Research Article

On an R-Randers m th-Root Space

P. N. Pandey and Shivalika Saxena

Department of Mathematics, University of Allahabad, Allahabad 211002, India

Correspondence should be addressed to Shivalika Saxena; mathshivalika@gmail.com

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We consider an n -dimensional Finsler space F^n ($n > 2$) with the metric $L(x, y) = F(x, y) + \alpha(x, y)$, where F is an m th-root metric and α is a Riemannian metric. We call such space as an R-Randers m th-root space. We obtain the expressions for the fundamental metric tensor, Cartan tensor, geodesic spray coefficients, and the coefficients of nonlinear connection in an R-Randers m th-root space. Some other properties of such space have also been discussed.

1. Introduction

The theory of an m th-root metric was introduced by Shimada [1] in 1979. By introducing the regularity of the metric, various fundamental quantities of a Finsler metric could be found. In particular, the Cartan connection of a Finsler space with m th-root metric was introduced from the theoretical standpoint. Matsumoto and Okubo [2] studied Berwald connection of a Finsler space with m th-root metric and gave main scalars in two dimensional cases and defined higher-order Christoffel symbols. The m th-root metric is used in many problems of theoretical physics [3]. Pandey et al. [4] studied three-dimensional Finsler space with m th-root metric. To discuss general relativity with the electromagnetic field, Randers [5] introduced a metric of the form $L(x, y) = \alpha(x, y) + \beta(x, y)$, where α is a square root metric and β is a differential one form. In his honor, this metric is called Randers metric, and it has been extensively studied by several geometers and physicists [6–8].

Munteanu and Purcaru [9] first defined complex Finsler spaces by reducing the scalar $\lambda \in \mathbb{R}$ in the homogeneity condition of fundamental function, that is, $F(z, \lambda\eta, \bar{z}, \bar{\lambda}\bar{\eta}) = |\lambda|F(z, \eta, \bar{z}, \bar{\eta})$, $\forall \lambda \in \mathbb{R}$, and named such space as an R-complex Finsler space. Aldea and Purcaru [10] introduced the concept of R-complex Finsler spaces with (α, β) -metrics. In 2011, Purcaru [11] also discussed the notion of R-complex Finsler space with Kropina metric. Lungu and Nimineş [12] studied a special Finsler space with the metric L of the form

$L(x, y) = F(x, y) + \alpha(x, y)$, $\forall y \in TM$, where F is a quartic root metric and α is a square root metric. They regarded this space as an R-Randers quartic space and obtained many results related to it. The aim of the present paper is to study a more general space with the metric $L(x, y) = F(x, y) + \alpha(x, y)$, where F is an m th-root metric and α is a Riemannian metric. We call the space endowed with this metric as an R-Randers m th-root space.

The paper is organized as follows. Section 2 deals with some preliminary concepts required for the discussion of the following sections. It includes the notion of an R-Randers m th-root space. In Section 3, we derive certain identities satisfied in an R-Randers m th-root space. We obtain the fundamental metric tensor g_{ij} , its inverse g^{ij} , and the Cartan tensor C_{ijk} for an R-Randers m th-root space. In Section 4, we obtain the spray coefficients of an R-Randers m th-root space. It includes the equations of the geodesics. Section 5 discusses the nonlinear connection in an R-Randers m th-root space.

2. Preliminaries

Let $F^n = (M, L(x, y))$ ($n > 2$) be an n -dimensional Finsler space. The m th-root metric on M is defined as $L^m = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, where $a_{i_1 i_2 \dots i_m}(x)$ are components of an m th-order covariant symmetric tensor.

In case of $m = 2$, the metric L is Riemannian, and in the cases $m = 3$ and $m = 4$ these metrics are called cubic and quartic, respectively.

