Research Article

A New Subclass of Meromorphic Close-to-Convex Functions

Amit Soni¹ and Shashi Kant²

¹ Department of Mathematics, Government Engineering College, Bikaner 334001, India

² Department of Mathematics, Government Dungar College, Bikaner 334001, India

Correspondence should be addressed to Amit Soni, aamittsonni@yahoo.com

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A new subclass $MK(t, A, B)$ of meromorphic close-to-convex functions, defined by means of subordination, is investigated. Some results such as inclusion relationship, coefficient inequality, convolution property, and distortion property for this class are derived. The results obtained here are extension of earlier known work.

1. Introduction

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For any two analytic function f and g in $\mathbb{U} = \mathbb{D} \setminus \{0\}$, we say that f is subordinate to g in $\mathbb{U}$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ such that $f(z) = g(w(z))$, for $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then f is subordinate to g , if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. If $f \in \Sigma$ is given by (1) and $g \in \Sigma$ is given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad (2)$$

then the Hadamard product (or convolution) of the function f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g * f)(z). \quad (3)$$

A function $f \in \Sigma$ is said to be in the class $MS^*(\alpha)$ of meromorphic starlike functions of order α if it satisfies the inequality

$$R\left(\frac{zf'(z)}{f(z)}\right) < -\alpha, \quad (z \in \mathbb{U}, 0 \leq \alpha < 1). \quad (4)$$

Moreover, a function $f \in \Sigma$ is said to be in the class MC of meromorphic close to convex function if it satisfies the condition

$$R\left(\frac{zf'(z)}{g(z)}\right) < 0 \quad (z \in \mathbb{U}, g \in MS^*(0) = MS^*). \quad (5)$$

Recently, Wang et al. [1] introduced and discussed the class MK of meromorphic functions $f(z) \in \Sigma$ which satisfies the inequality

$$R\left(\frac{f'(z)}{g(z)g(-z)}\right) > 0 \quad (z \in \mathbb{U}), \quad (6)$$

where $g(z) \in MS^*(1/2)$.

The class MK is very close to the interesting subclass K_s of close-to-convex function for analytic function introduced and studied by Gao and Zhou [2]. Many classes related to the class K_s have been further studied by some authors. Especially, Wang et al. [3, 4], Kowalczyk and Leś-Bomba [5], Xu et al. [6], Şeker [7], and Cho et al. [8] introduced generalization of K_s and obtained some properties for analytic functions in each class.

More recently, by means of subordination, Sim and Kwon [9] discussed a subclass $\Sigma(A, B)$ of the class MK . A function $f(z) \in \Sigma$ is said to be in the class $\Sigma(A, B)$ if it satisfies the following subordination relation:

$$\frac{f'(z)}{g(z)g(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad (7)$$

where $-1 \leq B < A \leq 1$ and $g(z) \in MS^*(1/2)$. Here, the assumption that $g(z)$ is meromorphic starlike function of order $1/2$ makes the function $-zg(z)g(-z)$ meromorphic starlike. So, instead of $-zg(z)g(-z)$ in (6) and (7), we can consider $tg(z)g(tz)$, $0 < |t| \leq 1$, because if $g(z) \in MS^*(1/2)$, then $tg(z)g(tz)$ is also a meromorphic starlike function, which motivates us to define a new subclass $MK(t, A, B)$ of meromorphic close-to-convex functions as follows.

Definition 1. A function $f \in \Sigma$ is said to be in the class $MK(t, A, B)$ if there exists $g \in MS^*(1/2)$, such that

$$\frac{-f'(z)}{tg(z)g(tz)} \prec \frac{1 + Az}{1 + Bz}, \quad (8)$$

$$(z \in \mathbb{U}, 0 < |t| \leq 1, -1 \leq B < A \leq 1).$$

The class $MK(t, A, B)$ provides a generalization of the classes given by Sim and Kwon [9] and Wang et al. [1]. The transformation $(1 + Az)/(1 + Bz)$ involved in the class $MK(t, A, B)$ is analytic and convex univalent in $\mathbb{U}$. Moreover,

$$\begin{aligned} 0 \leq \frac{1 - A}{1 - B} &< R\left(\frac{1 + Az}{1 + Bz}\right) \\ &< \frac{1 + A}{1 + B}, \quad (-1 < B < A \leq 1; z \in \mathbb{U}). \end{aligned} \quad (9)$$

In this paper, we aim at proving results such as inclusion relationship, coefficient bounds, and distortion theorem for the class $MK(t, A, B)$.

