Research Article

Fekete-Szegő Coefficient Functional for Certain Subclasses of Close-to-Star Functions

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We introduce some subclasses of close-to-star functions defined by subordination and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$, μ real, for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $|z| < 1$, belonging to these sub-classes.

1. Introduction

Let Ω be the class of bounded functions of the following form:

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, \quad (1)$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$ and satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$.

It is known (see [1]) that

$$|d_1| \leq 1, \quad |d_2| \leq 1 - |d_1|^2. \quad (2)$$

Let A denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (3)$$

which are analytic in E . Let S be the class of functions of the form (3) which are analytic univalent in E .

We will concentrate on the coefficient problem for the subclasses of A . In 1916, Bieberbach [2] proved that $|a_2| \leq 2$ for $f(z) \in S$ as an elementary corollary area theorem. He conjectured that, for each function $f(z) \in S$, $|a_n| \leq n$; the equality holds for the Koebe function $k(z) = z/(1-z)^2$, which maps the unit disc E onto the entire complex plane minus the slit along the negative real axis from $-1/4$ to $-\infty$. De Branges [3] solved the Bieberbach conjecture in 1985. The

contribution of Löwner [4] in proving that $|a_3| \leq 3$ for the class S was huge.

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and a_2^2 for the class S . This thought prompted Fekete and Szegő [5] and they used Löwner's method to prove the following well-known result for the class S : if $f(z) \in S$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases} \quad (4)$$

The inequality (4) plays a very important role in determining estimates of higher coefficients for some subclasses of S (see Chichra [6], Babalola [7]).

Next, we define some subclasses of A and obtain the analogy of (4).

We denote by S^* the class of univalent star-like functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ which satisfy the following condition:

$$\Re \left(\frac{zg'(z)}{g(z)} \right) > 0, \quad z \in E. \quad (5)$$

We denote by K the class of convex univalent functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in A$ which satisfy the condition

$$\Re \left(\frac{(zh'(z))'}{h'(z)} \right) > 0, \quad z \in E. \quad (6)$$

Let f and g be two analytic functions in E . Then, f is said to be subordinate to g (symbolically $f < g$) if there exists a bounded function $w(z) \in \Omega$, such that $f(z) = g(w(z))$.

A function $f(z) \in A$ is said to be close to star if there exists a function $g(z) \in S^*$ such that

$$\Re \left(\frac{f(z)}{g(z)} \right) > 0, \quad z \in E. \quad (7)$$

The class of close to star functions was introduced by Reade [8] and is denoted by C^* . A close to star function need not be univalent. The immediate shoot of C^* is the following subclass:

$$C_1^* = \left\{ f(z) \in A : \Re \left(\frac{f(z)}{h(z)} \right) > 0, h(z) \in K, z \in E \right\}. \quad (8)$$

In this paper, we consider the following subclasses of close-to-star functions:

$$C^*(A, B) = \left\{ f(z) \in A : \frac{f(z)}{g(z)} < \frac{1 + Az}{1 + Bz}, \right. \\ \left. g(z) \in S^*, -1 \leq B < A \leq 1, z \in E \right\}. \quad (9)$$

The class $C^*(A, B)$ was introduced and studied by Mehrok et al. [9]. In particular, $C^*(1, -1) \equiv C^*$,

$$C_1^*(A, B) = \left\{ f(z) \in A : \frac{f(z)}{h(z)} < \frac{1 + Az}{1 + Bz}, \right. \\ \left. h(z) \in K, -1 \leq B < A \leq 1, z \in E \right\}, \quad (10)$$

is the class introduced and studied by Mehrok et al. [10]. Obviously, $C_1^*(1, -1) \equiv C_1^*$.

Hummel [11] proved the conjecture of V. Singh that $|c_3 - c_2^2| \leq 1/3$ for the class K . Keogh and Merkes [12] obtained the estimates (4) for the classes S^* and K . We obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for functions belonging to the classes $C^*(A, B)$ and $C_1^*(A, B)$.