The covariant symmetric metric tensor g_{ij} of $F^n(M, L(x, y))$ is defined by

$$g_{ij} =: \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad \dot{\partial}_i \equiv \frac{\partial}{\partial \dot{x}^i}. \quad (1)$$

This tensor is positively homogeneous of degree zero in $\dot{x}^i$. From the metric tensor g_{ij} , we construct the Cartan tensor C_{ijk} by

$$C_{ijk} =: \frac{1}{2} \dot{\partial}_k g_{ij}. \quad (2)$$

The tensor C_{ijk} is symmetric in its lower indices and is positively homogeneous of degree -1 in $\dot{x}^i$. Due to its homogeneous and symmetric properties, it satisfies the following:

$$C_{ijk} \dot{x}^i = C_{kij} \dot{x}^i = C_{jki} \dot{x}^i = 0. \quad (3)$$

The angular metric tensor of a Finsler space is written as

$$h_{ij} = g_{ij} - l_i l_j, \quad (4)$$

where $l_i = \dot{\partial}_i L$.

The geodesic of a Finsler space is given by

$$\frac{d^2 x^i}{dt^2} + 2G^i = 0, \quad (5)$$

where G^i is the geodesic spray coefficients given by

$$2G^i = \frac{1}{2} g^{ij} \{ y^k \dot{\partial}_j \dot{\partial}_k L^2 - \dot{\partial}_j L^2 \}, \quad \dot{\partial}_j \equiv \frac{\partial}{\partial \dot{x}^j}. \quad (6)$$

The nonlinear connection of a Finsler space is defined as

$$N_j^i = \dot{\partial}_j G^i. \quad (7)$$

In the present paper, we study the space whose fundamental function is given by

$$L(x, y) = F(x, y) + \alpha(x, y), \quad (8)$$

where

$$F^m = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m} \quad (9)$$

is an m th-root metric and

$$\alpha^2 = b_{i_1 i_2}(x) y^{i_1} y^{i_2} \quad (10)$$

is a Riemannian metric. We call this space an R-Randers m th-root space.

3. Fundamental Metric Tensor and Cartan Tensor

In this section, we find the fundamental metric tensor g_{ij} , its inverse g^{ij} , angular metric tensor h_{ij} , and the Cartan tensor C_{ijk} for an R-Randers m th-root space.

Differentiating (9) partially with respect to $\dot{x}^i$, we get

$$F^{m-1}(\dot{\partial}_i F) = a_i, \quad (11)$$

where $a_i(x, y) = a_{i i_2 \dots i_m}(x) y^{i_2} y^{i_3} \dots y^{i_m}$.

Differentiating (11) partially with respect to $\dot{x}^j$, we find

$$(m-1) F^{m-2}(\dot{\partial}_j F)(\dot{\partial}_i F) + F^{m-1}(\dot{\partial}_j \dot{\partial}_i F) = (m-1) a_{ij}, \quad (12)$$

where $a_{ij}(x, y) = a_{i j i_3 \dots i_m}(x) y^{i_3} y^{i_4} \dots y^{i_m}$.

Take $\rho_{-(m-1)} = (m-1) F^{-(m-1)}$ and $\rho_{-(2m-1)} = (m-1) F^{-(2m-1)}$, where the subscripts denote the degree of homogeneity of the corresponding entities with respect to $\dot{x}^i$. In view of this, (11) and (12) give

$$\dot{\partial}_i F = \frac{\rho_{-(m-1)}}{(m-1)} a_i, \quad (13)$$

$$\dot{\partial}_j \dot{\partial}_i F = \rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j, \quad (14)$$

where

$$\dot{\partial}_j \rho_{-(m-1)} = -(m-1) \rho_{-(2m-1)} a_j. \quad (15)$$

Now, differentiating (10) partially with respect to $\dot{x}^i$, we get

$$\dot{\partial}_i \alpha = \mu_{-1} b_i, \quad (16)$$

where $b_i(x, y) = b_{i i_2}(x) y^{i_2}$ and $\mu_{-1} = \alpha^{-1}$.

Further differentiating (16) partially with respect to $\dot{x}^j$, we find

$$\dot{\partial}_j \dot{\partial}_i \alpha = \mu_{-1} b_{ij} + \mu_{-3} b_i b_j, \quad (17)$$

where $\mu_{-3} = -\alpha^{-3}$.