2. Properties of Meromorphic Starlike Functions

In beginning, we prove the following result of meromorphic starlike functions.

Theorem 2. Suppose that $\phi \in MS^*(\alpha_1)$ and $\psi \in MS^*(\alpha_2)$ with $0 \leq \alpha_1 + \alpha_2 - 1 < 1$. Then,

$$\begin{aligned} zt_1 t_2 \phi(t_1 z) \psi(t_2 z) &\in MS^*(\alpha_1 + \alpha_2 - 1), \\ \text{where } 0 < |t_1|, |t_2| &\leq 1. \end{aligned} \quad (10)$$

Proof. Let $\phi_i \in MS^*(\alpha_i)$ and $\psi \in MS^*(\alpha_2)$. By definition, we know that

$$\begin{aligned} R\left(\frac{t_1 z \phi'(t_1 z)}{\phi(t_1 z)}\right) &< -\alpha_1, \\ R\left(\frac{t_2 z \psi'(t_2 z)}{\psi(t_2 z)}\right) &< -\alpha_2. \end{aligned} \quad (11)$$

Next, we suppose that

$$h(z) = zt_1 t_2 \phi(t_1 z) \psi(t_2 z). \quad (12)$$

Then, we easily find that

$$\frac{zh'(z)}{h(z)} = 1 + \frac{t_1 z \phi'(t_1 z)}{\phi(t_1 z)} + \frac{t_2 z \psi'(t_2 z)}{\psi(t_2 z)}. \quad (13)$$

It follows that

$$R\left(\frac{zh'(z)}{h(z)}\right) < -(\alpha_1 + \alpha_2 - 1). \quad (14)$$

By noting that $0 \leq (\alpha_1 + \alpha_2 - 1) < 1$, which implies that $h(z) \in MS^*(\alpha_1 + \alpha_2 - 1)$, it completes the proof of Theorem 2. $\square$

For $\alpha_1 = \alpha_2 = 1/2$ and $t_1 = 1$, Theorem 2 gives the following corollary.

Corollary 3. If $g(z) \in MS^*(1/2)$ and $0 < |t| \leq 1$, then $tzg(z)g(tz) \in MS^*$.

Lemma 4 (see [10]). Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then,

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}. \quad (15)$$

Theorem 5. Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then,

$$MK(t, A_1, B_1) \subset MK(t, A_2, B_2). \quad (16)$$

Proof. Suppose that $f \in MK(t, A_1, B_1)$. Then,

$$\frac{-f'(z)}{tg(z)g(tz)} \prec \frac{1 + A_1 z}{1 + B_1 z}. \quad (17)$$

Since $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, by Lemma 4, we have

$$\frac{-f'(z)}{tg(z)g(tz)} \prec \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}. \quad (18)$$

It implies that $f(z) \in MK(t, A_2, B_2)$, which completes the proof. $\square$

3. Coefficient Estimates

In this section, we obtain the coefficient estimates of functions belonging to the class $MK(t, A, B)$. For this, we require the following Lemmas.

Lemma 6 (see [11]). Let

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \quad (19)$$

be analytic in $\mathbb{U}$ and

$$k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n \quad (20)$$

be analytic and convex in $\mathbb{U}$. If $h(z) \prec k(z)$, then

$$|h_n| \leq |k_n|, \quad (n \in \mathbb{N}). \quad (21)$$

Lemma 7. Let $g(z) \in MS^*(1/2)$ and be given by (2). Then,

$$\begin{aligned} & |b_n t^{n+1} + b_1 b_{n-2} t^{n-1} + b_2 b_{n-3} t^{n-2} \\ & + \cdots + b_{n-3} b_2 t^3 + b_1 b_{n-2} t^2 + b_n| \\ & \leq \frac{2}{n+1}, \quad (0 < |t| \leq 1). \end{aligned} \quad (22)$$