2. Preliminary Lemmas

Lemma 1. Let $g(z) \in S^*$. Then,

$$|b_3 - \mu b_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases} \quad (11)$$

This lemma is due to Keogh and Merkes [12].

Lemma 2. Let $h(z) \in K$. Then,

$$|c_3 - \mu c_2^2| \leq \begin{cases} 1 - \mu & \text{if } \mu \leq \frac{2}{3}, \\ \frac{1}{3} & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}, \\ \mu - 1 & \text{if } \mu \geq \frac{4}{3}. \end{cases} \quad (12)$$

This result was proved by Keogh and Merkes [12].

3. Main Results

Theorem 3. Let $f(z) \in C^*(A, B)$. Then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} [3 + (A - B)(2 - B)] - [2 + (A - B)]^2 \mu & \text{if } \mu \leq \frac{-B}{2 + (A - B)} \quad (B < 0, A + B < 0) \\ [3 + (A - B) - 4\mu] + \frac{(A - B)(1 - 2\mu)^2}{(1 + B) + \mu(A - B)} & \text{or } \mu \leq \frac{-B}{(A - B)} \quad (B > 0), \\ 1 + (A - B) + \frac{(A - B)(2\mu - 1)^2}{(1 - B) - \mu(A - B)} & \text{if } \frac{-B}{2 + (A - B)} \leq \mu \leq \frac{1}{2} \\ (A - B)(4 + A - B)\mu - [(A - B)(2 - B) - 1] & \text{or } \frac{-B}{2 + (A - B)} \leq \mu \leq \frac{-B}{(A - B)} \quad (B < 0), \\ [2 + (A - B)]^2 \mu - [3 + (A - B)(2 - B)] & \text{if } \frac{1}{2} \leq \mu \leq \frac{(2 - B)}{2 + (A - B)}, \\ & \text{if } \frac{2 - B}{2 + (A - B)} \leq \mu \leq \frac{1 - B}{A - B}, \\ & \text{if } \mu \geq \frac{1 - B}{A - B}. \end{cases} \quad (13)$$

These results are sharp.

Proof. By definition of $C^*(A, B)$,

$$f(z) = g(z) \frac{1 + Aw(z)}{1 + Bw(z)}, \quad g \in S^*, w(z) \in \Omega. \quad (14)$$

Identifying terms in the above expansion,

$$\begin{aligned} a_2 &= b_2 + (A - B)d_1, \\ a_3 &= b_3 + (A - B)b_2d_1 + (A - B)(d_2 - Bd_1^2). \end{aligned} \quad (15)$$

From (15) and by using (2), it is easily established that

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq (A - B) + |b_3 - \mu b_2^2| + (A - B)|1 - 2\mu|xy \\ &\quad + (A - B)(|B + \mu(A - B)| - 1)x^2, \end{aligned} \quad (16)$$

where $x = |d_1| \leq 1$ and $y = |b_2| \leq 2$.

Two cases arise: (a) $\mu < -B/(A - B)$ and (b) $\mu \geq -B/(A - B)$.

Consider the case $\mu < -B/(A - B)$.

Case I. Suppose that $\mu \leq 1/2$ so that $A + B \leq 0$.

By Lemma 1, (16) can be written as

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq [3 + (A - B)] - 4\mu + 2(A - B) \\ &\quad \times (1 - 2\mu)x - (A - B)[(1 + B) + \mu(A - B)]x^2 \\ &= H_1(x), \quad \text{say,} \\ H_1'(x) &= 2(A - B)(1 - 2\mu) - 2(A - B) \\ &\quad \times [(1 + B) + \mu(A - B)]x, \\ H_1''(x) &= -2(A - B)[(1 + B) + \mu(A - B)], \end{aligned} \quad (17)$$

$H_1'(x) = 0$ implies that $x = (1 - 2\mu)/((1 + B) + \mu(A - B)) = x_1$.
Subcase I(i). For $\mu < 0$, $H_1'(x) > 0$ provided $\mu \leq -(1 + B)/(A - B) < -B/(A - B)$.