Differentiating $\mu_{-1} = \alpha^{-1}$ partially with respect to $\dot{x}^j$, we get

$$\dot{\partial}_j \mu_{-1} = \mu_{-3} b_j. \quad (18)$$

Thus, we have the following.

Proposition 1. *In an R-Randers m th-root space, the following identities hold:*

$$\dot{\partial}_i F = \frac{\rho_{-(m-1)}}{(m-1)} a_i,$$

$$\dot{\partial}_j \dot{\partial}_i F = \rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j,$$

$$\dot{\partial}_i \alpha = \mu_{-1} b_i, \quad (19)$$

$$\dot{\partial}_j \dot{\partial}_i \alpha = \mu_{-1} b_{ij} + \mu_{-3} b_i b_j,$$

$$\dot{\partial}_j \rho_{-(m-1)} = -(m-1) \rho_{-(2m-1)} a_j,$$

$$\dot{\partial}_j \mu_{-1} = \mu_{-3} b_j.$$

Differentiating (8) partially with respect to $\dot{x}^i$ and using (13) and (16), we have

$$\dot{\partial}_i L = \frac{\rho_{-(m-1)}}{(m-1)} a_i + \mu_{-1} b_i. \quad (20)$$

Further, differentiating (20) partially with respect to $\dot{x}^j$ and using (14) and (17), we get

$$\begin{aligned} \dot{\partial}_j \dot{\partial}_i L &= \rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j \\ &\quad + \mu_{-1} b_{ij} + \mu_{-3} b_i b_j. \end{aligned} \quad (21)$$

From (1), we have

$$g_{ij} = (\dot{\partial}_i L)(\dot{\partial}_j L) + L(\dot{\partial}_i \dot{\partial}_j L). \quad (22)$$

Using (20) and (21), (22) yields

$$\begin{aligned} g_{ij} &= \left(\frac{\rho_{-(m-1)}}{(m-1)} a_i + \mu_{-1} b_i \right) \left(\frac{\rho_{-(m-1)}}{(m-1)} a_j + \mu_{-1} b_j \right) \\ &\quad + L(\rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j + \mu_{-1} b_{ij} + \mu_{-3} b_i b_j). \end{aligned} \quad (23)$$

Let us take $d_{ij} = \rho_{-(m-1)} a_{ij} + \mu_{-1} b_{ij}$ and $c_i = q_0 a_i + q_{-1} b_i$, where q_0 and q_{-1} satisfy

$$\begin{aligned} q_0^2 &= \left(\frac{\rho_{-(m-1)}^2}{(m-1)^2} - L\rho_{-(2m-1)} \right), \\ q_{-1}^2 &= (\mu_{-1}^2 + L\mu_{-3}), \\ q_0 q_{-1} &= \frac{\rho_{-(m-1)}}{(m-1)} \mu_{-1}. \end{aligned} \quad (24)$$

The matrix (d_{ij}) is nonsingular, in general, for a_{ij} 's are functions of x and y whereas b_{ij} 's are functions of x only.

Equation (23) takes the form

$$g_{ij} = Ld_{ij} + c_i c_j. \quad (25)$$

Thus, we have the following.

Theorem 2. *The fundamental metric tensor of an R-Randers m th-root space is given by (25).*

Theorem 3. *In an R-Randers m th-root space, the inverse g^{ij} of the fundamental metric tensor g_{ij} is given by*

$$g^{ij} = \frac{1}{L} \left(d^{ij} - \frac{1}{L+c^2} c^i c^j \right), \quad (26)$$

where $c^i = d^{ij} c_j$ and $c^2 = c_i c^i$.

Proof. Let (d^{ij}) be the inverse of nonsingular matrix (d_{ij}) . Suppose that (g^{ij}) is given by (26).

