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Research Article

Symmetric Tensor Rank and Scheme Rank: An Upper Bound in terms of Secant Varieties

E. Ballico

Department of Mathematics, University of Trento, Povo, 38123 Trento, Italy

Correspondence should be addressed to E. Ballico; ballico@science.unitn.it

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Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Let c be the minimal integer such that $\mathbb{P}^r$ is the c-secant variety of X, that is, the minimal integer c such that for a general $O \in \mathbb{P}^r$ there is $S \subset X$ with $\#(S) = c$ and $O \in \langle S \rangle$, where $\langle\ \rangle$ is the linear span. Here we prove that for every $P \in \mathbb{P}^r$ there is a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ and $\deg(Z) \le 2c$; we may take Z as union of points and tangent vectors of X_{reg}.

1. Introduction

There is a huge literature on the rank of tensors, on the symmetric tensor rank of symmetric tensors, and on the Waring decomposition of multivariate polynomials [1–14]. Most of the papers are over $\mathbb{C}$ (or over an algebraically closed field), but real tensors and real polynomials are also quite studied [6, 15]. In this paper we work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$ (e.g., $\mathbb{C}$), but for homogeneous polynomials we also work over $\mathbb{R}$ (see Corollary 3). Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Fix $P \in \mathbb{P}^r$. A *tangent vector of* X or a *tangent vector of* X_{reg} or a *smooth tangent vector of* X is a zero-dimensional connected subscheme of X whose support is a smooth point of X, that is, a point of X_{reg}, and with degree 2. Fix $O \in X_{\mathrm{reg}}$ and let m be the dimension of X at O. The set of all smooth tangent vectors of X with O as its support is parametrized by a projective space of dimension $m-1$. If $\mathbb{K} = \mathbb{C}$, X is defined over $\mathbb{R}$ and $O \in X_{\mathrm{reg}}(\mathbb{R})$, a smooth tangent vector $Z \subset X$ with $Z_{\mathrm{red}} = \{O\}$ is said to be *real* if it is defined over $\mathbb{R}$. A zero-dimensional scheme $Z \subset X$ is said to be *curvilinear* if for each connected component W of Z either W is a point of X or there is $W_{\mathrm{red}} \in X_{\mathrm{reg}}$ and W is contained in a smooth curve contained in an open neighborhood of W_{red} in X. A zero-dimensional scheme $Z \subset X$ is said to be *smoothable* if it is a flat limit of a flat family of finite subsets of X (a curvilinear scheme is smoothable). Fix $P \in \mathbb{P}^r$. The X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denote the linear span. The scheme X-rank (or X-cactus rank) $z_X(P)$ of P is the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ [16, Definition 5.1, page 135, Definition 5.66, page 198, 31, 12, 10, 11, 17, 18, 8, 9]. If we impose that Z is smoothable (curvilinear, resp.), then we get the smoothable X-rank $z'_X(P)$ (curvilinear X-rank $z''_X(P)$, resp.) of [17, 18] for wonderful uses of the scheme X-rank. Let $w_X(P)$ be the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ and each connected component of Z is either a point of X or a smooth tangent vector of X (any such Z is curvilinear). We have

$$z_X(P) \le z'_X(P) \le z''_X(P) \le w_X(P). \tag{1}$$

Hence to get an upper bound for the integer $z_X(P)$, it is sufficient to find an upper bound for the integer $w_X(P)$. We first state our upper bound in the case of the Veronese varieties (this case corresponds to the decomposition of homogeneous polynomials as a sum of powers of linear forms).

For all positive integers m and d let $\nu_d : \mathbb{P}^m \to \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, denote the order d Veronese embedding of $\mathbb{P}^m$, that is, the embedding of $\mathbb{P}^m$ given by the $\mathbb{K}$-vector space of all degree d homogeneous polynomials in $m+1$ variables.

Theorem 1. *Fix integers $m \ge 2$ and $d \ge 3$. If $m \le 4$, then assume $d \ge 5$. Let $X_{m,d} \subset \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$, be the order d Veronese embedding of $\mathbb{P}^m$. Set $c := \lceil \binom{m+d}{m} / (m+1) \rceil$. Let*

$\Omega \subset X_{m,d}$ be any nonempty open subset of $X_{m,d}$. Then there is a disjoint union $Z \subset \Omega$ of c tangent vectors such that $P \in \langle Z \rangle$.

Corollary 2. *In the setup of Theorem 1, one has $w_{X_{m,d}}(P) \leq 2\lceil \binom{m+d}{m} / (m+1) \rceil$ for all $P \in \mathbb{P}^r$.*

Corollary 3. *Let $X_{m,d}(\mathbb{R}) \subset \mathbb{P}^r(\mathbb{R})$, $r := \binom{m+d}{m} - 1$, be the order d Veronese embedding of $\mathbb{P}^m(\mathbb{R})$. Fix $P \in \mathbb{P}^r(\mathbb{R})$ and a nonempty open subset $U \subseteq \mathbb{P}^m(\mathbb{R})$ for the euclidean topology. Then there is $S \subset U$ and for each $Q \in S$ a real tangent vector v_Q of $\mathbb{P}^m(\mathbb{R})$ such that $\#(S) \leq \lceil \binom{m+d}{m} / (m+1) \rceil$ and $P \in \langle \cup_{Q \in S} \nu_d(v_Q) \rangle$.*

Theorem 1 is just a particular case of a general bound on $w_X(P)$ (see Theorem 4). We want to point out two features of these results.