Where $H_1(x)$ is an increasing function in $[0, 1]$ and $\max H_1(x) = H_1(1)$.

Subcase I(ii). Suppose $\mu \geq 0$ and $-(1 + B)/(A - B) \leq \mu \leq -B/(2 + (A - B))$.

If $\mu < -B/(2 + (A - B))$, then $H_1'(x) > 0$ and $\max H_1(x) = H_1(1)$.

Combining the above two subcases, we obtain the first result of (13). For $x_1 > 1$, $H_1''(x) < 0$ which holds for $\mu < -(1 + B)/(A - B)$.

For $-(1 + B)/(A - B) \leq \mu \leq -B/(2 + (A - B))$ ($B < 0$) or $-B/(2 + (A - B)) \leq \mu \leq 1/2$ (for all A and B), $\max H_1(x) = H_1(x_1)$ from which the second result of (13) follows.

We now consider the case $\mu \geq -B/(A - B)$.

Case II. Suppose that $1/2 \leq \mu \leq 1$. Then by Lemma 1, (16) takes the following form:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq 1 + (A - B) + 2(A - B)(2\mu - 1)x \\ &\quad - (A - B)[(1 - B) - \mu(A - B)]x^2 \\ &= H_2(x), \end{aligned} \quad (18)$$

Subcase II(i). For $1/2 \leq \mu \leq (2 - B)/(2 + (A - B))$,

$$\begin{aligned} H_2'(x) &= 2(A - B)(2\mu - 1) \\ &\quad - 2(A - B)[(1 - B) - \mu(A - B)]x, \end{aligned} \quad (19)$$

$$H_2''(x) = -2(A - B)[(1 - B) - \mu(A - B)],$$

where $H_2'(x) = 0$ implies that $x = (2\mu - 1)/((1 - B) - \mu(A - B)) = x_2$.

For $\mu \leq (2 - B)/(2 + (A - B))$, we have $x_2 < 1$ and $H_2'(x) > 0$. So $\max H_2(x) = H_2(x_2)$, which gives the third result of (13).

Subcase II(ii). For $(2 - B)/(2 + (A - B)) \leq \mu \leq (1 - B)/(A - B)$. For $x_2 \geq 1$, $(2 - B)/(2 + (A - B)) \leq \mu \leq (1 - B)/(A - B)$, $H_2'(x) > 0$, and $\max H_2(x) = H_2(1)$, which leads us to the fourth result of (13).

Case III. For $\mu \geq (1 - B)/(A - B) \geq 1$. By Lemma 1, (16) can be put in the following form:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq (A - B) + (4\mu - 3) + 2(A - B)(2\mu - 1)x \\ &\quad - (A - B)[(1 - B) - \mu(A - B)]x^2 \\ &= H_3(x), \quad \text{say,} \\ H_3'(x) &= 2(A - B)(2\mu - 1) - 2(A - B) \\ &\quad \times [(1 - B) - \mu(A - B)]x, \\ H_3''(x) &= -2(A - B)[(1 - B) - \mu(A - B)], \end{aligned} \quad (20)$$

where $H_3'(x) = 0$ shows that $x = (2\mu - 1)/((1 - B) - \mu(A - B)) = x_3$.

If $x_3 > 1$, then $(2 - B)/(2 + (A - B)) < \mu$, which is true.

$H_3'(x) > 0$ for $\mu \geq (1 - B)/(A - B) \geq 1$. So $\max H_3(x) = H_3(1)$, which takes us straight to the fifth result of (13).

The equality holds in the first and fifth results of (13) for the function $f_1(z)$ defined by

$$f_1(z) = \frac{z}{(1 - z)^2} \left(\frac{1 + Az}{1 + Bz} \right). \quad (21)$$

The equality holds in the second result of (13) for the function $f_2(z)$ defined by

$$f_2(z) = \frac{z}{(1 - z)^2} \left(\frac{1 + (A + x_1)z + Az^2}{1 + (B + x_1)z + Bz^2} \right), \quad (22)$$

where $x_1 = (1 - 2\mu)/((1 + B) + \mu(A - B))$.