Now,

$$\begin{aligned} g_{ij} g^{jk} &= (Ld_{ij} + c_i c_j) \frac{1}{L} \left(d^{jk} - \frac{1}{L+c^2} c^j c^k \right) \\ &= d_{ij} d^{jk} - \frac{d_{ij}}{L+c^2} c^j c^k + \frac{1}{L} c_i c_j d^{jk} \\ &\quad - \frac{1}{L(L+c^2)} c_i c_j c^j c^k \\ &= \delta_k^i - \frac{c_i c^k}{L+c^2} + \frac{1}{L} c_i c^k - \frac{1}{L(L+c^2)} c^2 c_i c^k \\ &= \delta_k^i. \end{aligned} \quad (27)$$

Therefore, (g^{ij}) given by (26) is the inverse of the matrix (g_{ij}) . This also shows that (g_{ij}) is nondegenerate. $\square$

Using (20) and (23) in (4), we get the angular metric tensor of an R-Randers m th-root space:

$$h_{ij} = L(\rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j + \mu_{-1} b_{ij} + \mu_{-3} b_i b_j). \quad (28)$$

Thus, we have the following.

Theorem 4. *In an R-Randers m th-root space, the angular metric tensor h_{ij} is given by (28).*

Differentiating $\rho_{-(2m-1)} = (m-1)F^{-(2m-1)}$ partially with respect to $\dot{x}^k$, we get

$$\dot{\partial}_k \rho_{-(2m-1)} = -(2m-1) \rho_{-(3m-1)} a_k, \quad (29)$$

where $\rho_{-(3m-1)} = (m-1)F^{-(3m-1)}$.

Also, we have

$$\dot{\partial}_k \mu_{-3} = 3\mu_{-5} b_k, \quad (30)$$

where $\mu_{-5} = \alpha^{-5}$.

Thus, we have the following.

Proposition 5. *In an R-Randers m th-root space, the following holds good:*

$$\begin{aligned} \dot{\partial}_k \rho_{-(2m-1)} &= -(2m-1) \rho_{-(3m-1)} a_k, \\ \dot{\partial}_k \mu_{-3} &= 3\mu_{-5} b_k. \end{aligned} \quad (31)$$

Differentiating (23) partially with respect to $\dot{x}^k$, we get

$$\begin{aligned} \dot{\partial}_k g_{ij} &= 2\rho_{-(m-1)} (\dot{\partial}_k \rho_{-(m-1)}) a_i a_j \\ &\quad + (\rho_{-(m-1)})^2 (a_j \dot{\partial}_k a_i + a_i \dot{\partial}_k a_j) \\ &\quad + [(\dot{\partial}_k \rho_{-(m-1)}) \mu_{-1} + \rho_{-(m-1)} (\dot{\partial}_k \mu_{-1})] (b_i a_j + b_j a_i) \\ &\quad + \rho_{-(m-1)} \mu_{-1} [a_j \dot{\partial}_k b_i + b_i \dot{\partial}_k a_j + a_i \dot{\partial}_k b_j + b_j \dot{\partial}_k a_i] \end{aligned}$$

$$\begin{aligned}
& + 2\mu_{-1} \left(\dot{\partial}_k \mu_{-1} \right) b_i b_j + \mu_{-1}^2 \left[b_j \dot{\partial}_k b_i + b_i \dot{\partial}_k b_j \right] \\
& + \left(\dot{\partial}_k L \right) \left[\rho_{-(m-1)} a_{ij} - \rho_{-(2m-1)} a_i a_j \right. \\
& \quad \left. + \mu_{-1} b_{ij} + \mu_{-3} b_i b_j \right] \\
& + L \left[\left(\dot{\partial}_k \rho_{-(m-1)} \right) a_{ij} + \left(\dot{\partial}_k a_{ij} \right) \rho_{-(m-1)} \right. \\
& \quad \left. - \left(\dot{\partial}_k \rho_{-(2m-1)} \right) a_i a_j - \rho_{-(2m-1)} \right. \\
& \quad \times \left(\left(\dot{\partial}_k a_i \right) a_j + \left(\dot{\partial}_k a_j \right) a_i \right) + \left(\dot{\partial}_k \mu_{-1} \right) b_{ij} \\
& \quad + \left(\dot{\partial}_k b_{ij} \right) \mu_{-1} + \left(\dot{\partial}_k \mu_{-3} \right) b_i b_j \\
& \quad \left. + \mu_{-3} \left(\dot{\partial}_k b_i \right) b_j + \left(\dot{\partial}_k b_j \right) b_i \right].
\end{aligned} \tag{32}$$