Proof. According to Corollary 3, we have $tzg(z)g(tz) \in MS^*$, and if

$$G(z) = tzg(z)g(tz) = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n, \quad (23)$$

then it is well known that

$$|c_n| \leq \frac{2}{n+1}. \quad (24)$$

Substituting the series expressions of $G(z)$ and $g(z)$ in (23) and comparing the coefficients of both sides of this equation, we get

$$\begin{aligned} c_n &= b_n t^{n+1} + b_1 b_{n-2} t^{n-1} + b_2 b_{n-3} t^{n-2} \\ &+ \cdots + b_{n-3} b_2 t^3 + b_1 b_{n-2} t^2 + b_n. \end{aligned} \quad (25)$$

On substituting the value from (25) in (24), we get the result. $\square$

Theorem 8. Let $-1 \leq B < A \leq 1$. If

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MK(t, A, B), \quad (26)$$

then

$$\begin{aligned} & |a_1| \leq 1, \\ & |a_n| \leq \frac{A-B}{n} \left(1 + \sum_{k=2}^n \frac{2}{k} \right) + \frac{2}{n(n+1)} \quad (n \in N \setminus \{1\}). \end{aligned} \quad (27)$$

Proof. Suppose that $f \in MK(t, A, B)$. Then, we know that

$$\frac{-zf'(z)}{G(z)} < \frac{1+Az}{1+Bz}, \quad (28)$$

where $G(z) = tzg(z)g(tz)$.

If we set

$$q(z) = \frac{-zf'(z)}{G(z)}, \quad (29)$$

it follows that

$$q(z) = 1 + \sum_{n=1}^{\infty} d_n z^n. \quad (30)$$

In view of Lemma 6, we know that

$$|d_n| \leq A - B \quad (n \in N). \quad (31)$$

By substituting the series expression of functions f , G , and q in (29), we obtain

$$\begin{aligned} & (1 + d_1 z + d_2 z^2 + \cdots + d_n z^n + \cdots) \\ & \times \left(\frac{1}{z} + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \right) \\ & = \frac{1}{z} - a_1 z - 2a_2 z^2 - \cdots - na_n z^n - \cdots. \end{aligned} \quad (32)$$

Since f is univalent in $\mathbb{D}$, it is well known that

$$|a_1| \leq 1. \quad (33)$$

On comparing the coefficient of z^n in both sides of (32), we have

$$\begin{aligned} -na_n &= c_n + d_1 c_{n-1} + d_2 c_{n-2} \\ &+ \cdots + d_{n-1} c_1 + d_{n+1} \quad (n \in N). \end{aligned} \quad (34)$$

Substituting the values from (24) and (31) to (34), we have

$$\begin{aligned} n|a_n| &\leq (A-B) \left(1 + \frac{2}{2} + \frac{2}{3} + \cdots + \frac{2}{n-1} + \frac{2}{n} \right) \\ &+ \frac{2}{n+1}, \end{aligned} \quad (35)$$

$$|a_n| \leq \frac{(A-B)}{n} \left(1 + \sum_{k=2}^n \frac{2}{k} \right) + \frac{2}{n(n+1)}. \quad (36)$$

This evidently completes the proof of Theorem 8. $\square$

Theorem 9. Let $-1 \leq B < A \leq 1$. If

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in MS^* \left(\frac{1}{2} \right) \quad (37)$$

and $f \in \Sigma$ given by (1), satisfying the condition

$$(1 + |B|) \sum_{n=1}^{\infty} n|a_n| + (1 + |A|) \sum_{n=1}^{\infty} |c_n| \leq (A - B), \quad (38)$$

where c_n is given by (25), then $f \in MK(t, A, B)$.

