole is in x_0 . Solve the $Fp(3; a_1, a_1, a_1)$ subgraph, ending in x_0 , as in Case 1. Now use the method described in Configuration A to eliminate the $a_0 - a_1$ remaining pegs in A_0 . If $a_0 - a_1$ is even, then the final peg is in x_0 . However, the final jump could also be $y_{0,a_0} \cdot \vec{x}_1 \cdot y_{0,1}$. If $a_0 - a_1$ is odd, then the final peg is in x_1 . However the final jump could also be $y_{0,a_0} \cdot \vec{x}_0 \cdot y_{0,1}$. In either case, the graph can be solved with the initial hole in $y_{0,1}$ by Theorem 1.

Suppose the initial hole is in x_2 . Jump $x_1 \cdot y_{1,1} \cdot x_2, x_0 \cdot y_{0,a_0} \cdot x_1, x_2 \cdot y_{2,1} \cdot x_0$, and $y_{0,1} \cdot \vec{x}_1 \cdot y_{0,a_0}$. Use the method of Configuration A to eliminate the remaining $a_0 - 1$ pegs in A_0 . If $a_1 = 1$, then this solves the graph. If $a_0 - 1$ is even, then the final peg is in x_0 . If $a_0 - 1$ is odd, then the final peg is in x_1 . If $a_1 \geq 2$, then use the method described in Configuration B to eliminate the remaining $a_1 - 1$ pegs in A_1 and A_2 . If $a_0 - 1$ and $a_1 - 1$ have the same parity, then the final peg is in x_0 . Otherwise, the final peg is in x_1 .

Suppose the initial hole is in $y_{2,1}$. Jump $y_{0,1} \cdot \vec{x}_0 \cdot y_{2,1}$. If $a_0 = a_1 + 1$, then this reduces the graph to Case 1. If $a_0 \geq a_1 + 2$, then this reduces the graph to the case of the initial hole in x_0 .

Case 3. If $a_0 = a_1 > a_2$, then it suffices to show solutions with the initial hole in vertices $x_0, y_{0,1}, y_{2,1}$, and x_1 .

Suppose the initial hole is in x_0 . Solve the $Fp(3; a_2, a_2, a_2)$ subgraph, ending in x_0 , as in Case 1. Now use the method described in Configuration B to eliminate the remaining $a_0 - a_2$ pegs in A_0 and A_1 . As in Case 2, the graph may also be solved with the initial hole in $y_{0,1}$.

Suppose the initial hole is in vertex $y_{2,1}$. First jump $y_{0,a_0} \cdot \vec{x}_0 \cdot y_{2,1}$. If $a_0 = 2$, then this reduces the graph to Case 1. Assume that $a_0 \geq 3$. Jump $x_1 \cdot y_{0,1} \cdot x_0, y_{0,2} \cdot \vec{x}_0 \cdot y_{0,a_0}, x_2 \cdot y_{1,1} \cdot x_1$, and $x_1 \cdot y_{1,2} \cdot x_2$. Now use the method described in Configuration A to eliminate the a_2 pegs in A_2 . If a_2 is even, then the ending peg is in x_2 . If a_2 is odd, then the ending peg is in x_0 . Next use the method described in Configuration B to eliminate the remaining $a_0 - 2$ pegs from A_0 and A_1 . If $a_0 - 2$ and a_2 have the same parity, then the final peg is in x_2 . Otherwise, the final peg is in x_0 .

Suppose the initial hole is in x_1 . Jump $x_0 \cdot y_{0,1} \cdot x_1, x_2 \cdot y_{2,1} \cdot x_0, x_1 \cdot y_{1,1} \cdot x_2$, and $y_{0,2} \cdot \vec{x}_0 \cdot y_{0,1}$. If necessary, use the method described in Configuration A to eliminate the remaining $a_2 - 1$ pegs in A_2 . If $a_2 - 1$ is even, then the ending peg is in x_2 . If $a_2 - 1$ is odd, then the ending peg is in x_0 . Next use the method described in Configuration B to eliminate the remaining $a_0 - 1$ pegs from A_0 and A_1 . If $a_2 - 1$ and $a_0 - 1$ have the same parity, then the final peg is in x_2 . Otherwise, the final peg is in x_0 .

Case 4. If $a_0 > a_1 > a_2 \geq 1$, then the graph may be reduced to a solvable configuration with the initial hole in each vertex.

- (i) Suppose the initial hole is in x_0 . Use the method described in Configuration A to eliminate $a_0 - a_1$ pegs from A_0 . If $a_0 - a_1$ is even, then the ending peg is in x_1 . If $a_0 - a_1$ is odd, then the ending peg is in x_0 . This reduces the graph to Case 3. The case where the initial hole is in x_1 is analogous.
- (ii) Suppose the initial hole is in x_2 . Use the method described in Configuration A to eliminate $a_1 - a_2$ pegs from A_1 . This reduces the graph to Case 2.
- (iii) Suppose the initial hole is in $y_{0,1}$. Jump $y_{0,a_0} \cdot \vec{x}_0 \cdot y_{0,1}$. This reduces the graph to Case 3 or 4(i). The cases where the initial hole is in $y_{1,1}$ or $y_{2,1}$ are analogous. $\square$

5. Open Problems

We end this paper by listing a few open problems as possible avenues for future research. (i) What other graphs are freely solvable? (ii) Is the corona of a (freely) solvable graph likewise (freely) solvable? (iii) What is the solvability of a cycle with a chord that is subdivided multiple times? (iv) For what graphs is it possible to start with a specific initial jump and end with another specific jump? (v) Suppose that we want to start with our initial hole in any vertex s and end with our final peg in any vertex t . For what graphs is this possible? (vi) What is the solvability of other fat polygons?

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