Partial differentiation of a_i and b_i , with respect to $\dot{x}^k$, yields

$$\dot{\partial}_k a_i = (m-1) a_{ik}, \quad \dot{\partial}_k b_i = b_{ik}. \tag{33}$$

Further, differentiation of a_{ij} and b_{ij} with respect to $\dot{x}^k$ gives

$$\dot{\partial}_k a_{ij} = (m-2) a_{ijk}, \quad \dot{\partial}_k b_{ij} = 0, \tag{34}$$

where $a_{ijk}(x, y) = a_{ijk i_1 \dots i_m}(x) y^{i_1} \dots y^{i_m}$.

Thus, we have the following.

Proposition 6. *In an R-Randers m th-root space the following holds good:*

$$\begin{aligned}
\dot{\partial}_k a_i &= (m-1) a_{ik}, \\
\dot{\partial}_k a_{ij} &= (m-2) a_{ijk}, \\
\dot{\partial}_k b_i &= b_{ik}, \quad \dot{\partial}_k b_{ij} = 0.
\end{aligned} \tag{35}$$

If we use (2), (15), (18), (20), (29), (30), (33), and (34) in (32), on simplification it follows that

$$\begin{aligned}
2C_{ijk} &= \rho_{-(2m-2)} a_k a_{ij} \\
&+ \xi_{-(2m-2)} \left(a_j a_{ik} + a_i a_{jk} \right) \\
&+ \mu_{-2} \left(b_i b_{jk} + b_j b_{ik} + b_k b_{ij} \right) \\
&+ \mu_{-4} b_i b_j b_k + \rho_{-(3m-2)} a_i a_j a_k \\
&+ \rho_{-(m-2)} a_{ijk} + \rho_{-(m-1)} \mu_{-1} \\
&\times \left\{ \frac{1}{m-1} a_k b_{ij} + b_k a_{ij} + a_j b_{ik} \right. \\
&\quad \left. + (m-1) \left(b_i a_{jk} + b_j a_{ik} \right) + a_i b_{jk} \right\} \\
&- (m-1) \rho_{-(2m-1)} \mu_{-1} \left(b_i a_j + b_j a_i \right) a_k \\
&+ \rho_{-(m-1)} \mu_{-3} \left(b_i a_j + b_j a_i \right) b_k,
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\rho_{-(2m-2)} &= \frac{\rho_{-(m-1)}^2}{m-1} - L(m-1) \rho_{-(2m-1)}, \\
\xi_{-(2m-2)} &= (m-1) \left(\rho_{-(m-1)}^2 - L \rho_{-(2m-1)} \right), \\
\mu_{-2} &= \left(\mu_{-1}^2 + L \mu_{-3} \right), \\
\mu_{-4} &= \left(3L \mu_{-5} + 2\mu_{-1} \mu_{-3} \right), \\
\rho_{-(m-2)} &= L(m-2) \rho_{-(m-1)}, \\
\rho_{-(3m-2)} &= (2m-1) L \rho_{-(3m-1)} \\
&- 2(m-1) \rho_{-(m-1)} \rho_{-(2m-1)}.
\end{aligned} \tag{37}$$

Thus, we have the following.

Theorem 7. *In an R-Randers m th-root space, the Cartan tensor C_{ijk} is given by (36).*

4. Spray and Equation of Geodesics

In this section, we discuss the spray of an R-Randers m th-root space and obtain its local coefficients. We also obtain the equation of geodesics in such space.