Proof. To prove $f \in MK(t, A, B)$, it suffices to show that

$$\left| \frac{zf'(z)/G(z) + 1}{A + B(zf'(z)/G(z))} \right| < 1, \quad (39)$$

where $G(z)$ is given by (23). From (38), we know that

$$\begin{aligned} & (A - B) - |A| \sum_{n=1}^{\infty} |c_n| - |B| \sum_{n=1}^{\infty} n|a_n| \\ & \geq \sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} |c_n|. \end{aligned} \quad (40)$$

Now, by maximum modulus principle and (40), we deduce that

$$\begin{aligned} & \left| \frac{zf'(z)/G(z) + 1}{A + B(zf'(z)/G(z))} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} na_n z^{n+1} + \sum_{n=1}^{\infty} c_n z^{n+1}}{(A-B) + A \sum_{n=1}^{\infty} c_n z^{n+1} + B \sum_{n=1}^{\infty} na_n z^{n+1}} \right| \quad (41) \\ &\leq \frac{\sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} |c_n|}{|A-B| - |A| \sum_{n=1}^{\infty} |c_n| - |B| \sum_{n=1}^{\infty} n|a_n|} \leq 1. \end{aligned}$$

This evidently completes the proof of Theorem 9. $\square$

Theorem 10. Let $-1 \leq B < A \leq 1$. If $f(z) \in MK(t, A, B)$, then

$$\begin{aligned} & f(z) * \left((1 + Be^{i\theta}) \frac{(2z-1)}{z(1-z)^2} \right) \\ &+ G(z) * \left((1 + Ae^{i\theta}) \frac{(z^2 - z + 1)}{z(1-z)} \right) \neq 0, \quad (42) \\ & (z \in \mathbb{D}, 0 < \theta < 2\pi). \end{aligned}$$

Proof. Suppose that $f \in MK(t, A, B)$. Then, we know that

$$\frac{-zf'(z)}{G(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{D}, 0 < \theta < 2\pi), \quad (43)$$

where G is given by (23). It is easy to see that the condition (43) can be written as

$$\begin{aligned} & zf'(z)(1 + Be^{i\theta}) + G(z)(1 + Ae^{i\theta}) \neq 0, \quad (44) \\ & (z \in \mathbb{D}, 0 < \theta < 2\pi). \end{aligned}$$

Note that

$$\begin{aligned} & zf'(z) = f(z) * \frac{2z-1}{z(1-z)^2}, \\ & G(z) = G(z) * \frac{z^2 - z + 1}{z(1-z)}. \end{aligned} \quad (45)$$

By substituting (45) into (44), we get the desired assertion (42) of Theorem 10. $\square$

4. Distortion Theorem

Lemma 11 (see [12]). Let $\phi \in MS^*$. Then,

$$\frac{(1-r)^2}{r} \leq |\phi(z)| \leq \frac{(1+r)^2}{r} \quad (|z| = r, 0 < r < 1). \quad (46)$$

Theorem 12. Let $-1 \leq B < A \leq 1$. If $f(z) \in MK(t, A, B)$, then

$$\begin{aligned} & \frac{(1-Ar)(1-r)^2}{(1-Br)r^2} \leq |f'(z)| \leq \frac{(1+Ar)(1+r)^2}{(1+Br)r^2}, \quad (47) \\ & (|z| = r; 0 < r < 1). \end{aligned}$$

Proof. Suppose that $f(z) \in MK(t, A, B)$. By definition, we have the transformation

$$\frac{-zf'(z)}{G(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in \mathbb{U}, \quad (48)$$

which maps $|w(z)| \leq r$ in the closed disk

$$\left| \frac{-zf'(z)}{G(z)} - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{(A-B)r}{1 - B^2 r^2}, \quad |z| = r. \quad (49)$$

This implies that

$$\frac{1 - Ar}{1 - Br} \leq \left| \frac{-zf'(z)}{G(z)} \right| \leq \frac{1 + Ar}{1 + Br}. \quad (50)$$

Thus, by virtue of Lemma 11, we readily get the desired result. $\square$

Remark 13. Taking $A = 1, B = -1$, and $t = -1$, we obtain the results such as inclusion relation, coefficient inequality, convolution property, and distortion theorem derived by Wang et al. [1].

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