The equality holds in the third result of (13) for the function $f_3(z)$ defined by

$$f_3(z) = \frac{z}{(1-z)^2} \left(\frac{1 + (A + x_2)z + Az^2}{1 + (B + x_2)z + Bz^2} \right), \quad (23)$$

where $x_2 = (2\mu - 1)/((1 - B) - \mu(A - B))$.

The equality holds in the fourth result of (13) for the function $f_4(z)$ defined by

$$f_4(z) = \left(\int_0^z \left[1 + \left(\frac{M^2 - 2L}{M} \right) z \right]^{M^2/(M^2 - 2L)} dz \right), \quad (24)$$

where $L = 3[(A - B)(2 - B) - 1]$ and $M = 2[(A - B)(4 + A - B)]^{1/2}$ for $|(M^2 - 2L)/M| \leq 1$.

Proof of Theorem 3 is completed. $\square$

For $A = 1$ and $B = -1$, Theorem 3 gives the following result.

Corollary 4. *If $f(z) \in C^*$, then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 9 - 16\mu & \text{if } \mu \leq \frac{1}{4}, \\ 1 + \frac{1}{\mu} & \text{if } \frac{1}{4} \leq \mu \leq \frac{1}{2}, \\ 3 + \frac{(2\mu - 1)^2}{1 - \mu} & \text{if } \frac{1}{2} \leq \mu \leq \frac{3}{4}, \\ 12\mu - 5 & \text{if } \frac{3}{4} \leq \mu \leq 1, \\ 16\mu - 9 & \text{if } \mu \geq 1. \end{cases} \quad (25)$$

Theorem 5. *Let $f(z) \in C_1^*(A, B)$. Then,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} [1 + (A - B)(1 - B)] - [1 + A - B]^2 \mu & \text{if } \mu \leq \frac{-(1 + 2B)}{2(1 + A - B)} (A + B > 0), \\ [1 + (A - B) - \mu] + \frac{(A - B)(1 - 2\mu)^2}{4[(1 + B) + \mu(A - B)]} & \text{if } \frac{-(1 + 2B)}{2(1 + A - B)} \leq \mu \leq \frac{-B}{A - B} \\ \text{or } \frac{-(1 + 2B)}{2(1 + A - B)} \leq \mu \leq \frac{1}{2}, \\ [1 + (A - B) - \mu] + \frac{(A - B)(2\mu - 1)^2}{4[(1 - B) - \mu(A - B)]} & \text{if } \frac{-B}{A - B} \leq \mu \leq \frac{2}{3}, \\ \frac{1}{3}[1 + 3(A - B)] + \frac{(A - B)(2\mu - 1)^2}{4[(1 - B) - \mu(A - B)]} & \text{if } \frac{2}{3} \leq \mu \leq \frac{3 - 2B}{2(1 + A - B)}, \\ \frac{1}{3} - (A - B)(1 - B) + (A - B)(2 + A - B)\mu & \text{if } \frac{3 - 2B}{2(1 + A - B)} \leq \mu \leq \frac{4}{3}, \\ [1 + A - B]^2 \mu - [1 + (A - B)(1 - B)] & \text{if } \mu \geq \frac{4}{3}. \end{cases} \quad (26)$$

The results are sharp.

Proof. Proceeding as in Theorem 3, we have

$$|a_3 - \mu a_2^2| \leq (A - B) + |c_3 - \mu c_2^2| + (A - B)|1 - 2\mu|xy + (A - B)(|B + \mu(A - B)| - 1)x^2, \quad (27)$$

where $x = |d_1| \leq 1$ and $y = |c_2| \leq 1$.