If we differentiate (9) partially with respect to x^j , we get

$$\partial_j F = \frac{\rho_{-(m-1)}}{m(m-1)} A_j, \tag{38}$$

where

$$A_j = \left(\partial_j a_{i_1 \dots i_m} \right) y^{i_1} \dots y^{i_m}. \tag{39}$$

Further, differentiating (13) partially with respect to x^k and utilizing $\partial_k \rho_{-(m-1)} = -(m-1)/m \rho_{-(2m-1)} A_k$, we have

$$\partial_k \dot{\partial}_i F = -\frac{1}{m} \rho_{-(2m-1)} A_k a_i + \frac{\rho_{-(m-1)}}{(m-1)} \partial_k a_i. \tag{40}$$

Differentiating (10) partially with respect to x^j , we get

$$\partial_j \alpha = \frac{1}{2} \mu_{-1} B_j, \tag{41}$$

where

$$B_j = \left(\partial_j b_{i_1 i_2} \right) y^{i_1} y^{i_2}. \tag{42}$$

Also, we have

$$\partial_k \mu_{-1} = \frac{1}{2} \mu_{-3} B_k. \tag{43}$$

Next, differentiating (16) partially with respect to x^k and using (43), we obtain

$$\partial_k \dot{\partial}_i \alpha = \frac{1}{2} \mu_{-3} b_i B_k + \mu_{-1} \partial_k b_i. \tag{44}$$

Thus, we have the following.

Proposition 8. In an R -Randers m th-root space, the following holds good:

$$\begin{aligned}\partial_j F &= \frac{\rho_{-(m-1)}}{m(m-1)} A_j, & \partial_j \alpha &= \frac{1}{2} \mu_{-1} B_j, \\ \partial_k \dot{\partial}_i F &= -\frac{1}{m} \rho_{-(2m-1)} A_k a_i + \frac{\rho_{-(m-1)}}{(m-1)} \partial_k a_i, \\ \partial_k \dot{\partial}_i \alpha &= \frac{1}{2} \mu_{-3} b_i B_k + \mu_{-1} \partial_k b_i.\end{aligned}\quad (45)$$

If we differentiate (8) partially with respect to x^k and use (38) and (41), it follows that

$$\partial_k L = \frac{\rho_{-(m-1)}}{m(m-1)} A_k + \frac{1}{2} \mu_{-1} B_k. \quad (46)$$

Next, differentiating (20) partially with respect to x^k and using (40) and (44), we get

$$\begin{aligned}\partial_k \dot{\partial}_i L &= -\frac{1}{m} \rho_{-(2m-1)} A_k a_i + \frac{\rho_{-(m-1)}}{(m-1)} \partial_k a_i \\ &\quad + \frac{1}{2} \mu_{-3} b_i B_k + \mu_{-1} \partial_k b_i.\end{aligned}\quad (47)$$

In view of (46) and (47), (6) gives

$$\begin{aligned}G^i &= \frac{1}{4} g^{ij} \left[\frac{2}{m} \left(\frac{\rho_{-(m-1)}^2}{(m-1)^2} - L \rho_{-(2m-1)} \right) a_j A_k y^k \right. \\ &\quad + (\mu_{-1}^2 + L \mu_{-3}) b_j B_k y^k \\ &\quad + \frac{\rho_{-(m-1)}}{(m-1)} \mu_{-1} (2b_j A_k + a_j B_k) y^k \\ &\quad + 2L \frac{\rho_{-(m-1)}}{(m-1)} (\partial_k a_j) y^k \\ &\quad + 2L \mu_{-1} (\partial_k b_j) y^k \\ &\quad \left. - 2L \frac{\rho_{-(m-1)}}{(m-1)} A_j - L \mu_{-1} B_j \right],\end{aligned}\quad (48)$$

that is

$$\begin{aligned}G^i &= \frac{1}{4} g^{ij} \left[\frac{2}{m(m-1)} \rho_{-(2m-2)} a_j A_0 \right. \\ &\quad + \mu_{-2} b_j B_0 + \frac{1}{m-1} \rho_{-(m-1)} \mu_{-1} (2b_j A_0 + a_j B_0) \\ &\quad + \frac{2}{(m-1)(m-2)} \rho_{-(m-2)} ((\partial_k a_j) y^k - A_j) \\ &\quad \left. + \eta_{-2} (2(\partial_k b_j) y^k - B_j) \right],\end{aligned}\quad (49)$$

where

$$A_0 = A_k y^k, \quad B_0 = B_k y^k, \quad \eta_{-2} = L \mu_{-1}, \quad (50)$$

and g^{ij} is given by (26). Thus, we have the following.