(i) The use of an arbitrary nonempty open subset Ω (U, resp.) of $X_{m,d}$ ($\mathbb{P}^m(\mathbb{R})$, resp.). This is not just to get a formally stronger statement. In many cases, the inductive proofs require the existence of sets (or schemes) bounding $r_X(P)$ or $z_X(P)$ and with supports away from some bad varieties [10, 11, 14, 19]. For instance, in [19] Jelisiejew takes as Ω the image by the Veronese embedding ν_d of the complement of finitely many hyperplanes; he calls it the "open rank."

(ii) We use very particular curvilinear schemes, just disjoint unions of tangent vectors. One should find algorithms to find the support and the direction of tangent vectors needed to compute a good upper bound for the integer $w_X(P)$.

Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Set $m := \dim(X)$. For each integer $b > 0$ the b-secant variety $\sigma_b(X) \subseteq \mathbb{P}^r$ of X is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle S \rangle$, where $S \subset X$ is a subset with cardinality b. The set $\sigma_b(X)$ is an integral variety of dimension at most $\min\{r, (m+1)b - 1\}$. In many important cases, the integer $\dim(\sigma_b(X))$ is known and either $\dim(\sigma_b(X)) = \min\{r, (m+1)b - 1\}$ or $\dim(\sigma_b(X)) - \min\{r, (m+1)b - 1\}$ is very small [20, 21]. Hence it is usually easy to find an integer c with $c - \lceil (r+1)/(m+1) \rceil$ small such that $\sigma_c(X) = \mathbb{P}^r$.

Theorem 4. *Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Let c be the first positive integer such that $\sigma_c(X) = \mathbb{P}^r$. Fix any nonempty open subset U of X and any $P \in \mathbb{P}^r$. Then one has the following.*

(a) *There is a disjoint union $Z \subset U$ of c smooth tangent vectors such that $P \in \langle Z \rangle$;*

(b) $w_X(P) \leq 2c$.

Remark 5. Take X and c as in Theorem 4. We have $c = r_X(O)$ with O a general element of $\mathbb{P}^r$. Hence the scheme X-rank of the worst point of $\mathbb{P}^r$ is at most twice the rank of almost all points of $\mathbb{P}^r$.

2. The Proofs

Proof of Theorem 4. Fix a general $S \subset U \cap X_{\mathrm{reg}}$ such that $\#(S) = c$. For each $O \in X_{\mathrm{reg}}$ let $T_O X \subset \mathbb{P}^r$ denote the Zariski tangent space of X at O. Since $\sigma_c(X) = \mathbb{P}^r$, Terracini's lemma gives $\mathbb{P}^r = \langle \cup_{O \in S} T_O X \rangle$ [20, Corollary 1.11]. Hence for each $O \in S$, there is $P_O \in T_O X$ such that $P \in \langle \cup_{O \in S} P_O \rangle$. Fix $O \in S$. If $P_O = O$, then let v_O be any tangent vector of X at O. Now assume that $P_O \neq O$. Since the line $L := \langle \{P_O, O\} \rangle$ is contained in $T_O X$, the scheme $L \cap X$ contains the tangent vector v_O of L at O. Set $Z := \cup_{O \in S} v_O$. Since $P_O \in \langle v_O \rangle$ for all $O \in S$, we have $P \in \langle Z \rangle$. □

Proof of Theorem 1. Since either $m \geq 5$ and $d \geq 3$ or $d \geq 5$, a theorem of Alexander and Hirschowitz says that c is the first positive integer t such that $\sigma_t(X_{m,d}) = \mathbb{P}^r$ [22–25]. Apply Theorem 4. □

Proof of Corollary 2. This is a consequence of Theorem 1. □

Proof of Corollary 3. For any subset A of $\mathbb{P}^r(\mathbb{R})$ we write $\langle A \rangle$ for its linear span over $\mathbb{C}$ and $\langle A \rangle_{\mathbb{R}}$ for its linear span over $\mathbb{R}$. For each $Q \in X_{m,d}(\mathbb{R})$, we write $T_Q X_{m,d}(\mathbb{C})$ for the Zariski tangent space over $\mathbb{C}$ and write $T_Q X_{m,d}(\mathbb{R})$ for the real tangent space (hence both projective spaces have dimension m, the first one over $\mathbb{C}$, the second one over $\mathbb{R}$). We have $T_Q X_{m,d}(\mathbb{C}) \cap \mathbb{P}^r(\mathbb{R}) = T_Q X_{m,d}(\mathbb{R})$. The set $\nu_d(U)$ is Zariski dense in $X_{m,d}(\mathbb{C})$. Hence Terracini's lemma gives the existence of $S \subset U$ such that $\#(S) = a$ and $\langle \cup_{Q \in \nu_d(S)} T_Q X_{m,d}(\mathbb{C}) \rangle = \mathbb{P}^r(\mathbb{C})$. Since $\nu_d(S) \subset X_{m,d}(\mathbb{R})$ and $P \in \mathbb{P}^r(\mathbb{R})$, we get $P \in \langle \cup_{O \in \nu_d(S)} T_O X_{m,d}(\mathbb{R}) \rangle_{\mathbb{R}}$. Hence for each $O \in \nu_d(S)$ there is $P_O \in T_O X_{m,d}(\mathbb{R})$ such that $P \in \langle \cup_{O \in \nu_d(S)} P_O \rangle_{\mathbb{R}}$. Continue as in the proof of Theorem 4. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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