Two cases arise: (a) $\mu < -B/(A - B)$, (b) $\mu \geq -B/(A - B)$.

Consider the case $\mu < -B/(A - B)$.

By Lemma 2, (27) can be written as

$$|a_3 - \mu a_2^2| \leq [1 + (A - B)] - \mu + (A - B)|1 - 2\mu|x - (A - B)[(1 + B) + \mu(A - B)]x^2. \quad (28)$$

Subcase I(i). For $\mu < 0$,

$$|a_3 - \mu a_2^2| \leq [1 + (A - B)] - \mu + (A - B)(1 - 2\mu)x - (A - B) \times [(1 + B) + \mu(A - B)]x^2 = H_1(x), \quad (29)$$

$$H_1'(x) = (A - B)(1 - 2\mu) - 2(A - B)[(1 + B) + \mu(A - B)]x,$$

$$H_1''(x) = -2(A - B)[(1 + B) + \mu(A - B)].$$

For $\mu < 0$, $H_1'(x) > 0$ which shows that $H_1(x)$ is an increasing function in $[0, 1]$ and $\max H_1(x) = H_1(1)$.

Subcase I(ii). Suppose $\mu \geq 0$.

$H'_1(x) = 0$ implies that $x = (1 - 2\mu)/(2[(1 + B) + \mu(A - B)]) = x_1$.

For $x_1 \geq 1$, $-(1 + B)/(A - B) \leq \mu \leq -(1 + 2B)/(2(1 + (A - B)))$. So $H'_1(x) > 0$ and $\max H_1(x) = H_1(1)$.

Combining the above two subcases, we obtain the first result of (26).

For $x_1 > 1$, $H''_1(x) < 0$, which holds for $-(1 + 2B)/(2(1 + (A - B))) \leq \mu \leq -B/(A - B)$.

So $\max H_1(x) = H_1(x_1)$, which leads us to the second result of (26).

We now consider the case $\mu \geq -B/(A - B)$.

Subcase I(iii). For $1/2 \leq -B/(A - B) \leq \mu \leq (2/3)$ ($B < 0, 2A + B > 0$), (27) takes the following form:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq 1 + (A - B) - \mu + (A - B)(2\mu - 1)x - (A - B) \\ &\quad \times [(1 - B) - \mu(A - B)]x^2 = H_2(x), \\ H'_2(x) &= (A - B)(2\mu - 1) \\ &\quad - 2(A - B)[(1 - B) - \mu(A - B)]x, \\ H''_2(x) &= -2(A - B)[(1 - B) - \mu(A - B)], \end{aligned} \quad (30)$$

where $H'_2(x) = 0$ implies that $x = (2\mu - 1)/(2[(1 - B) - \mu(A - B)]) = x_2$.

It is easy to verify that $H'_2(x) < 0$. So $\max H_2(x) = H_2(x_2)$, which gives the third result of (26).

Case II. For $2/3 \leq \mu \leq 4/3$.

Subcase II(i). For $2/3 \leq \mu \leq (3 - 2B)/(2(1 + A - B))$, (27) reduces to

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{3} + (A - B) + (A - B)(2\mu - 1)x \\ &\quad - (A - B)[(1 - B) - \mu(A - B)]x^2 = H_3(x), \\ H'_3(x) &= (A - B)(2\mu - 1) \\ &\quad - 2(A - B)[(1 - B) - \mu(A - B)]x, \\ H''_3(x) &= -2(A - B)[(1 - B) - \mu(A - B)], \end{aligned} \quad (31)$$

where $H'_3(x) = 0$ implies that $x = (2\mu - 1)/(2[(1 - B) - \mu(A - B)]) = x_3$.

For $x_3 < 1$, $(2/3) \leq \mu \leq (3 - 2B)/(2(1 + A - B))$, $H''_3(x) < 0$, and $\max H_3(x) = H_3(x_3)$, which leads us to the fourth result of (26).