Theorem 9. In an R -Randers m th-root space, the spray coefficients are given by (49).

In view of (5) and Theorem 9, we have the following.

Corollary 10. In an R -Randers m th-root space, the equation of geodesics is given by

$$\frac{d^2 x^i}{dt^2} + 2G^i = 0, \quad (51)$$

where the spray coefficients G^i are given by (49).

5. Nonlinear Connection

In this section, we obtain the coefficients of nonlinear connection of the space under consideration.

Differentiating (49) partially with respect to $\dot{x}^k$, we have

$$\begin{aligned}\dot{\partial}_k G^i &= \frac{1}{4} (\dot{\partial}_k g^{ij}) \left[\frac{2}{m(m-1)} \rho_{-(2m-2)} a_j A_0 + \mu_{-2} b_j B_0 \right. \\ &\quad + \frac{1}{m-1} \rho_{-(m-1)} \mu_{-1} (2b_j A_0 + a_j B_0) \\ &\quad + \frac{2}{(m-1)(m-2)} \rho_{-(m-2)} ((\partial_k a_j) y^j - A_j) \\ &\quad \left. + \eta_{-2} (2(\partial_k b_j) y^k - B_j) \right] \\ &\quad + \frac{1}{4} g^{ij} \left[\frac{2}{m(m-1)} \{ (\dot{\partial}_k \rho_{-(2m-2)}) a_j A_0 \right. \\ &\quad \left. + \rho_{-(2m-2)} (A_0 \dot{\partial}_k a_j + a_j \dot{\partial}_k A_0) \} \right. \\ &\quad + (\dot{\partial}_k \mu_{-2}) b_j B_0 + \mu_{-2} (b_j \dot{\partial}_k B_0 + B_0 \dot{\partial}_k b_j) \\ &\quad + \frac{1}{m-1} \{ (\dot{\partial}_k \rho_{-(m-1)}) \mu_{-1} \\ &\quad \left. + \rho_{-(m-1)} (\dot{\partial}_k \mu_{-1}) \} (2b_j A_0 + a_j B_0) \right. \\ &\quad + \frac{1}{m-1} \rho_{-(m-1)} \mu_{-1} \\ &\quad \times \{ 2(\dot{\partial}_k b_j) A_0 + 2b_j (\dot{\partial}_k A_0) \\ &\quad \left. + (\dot{\partial}_k a_j) B_0 + a_j (\dot{\partial}_k B_0) \} \right. \\ &\quad + \frac{2}{(m-1)(m-2)} \\ &\quad \times \{ (\dot{\partial}_k \rho_{-(m-2)}) ((\partial_l a_j) y^l - A_j) + \rho_{-(m-2)} \\ &\quad \times ((\dot{\partial}_k \partial_l a_j) y^l + (\partial_l a_j) (\dot{\partial}_k y^l) - \dot{\partial}_k A_j) \} \\ &\quad + (\dot{\partial}_k \eta_{-2}) (2(\partial_k b_j) y^k - B_j) + \eta_{-2} \\ &\quad \left. \times (2(\dot{\partial}_k \partial_k b_j) y^k + 2(\partial_k b_j) (\dot{\partial}_k y^k) - \dot{\partial}_k B_j) \right].\end{aligned}\quad (52)$$

Differentiation of $g^{ij}g_{jm} = \delta_m^i$ partially with respect to $\dot{x}^k$ yields

$$\dot{\partial}_k g^{il} = -2C_{km}^i g^{ml}, \quad (53)$$

where $C_{km}^i = C_{jkm} g^{ij}$.