Subcase II(ii). For $(3 - 2B)/(2(1 + A - B)) \leq \mu \leq 1$, (27) reduces to the following form:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{3} + (A - B) + (A - B)(2\mu - 1)x \\ &\quad - (A - B)[(1 - B) - \mu(A - B)]x^2 = H_4(x), \\ H'_4(x) &= (A - B)(2\mu - 1) \\ &\quad - 2(A - B)[(1 - B) - \mu(A - B)]x; \end{aligned} \quad (32)$$

$H'_4(x) = 0$ implies that $x = (2\mu - 1)/(2[(1 - B) - \mu(A - B)])$. $\max H_4(x) = H_4(1)$.

Combining the above two subcases, $\max H_4(x) = H_4(1)$, which take us straight to the fifth result of (26).

Case III. For $\mu \geq 4/3$. By Lemma 1, (27) can be put in the form

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq (A - B) - 1 + \mu + (A - B)(2\mu - 1)x \\ &\quad - (A - B)[(1 - B) - \mu(A - B)]x^2 \\ &= H_5(x), \\ H'_5(x) &= (A - B)(2\mu - 1) \\ &\quad - 2(A - B)[(1 - B) - \mu(A - B)]x, \end{aligned} \quad (33)$$

where $H'_5(x) = 0$ shows that $x = (2\mu - 1)/(2[(1 - B) - \mu(A - B)]) = x_5$.

$x_5 > 1$ and $\max H_5(x) = H_5(1)$, which yield the sixth result of (26).

The equality holds in the first and sixth results of (26) for the function $f_1(z)$ defined by

$$f_1(z) = \frac{z}{1 - z} \left(\frac{1 + Az}{1 + Bz} \right). \quad (35)$$

The equality holds in the second result of (26) for the function $f_2(z)$ defined by

$$f_2(z) = \frac{z}{1 - z} \left(\frac{1 + (A + x_1)z + Az^2}{1 + (B + x_1)z + Bz^2} \right), \quad (36)$$

where $x_1 = (1 - 2\mu)/(2[(1 + B) + \mu(A - B)])$.

The equality holds in the third and fourth results of (26) for the function $f_3(z)$ defined by

$$f_3(z) = \frac{z}{1 - z} \left(\frac{1 + (A + x_2)z + Az^2}{1 + (B + x_2)z + Bz^2} \right), \quad (37)$$

where $x_2 = (2\mu - 1)/(2[(1 - B) - \mu(A - B)])$.

The equality holds in the fifth result of (26) for the function $f_4(z)$ defined by

$$f_4(z) = \left(\int_0^z \left[1 + \left(\frac{M^2 - 2L}{M} \right) z \right]^{M^2/(M^2 - 2L)} dz \right), \quad (38)$$

where $L = 1 - 3(A - B)(1 - B)$ and $M = 2[(A - B)(2 + A - B)]^{1/2}$ for $|(M^2 - 2L)/M| \leq 1$.

Proof of Theorem 5 is completed. $\square$

For $A = 1$, $B = -1$, Theorem 5 gives the following result.

Corollary 6. *If $f(z) \in C_1^*$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 5 - 9\mu & \text{if } \mu \leq \frac{1}{6}, \\ 2 + \frac{1}{4\mu} & \text{if } \frac{1}{6} \leq \mu \leq \frac{1}{2}, \\ 3 - \mu + \frac{(2\mu - 1)^2}{4(1 - \mu)} & \text{if } \frac{1}{2} \leq \mu \leq \frac{2}{3}, \\ \frac{7}{3} + \frac{(2\mu - 1)^2}{4(1 - \mu)} & \text{if } \frac{2}{3} \leq \mu \leq \frac{5}{6}, \\ 8\mu - \frac{11}{3} & \text{if } \frac{5}{6} \leq \mu \leq \frac{4}{3}, \\ 9\mu - 5 & \text{if } \mu \geq \frac{4}{3}. \end{cases} \quad (39)$$

The results of Corollaries 4 and 6 were proved by the authors of [13].

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