Next, differentiating $\rho_{-(2m-2)} = (\rho_{-(m-1)}^2 / (m-1)) - L(m-1)\rho_{-(2m-1)}$ partially with respect to $\dot{x}^k$ and using (15), (20), and (29), we get

$$\begin{aligned} \dot{\partial}_k \rho_{-(2m-2)} = & (m-1) (-3\rho_{-(3m-2)} a_k - \rho_{-(2m-1)} \mu_{-1} b_k \\ & + (2m-1) L \rho_{-(3m-1)} a_k), \end{aligned} \quad (54)$$

where $\rho_{-(3m-2)} = (m-1)F^{-(3m-2)}$.

Differentiating $A_0 = A_l y^l$ and $B_0 = B_l y^l$ partially with respect to $\dot{x}^k$, we, respectively, have

$$\begin{aligned} \dot{\partial}_k A_0 &= m (\partial_l a_k) y^l + A_k, \\ \dot{\partial}_k B_0 &= 2 (\partial_l b_k) y^l + B_k. \end{aligned} \quad (55)$$

Also, we have

$$\begin{aligned} \dot{\partial}_k (\partial_l a_j) &= (m-1) \partial_l a_{jk}, \\ \dot{\partial}_k (\partial_l b_j) &= \partial_l b_{jk}, \\ \dot{\partial}_k A_j &= m \partial_j a_k, \\ \dot{\partial}_k B_j &= 2 \partial_j b_k. \end{aligned} \quad (56)$$

Using (7), (15), (18), (29), (30), (33), and (53)–(56) in (52), we have

$$\begin{aligned} N_k^i = & -2C_{km}^i G^m \\ & + \frac{1}{4} g^{ij} \left[\frac{2}{m} (-3\rho_{-(3m-2)} - (2m-1) \rho_{-(3m-1)}) a_k a_j A_0 \right. \\ & - \rho_{-(2m-1)} \mu_{-1} (b_k a_j A_0 + 2a_k b_j A_0 + a_k a_j B_0) \\ & + \frac{2\rho_{-(2m-2)}}{m(m-1)} (m (\partial_l a_k) y^l a_j \\ & \quad + A_k a_j + (m-1) A_{jk} A_0 \\ & \quad + (\partial_l a_j) y^l a_k - A_j a_k) \\ & + (3\mu_{-4} - \mu_{-1} \mu_{-3}) b_k b_j B_0 \\ & \left. + \frac{1}{(m-1)} \rho_{-(m-1)} \mu_{-3} (a_k b_j B_0 \right. \\ & \quad \left. + (2b_j A_0 + a_j B_0) b_k) \right] \end{aligned}$$

$$\begin{aligned} & + \mu_{-2} (b_{jk} B_0 + 2 (\partial_l b_k) y^l b_j \\ & \quad + B_k b_j + b_k (2 (\partial_l b_j) y^l - B_j)) \\ & + \frac{\rho_{-(m-1)} \mu_{-1}}{(m-1)} \{ 2b_{jk} A_0 + 2mb_j (\partial_l a_k) y^l \\ & \quad + 2b_j A_k + (m-1) a_{jk} B_0 \\ & \quad + 2a_j (\partial_l b_k) y^l + B_k a_j \\ & \quad + 2 (\partial_l b_j) y^l a_k - B_j a_k \\ & \quad + 2 (\partial_l a_j) y^l b_k - 2A_j b_k \} \\ & + \frac{2\rho_{-(m-2)}}{(m-1)(m-2)} \{ (m-1) (\partial_l a_{jk}) y^l \\ & \quad + (\partial_k a_j) - m (\partial_j a_k) \} \\ & + \eta_{-2} \{ 2 (\partial_l b_{jk}) y^l + 2 (\partial_k b_j) - (\partial_j b_k) \} \}. \end{aligned} \quad (57)$$

Thus, we have the following.

Theorem 11. *The local coefficients of the nonlinear connection of an R-Randers mth-root space are given by (57).*

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